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SYNCHRONIZATION OF CHUA'S OSCILLATORS WITH THE THIRD STATE AS THE DRIVING SIGNAL

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In this paper, we study the use of the *third* state as the driving signal to achieve synchronization of two Chua's oscillators. We prove that globally asymptotic synchronization can be attained via linear error feedback. Simulation results are presented to verify the operation of the design.

1. Introduction

Although research concerning the synchronization of chaotic systems is still at an early stage, the literature on the subject is growing at a fast pace, as documented by the surveys of Ogorzalek [1993] and Hasler [1997], and the bibliography by Chen [1997].

In most of the synchronization schemes, the driving signal can be taken as one of the state variables of the drive system. The driving signal is used to synchronize the response to the drive in the sense that the trajectory of the response tracks the trajectory of the drive. The synchronization stability is proven by numerically computing the conditional Lyapunov exponents of the response system [Pecora & Carroll, 1991] or finding a Lyapunov function [Wu & Chua, 1994].

A widely used system for generating a chaotic carrier signal is Chua's oscillator [Chua *et al.*, 1993a] described by Eq. (1). Many authors have considered the synchronization of two Chua's oscillators through linear feedback (coupling) [Chua *et al.*, 1993b; Wu & Chua, 1994; Chua *et al.*, 1996]. Wu and Chua proved that synchronization can be attained with x_1, the first state of the drive system as the driving signal when the feedback gain is large enough. However, the single nonlinearity in Chua's oscillator is a piecewise linear function which depends only on x_1. If x_1 is taken as the driving signal, one could find out in which linear region is the drive system at any time; this in turn would enable an unauthorized receiver to estimate system parameters with conventional linear short term system identification methods.

In the computer simulations, we can see that synchronization can be achieved with x_2, the second state of the drive system as the driving signal when the feedback gain is large enough. However, we were not able to prove this conjecture rigorously [Wu & Chua, 1994]. Schweizer *et al.* [1995] show that, if we can get x_2 and its derivative, $\dot{x}_2$ simultaneously at the response, then we can develop an error feedback synchronization scheme such that a global synchronization proof is possible. However, a large error can potentially occur when one calculates the derivative of x_2.

To the best of our knowledge, no one have used x_3, the third state as the driving signal. In this work, we consider the linear feedback method for the synchronization of two Chua's oscillators with x_3 as the driving signal. The feedback vector depends on a free parameter. We prove that one can achieve globally asymptotic synchronization when the parameter is large enough.

2. Main Result

We consider the drive Chua's oscillator described by the state equation

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} \alpha(x_2 - x_1 + f(x_1)) \\ x_1 - x_2 + x_3 \\ -\beta x_2 - \gamma x_3 \end{pmatrix},$$

$$f(x_1) = \begin{cases} -bx_1 - a + b & x_1 > 1 \\ -ax_1 & |x_1| \le 1 \\ -bx_1 + a - b & x_1 < -1 \end{cases}. \quad (1)$$

This equation represents a piecewise linear third order autonomous continuous-time dynamical system. Chua's oscillator can exhibit a large variety of steady-state behavior (including chaos) in the region of its parameter space defined by [Chua *et al.*, 1993a]

$$\alpha > 0, \quad \beta > 0, \quad \gamma > 0, \quad a < b < 0. \quad (2)$$

We use the third state to synchronize the drive system (1) with the response system characterized by the equation

$$\begin{pmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \end{pmatrix} = \begin{pmatrix} \alpha(\hat{x}_2 - \hat{x}_1 + f(\hat{x}_1)) \\ \hat{x}_1 - \hat{x}_2 + \hat{x}_3 \\ -\beta \hat{x}_2 - \gamma \hat{x}_3 \end{pmatrix} + \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} (x_3 - \hat{x}_3). \quad (3)$$

Note that if $k_1 = k_2 = 0$, then no synchronization is observed for any value of k_3. That is possibly the reason why x_3 has not been used as the driving signal. However, we will show that one can select the feedback vector $\mathbf{k} = [k_1, k_2, k_3]^T$ to achieve global synchronization of the drive (1) and the response (3). We state the main result in the following synchronization theorem.

Theorem 1. *Suppose that the feedback vector in the response system (3) is chosen as follows:*

$$\mathbf{k} = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{bmatrix} -1 - \dfrac{\gamma}{\beta} & -\dfrac{1+\gamma}{\beta} & -\dfrac{1}{\beta} \\ -\dfrac{\gamma}{\beta} & -\dfrac{1}{\beta} & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3\theta \\ 3\theta^2 \\ \theta^3 \end{bmatrix}. \quad (4)$$

If the parameter θ is large enough, then the drive system (1) and the response system (3) are globally asymptotically synchronized in the sense that

$$\lim_{t\to\infty} \hat{\mathbf{X}}(t) = \mathbf{X}(t), \quad \textit{for any } \mathbf{X}(0) \textit{ and } \hat{\mathbf{X}}(0).$$

3. Proof of the Synchronization Theorem

We denote

$$\mathbf{z} = H(\mathbf{x}) = \begin{pmatrix} x_3 \\ \dot{x}_3 \\ \ddot{x}_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ -\beta x_2 - \gamma x_3 \\ -\beta x_1 + \beta(1+\gamma)x_2 + (\gamma^2 - \beta)x_3 \end{pmatrix},$$

$$\hat{\mathbf{z}} = H(\hat{\mathbf{x}}) = \begin{pmatrix} \hat{x}_3 \\ -\beta \hat{x}_2 - \gamma \hat{x}_3 \\ -\beta \hat{x}_1 + \beta(1+\gamma)\hat{x}_2 + (\gamma^2 - \beta)\hat{x}_3 \end{pmatrix}.$$

We can see that $\mathbf{z} = \hat{\mathbf{z}}$ if and only if $\mathbf{x} = \hat{\mathbf{x}}$. In the following, we will prove that $\mathbf{e}(t) = \hat{\mathbf{z}}(t) - \mathbf{z}(t) \to \mathbf{0}$ by introducing the Lyapunov function $V = \mathbf{e}^T \mathbf{P}(\theta)\mathbf{e}$, where

$$\mathbf{P}(\theta) = \begin{bmatrix} \theta^{-1} & -\theta^{-2} & \theta^{-3} \\ -\theta^{-2} & 2\theta^{-3} & -3\theta^{-4} \\ \theta^{-3} & -3\theta^{-4} & 6\theta^{-5} \end{bmatrix}, \quad \theta > 0.$$

Since

$$\theta^{-1} > 0, \quad \det \begin{pmatrix} \theta^{-1} & -\theta^{-2} \\ -\theta^{-2} & 2\theta^{-3} \end{pmatrix} = \theta^{-4} > 0,$$

$$\det \mathbf{P}(\theta) = \theta^{-9} > 0.$$

$\mathbf{P}(\theta)$ is a positive definite matrix.

From the definition of $\mathbf{z}$ and $\hat{\mathbf{z}}$, we have

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \begin{pmatrix} 0 \\ 0 \\ h(\mathbf{x}) \end{pmatrix}, \quad (5)$$

$$\dot{\hat{\mathbf{z}}} = \mathbf{A}\hat{\mathbf{z}} + \begin{pmatrix} 0 \\ 0 \\ h(\hat{\mathbf{x}}) \end{pmatrix} + \left(\frac{\partial H(\hat{\mathbf{x}})}{\partial \hat{\mathbf{x}}}\right) \mathbf{k}(x_3 - \hat{x}_3), \quad (6)$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

and

$$h(\mathbf{x}) = \dddot{x}_3 = c_1 x_1 + c_2 x_2 + c_3 x_3 - \alpha\beta f(x_1),$$

$$h(\hat{\mathbf{x}}) = c_1 \hat{x}_1 + c_2 \hat{x}_2 + c_3 \hat{x}_3 - \alpha\beta f(\hat{x}_1),$$

where c_i $(i = 1, 2, 3)$ are constants which depend on α, β and γ. Thus

$$|h(\hat{\mathbf{x}}) - h(\mathbf{x})| \leq (c_1^2 + c_2^2 + c_3^2)^{1/2}\|\hat{\mathbf{x}} - \mathbf{x}\| + \alpha\beta|f(\hat{\mathbf{x}}) - f(\mathbf{x})| \leq \eta\|\hat{\mathbf{x}} - \mathbf{x}\|\,.$$

$$\eta = (c_1^2 + c_2^2 + c_e^2)^{1/2} - a\alpha\beta\,.$$

An explicit computation show that

$$\frac{\partial H(\hat{\mathbf{x}})}{\partial \hat{\mathbf{x}}} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -\beta & -\gamma \\ -\beta & \beta(\gamma+1) & \gamma^2 - \beta \end{bmatrix},$$

$$\left(\frac{\partial H(\hat{\mathbf{x}})}{\partial \hat{\mathbf{x}}}\right)^{-1} = \begin{bmatrix} -1 - \dfrac{\gamma}{\beta} & -\dfrac{1+\gamma}{\beta} & -\dfrac{1}{\beta} \\ -\dfrac{\gamma}{\beta} & -\dfrac{1}{\beta} & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{P}(\theta)^{-1} = \begin{bmatrix} 3\theta & 3\theta^2 & \theta^3 \\ 3\theta^2 & 5\theta^3 & 2\theta^4 \\ \theta^3 & 2\theta^4 & \theta^5 \end{bmatrix}.$$

Hence, from (4), the feedback vector is

$$\mathbf{k} = \left(\frac{\partial H(\hat{\mathbf{x}})}{\partial \hat{\mathbf{x}}}\right)^{-1} \mathbf{P}(\theta)^{-1}\mathbf{d}\,, \quad \mathbf{d} = [1 \quad 0 \quad 0]^T\,. \tag{7}$$

Now Eq. (6) becomes

$$\dot{\hat{\mathbf{z}}} = \mathbf{A}\hat{\mathbf{z}} + \begin{pmatrix} 0 \\ 0 \\ h(\hat{\mathbf{x}}) \end{pmatrix} + \mathbf{P}(\theta)^{-1}\mathbf{d}\mathbf{d}^T(\mathbf{z} - \hat{\mathbf{z}})\,. \tag{8}$$

From (5) and (8), we get the equation for the error $\mathbf{e}$

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{P}(\theta)^{-1}\mathbf{d}\mathbf{d}^T)\mathbf{e} + \begin{pmatrix} 0 \\ 0 \\ h(\hat{\mathbf{x}}) - h(\mathbf{x}) \end{pmatrix}. \tag{9}$$

$\dot{V}$ will then be

$$\dot{V} = \mathbf{e}^T(\mathbf{A}^T\mathbf{P}(\theta) + \mathbf{P}(\theta)\mathbf{A} - 2\mathbf{d}\mathbf{d}^T)\mathbf{e} + 2\mathbf{e}^T\mathbf{P}(\theta)\begin{pmatrix} 0 \\ 0 \\ h(\hat{\mathbf{x}}) - \mathbf{h}(\mathbf{x}) \end{pmatrix}.$$

It is easy to verify that

$$\mathbf{A}^T\mathbf{P}(\theta) + \mathbf{P}(\theta)\mathbf{A} - \mathbf{d}\mathbf{d}^T = -\theta\mathbf{P}(\theta)\,.$$

Thus

$$\dot{V} = -\theta\mathbf{e}^T\mathbf{P}(\theta)\mathbf{e} - (\mathbf{d}^T\mathbf{e})^2 + 2\mathbf{e}^T\mathbf{P}(\theta)\begin{pmatrix} 0 \\ 0 \\ h(\hat{\mathbf{x}}) - \mathbf{h}(\mathbf{x}) \end{pmatrix} \leq -\theta\|\mathbf{e}\|^2_{\mathbf{P}(\theta)} + 2\|\mathbf{e}\|_{\mathbf{P}(\theta)}\sqrt{p_{33}(\theta)}|h(\hat{\mathbf{x}}) - \mathbf{h}(\mathbf{x})|$$

where we denote $\|\mathbf{e}\|_{\mathbf{P}(\theta)} = \sqrt{\mathbf{e}^T\mathbf{P}(\theta)\mathbf{e}}$ and $\mathbf{P}(\theta) = (p_{ij}(\theta))$. But

$$\dot{V} = 2\|\mathbf{e}\|_{\mathbf{P}(\theta)}\frac{d}{dt}(\|\mathbf{e}\|_{\mathbf{P}(\theta)})\,,$$

we have

$$\frac{d}{dt}(\|\mathbf{e}\|_{\mathbf{P}(\theta)}) \leq -\frac{\theta}{2}\|\mathbf{e}\|_{\mathbf{P}(\theta)} + \sqrt{p_{33}(\theta)}|h(\hat{\mathbf{x}}) - \mathbf{h}(\mathbf{x})|\,. \tag{10}$$

Denote $\mu(\beta, \gamma) = \|(\partial H(\hat{\mathbf{x}})/\partial\hat{\mathbf{x}})^{-1}\|$, we have

$$|h(\hat{\mathbf{x}}) - \mathbf{h}(\mathbf{x})| \leq \eta\|\hat{\mathbf{x}} - \mathbf{x}\| \leq \eta\mu(\beta, \gamma)\|\mathbf{e}\|\,.$$

The smallest eigenvalue of $\mathbf{P}(1)$ is $\lambda_{\min}(\mathbf{P}(1)) = 0.127$. Since

$$\|\mathbf{e}\| \leq \sqrt{\frac{1}{\lambda_{\min}(\mathbf{P}(1))}}\|\mathbf{e}\|_{\mathbf{P}(1)} = 2.8061\|\mathbf{e}\|_{\mathbf{P}(1)}\,,$$

$$p_{ij}(\theta) = \frac{1}{\theta^{i+j-1}}p_{ij}(1)\,, \quad i, j = 1, 2, 3\,.$$

and if $\theta \geq 1$,

$$\|\mathbf{e}\|^2_{\mathbf{P}(1)} \leq \theta^5\sum_{i,j=1}^{3} e_i p_{ij}(1)e_j\frac{1}{\theta^{i+j-1}} = \theta^5\|\mathbf{e}\|^2_{\mathbf{P}(\theta)}\,.$$

We get from (10) that

$$\frac{d}{dt}(\|\mathbf{e}\|_{\mathbf{P}(\theta)}) \leq -\left(\frac{\theta}{2} - \eta\mu(\beta, \gamma)\sqrt{\frac{p_{33}(1)}{\lambda_{\min}(\mathbf{P}(1))}}\right)\|\mathbf{e}\|_{\mathbf{P}(\theta)} = -\left(\frac{\theta}{2} - 6.8734\eta\mu(\beta, \gamma)\right)\|\mathbf{e}\|_{\mathbf{P}(\theta)} \tag{11}$$

Denote $\rho \equiv 13.7468\eta\mu(\beta, \gamma)$. Note that ρ is a constant which depends only on the system parameters. If $\theta > \rho$, then (11) implies that

$$\|\mathbf{e}(t)\|_{\mathbf{P}(\theta)} \leq \|\mathbf{e}(0)\|_{\mathbf{P}(\theta)}\exp\left\{-\frac{1}{2}(\theta - \rho)t\right\} \to 0\,.$$

Therefore

$$\|\hat{\mathbf{x}}(t) - \mathbf{x}(t)\| \le \mu(\beta, \gamma)\|\mathbf{e}(t)\|$$
$$\le \mu(\beta, \gamma)(\lambda_{\max}(\mathbf{P}(\theta)))^{-1/2}\|\mathbf{e}(0)\|_{\mathbf{P}(\theta)}$$
$$\times \exp\left\{-\frac{1}{2}(\theta - \rho)t\right\} \to 0\,. \qquad (12)$$

4. Simulation Results

In all computer simulations, the system parameters of Chua's oscillator are:

$$\alpha = 10.0000\,, \quad \beta = 15.0000\,, \quad \gamma = 0.0385\,,$$
$$a = -1.2700\,, \quad b = -0.6800\,,$$

the initial states of the drive and the response system are

$$\mathbf{x}(0) = [-2,\, 0.002,\, 4]^T, \quad \hat{\mathbf{x}}(0) = [0.7,\, 0.4,\, -0.8]^T.$$

The driving signal $x_3(t)$ is shown in Fig. 1. By (4), the feedback vector in the response system is

$$\mathbf{k} = \begin{bmatrix} -1.0026 & -0.0692 & -0.0667 \\ -0.0026 & -0.0667 & 0 \\ 1.0000 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3\theta \\ 3\theta^2 \\ \theta^3 \end{bmatrix}.$$

We find that synchronization occurs when $\theta \ge 1$. Figure 2 shows the magnitude of the feedback gain, while Fig. 3 shows the time needed to achieve synchronization ($\|\hat{\mathbf{x}} - \mathbf{x}\| \le 10^{-4}$) for different values of θ. Note that we are using the normalized form of the equations [Chua *et al.*, 1993a]. With respect to the original parameters of the electronic circuit described in [Dedieu *et al.*, 1993], one second of normalized time corresponds to about 0.2 milliseconds real time. Figure 4 shows the synchronization process of x_3 (solid line) and $\hat{x}_3$ (dashed line) performed with $\theta = 1$.

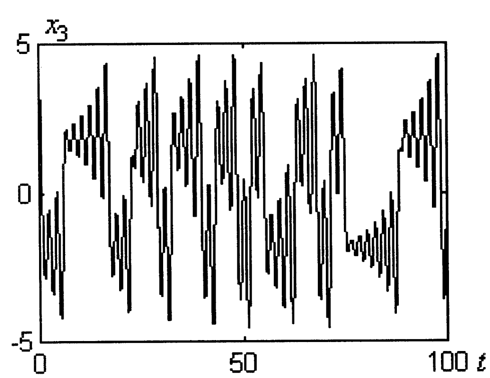

Fig. 1. Driving signal $x_3(t)$.

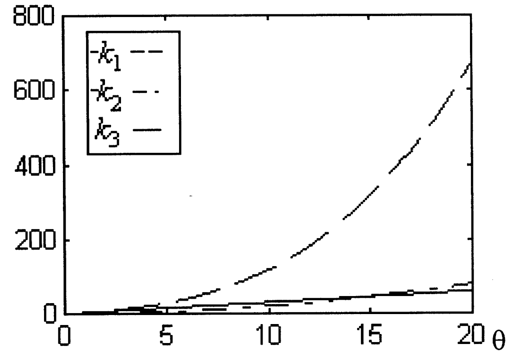

Fig. 2. The feedback gain for different values of θ.

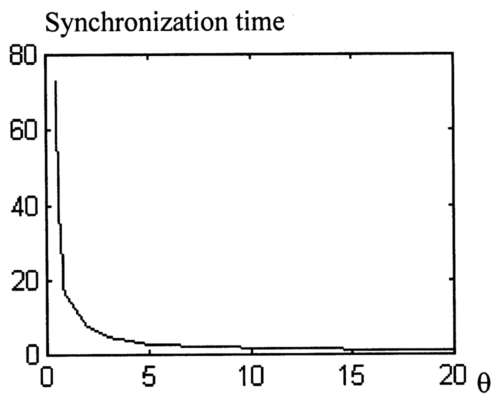

Fig. 3. Synchronization time for different values of θ.

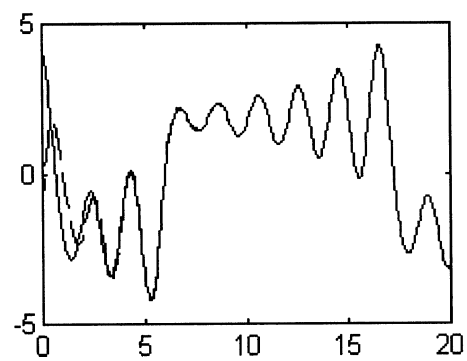

Fig. 4. Synchronization process of x_3 (solid line) and $\hat{x}_3$ (dashed line) performed with $\theta = 1$.

5. Conclusions

We have proposed a linear feedback approach to synchronizing two Chua's oscillators with the *third* state as the driving signal. The feedback vector is chosen as in (4). We have proven the globally asymptotic synchronization when the parameter θ is large enough. Simulation results show that the proposed scheme may be useful for developing synchronized Chua's oscillators.

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