NONLINEAR H_∞ SYNCHRONIZATION: CASE STUDY FOR A HYPERCHAOTIC SYSTEM

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ABSTRACT

In this paper we apply the method of nonlinear H_∞ synchronization to a hyperchaotic system, which consists of two unidirectionally coupled Chua's circuits. This method makes use of a vector field modulation at the master system by a filtered binary valued message signal. Static output feedback is applied to the slave system. The original message is recovered from a tracking error, which is defined after representing the synchronization scheme in standard plant form according to modern control theory. The controller is designed based upon a matrix inequality, in order to minimize the L_2-gain from the exogenous input (message signal and channel noise) to the regulated output (tracking error).

Keywords. Lur'e systems, Chua's circuit, master-slave scheme, dissipativity, L_2-gain, nonlinear H_∞ control, matrix inequality.

1. INTRODUCTION

Recently, a method of nonlinear H_∞ synchronization [12, 13] has been proposed for secure communications with chaotic Lur'e systems. The method makes use of a master-slave synchronization scheme with vector field modulation [16] of the master system by a binary valued message signal. Both full static state feedback [12] and dynamic output feedback control laws [13] have been proposed. In this scheme the message signal is recovered from a so-called tracking error after representing the synchronization scheme in standard plant form [9]. The message signal is interpreted then as an external reference input to the control scheme which has to be tracked. In the scheme channel noise has been taken into account as part of the exogenous input. Input-output properties of the standard plant have been analyzed based upon dissipativity with finite L_2-gain and quadratic storage functions [6]. The name *nonlinear H_∞ synchronization* of the method stems from the fact that one minimizes the L_2-gain from the exogenous input to the tracking error, which is very much related to methods in nonlinear H_∞ control [7, 15]. Control laws are designed by solving a non-convex optimization problem which involves a matrix inequality [1], characterizing dissipativity with finite L_2-gain of the synchronization scheme.

Previously, the method has been illustrated on chaotic Lur'e systems such as Chua's circuit [3, 4, 10] and generalized Chua's circuits exhibiting n-scroll attractors [14]. In this paper we successfully apply the method to the more complex example of a hyperchaotic system which consists of two unidirectionally coupled Chua's circuits [8] possessing a double-double scroll attractor. The use of a double-double scroll attractor for secure communications purposes has been previously investigated in [2] for a different synchronization scheme. Our approach on the other hand enables to take into channel noise in designing the controller. Two outputs and four control inputs are considered for the hyperchaotic system with six state variables, for which we apply static output feedback instead of full static state feedback as in the examples of [12].

This paper is organized as follows. In Section 2 we present the master-slave synchronization scheme. In Section 3 we derive a matrix inequality related to nonlinear H_∞ synchronization. In Section 4 the theory is applied to the hyperchaotic system with Chua's circuits.

2. SYNCHRONIZATION SCHEME

Consider the synchronization scheme [12]

$$\mathcal{R}: \begin{cases} \dot{m} &= Rm + Sr \\ d &= Tm + Ur \end{cases}$$

$$\mathcal{M}: \begin{cases} \dot{x} &= Ax + B\sigma(Cx) + Dd \\ p &= Hx \end{cases}$$

$$\mathcal{S}: \begin{cases} \dot{z} &= Az + B\sigma(Cz) + F(p-q) + F\epsilon \\ q &= Hz \end{cases} \tag{1}$$

with master system $\mathcal{M}$, slave system $\mathcal{S}$ and low pass filter $\mathcal{R}$. At the slave system static output feedback is applied using the output difference $p-q$. The subsystems have state vectors $x, z \in \mathbb{R}^n$, $m \in \mathbb{R}^{n_r}$ and output vectors $p, q \in \mathbb{R}^l$, $d \in \mathbb{R}$, where $l, m \leq n$. The message signal $r \in \mathbb{R}$ is assumed to be binary valued. Vector field modulation is applied at the master system by the filtered message. The output vector p is transmitted along the channel. The original message is not recovered from $e = x - z$ but from taking the sign of a tracking error $\nu = d - \beta^T e$ with $\beta = [1; 0; ...; 0]$. The transmitted signal is corrupted by the signal ϵ (which is theoretically treated as a deterministic disturbance input instead of a stochastic input here). The master-slave systems are identical Lur'e systems with system matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n_h}$ and $C \in \mathbb{R}^{n_h \times n}$ where n_h corresponds to the number of hidden units if one interprets the Lur'e system as a class of recurrent neural networks [11]). The diagonal nonlinearity $\sigma(.) : \mathbb{R}^{n_h} \mapsto \mathbb{R}^{n_h}$ is assumed to belong to sector $[0, k]$ (saturation characteristic in the case of Chua's circuit and n-scroll attractors or arrays containing such cells [14]).

According to [12, 13] a control theoretical interpretation is given to (1) by representing it in standard plant form [9, 11]

$$\begin{cases} \begin{bmatrix} \dot{e} \\ \dot{m} \end{bmatrix} = \begin{bmatrix} A - FH & DT \\ 0 & R \end{bmatrix} \begin{bmatrix} e \\ m \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \eta(Ce; z) \\ \qquad + \begin{bmatrix} DU & -F \\ S & 0 \end{bmatrix} \begin{bmatrix} r \\ \epsilon \end{bmatrix} \\ \nu = [-\beta^T \; T] \begin{bmatrix} e \\ m \end{bmatrix} + [U \; 0] \begin{bmatrix} r \\ \epsilon \end{bmatrix} \end{cases} \tag{2}$$

with exogenous input $w = [r; \epsilon]$ and regulated output ν. The nonlinearity is given by $\eta(Ce; z) = \sigma(Ce + Cz) - \sigma(Cz)$ and is assumed to satisfy a sector condition $[0, k]$.

3. NONLINEAR H_∞ SYNCHRONIZATION AND MATRIX INEQUALITY

According to [12, 13] we analyze the I/O properties of the standard plant form (2) by means of the quadratic storage function

$$\phi(e, m) = [e^T \; m^T] \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} e \\ m \end{bmatrix} \tag{3}$$

with $P = P^T > 0$ and the supply rate with finite L_2-gain γ

$$s(w, \nu) = \gamma^2 w^T w - \nu^T \nu. \tag{4}$$

The system is said to be dissipative with respect to ϕ and $s(w, \nu)$ if $\dot{\phi} \leq s(w, \nu)$ for all w, ν [6].

The latter condition is checked by applying the S-procedure [1] (by using the matrix inequality $\eta^T \Lambda(\eta - kCe) \leq 0 \; \forall e, z$ with Λ a diagonal matrix) and writing $\dot{\phi} - s(w, \nu) - 2\eta^T \Lambda(\eta - kCe) < 0$ as a quadratic form in the variable $[e; m; \eta; r; \epsilon]$. This yields the following matrix inequality which is sufficient to prove dissipativity of (2) with respect to (3) and (4):

$$Z = Z^T = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} & Z_{15} \\ . & Z_{22} & Z_{23} & Z_{24} & Z_{25} \\ . & . & Z_{33} & 0 & 0 \\ . & . & . & Z_{44} & 0 \\ . & . & . & . & Z_{55} \end{bmatrix} < 0 \tag{5}$$

with

$$\begin{aligned} Z_{11} &= A_*^T P_{11} + P_{11} A_* + \beta\beta^T \\ Z_{22} &= T^T D^T P_{12} + P_{21} DT + P_{22} R \\ &\quad + R^T P_{22} + T^T T \\ Z_{33} &= -2\Lambda \\ Z_{44} &= -\gamma^2 I + U^T U \\ Z_{55} &= -\gamma^2 I \\ Z_{12} &= A_*^T P_{12} + P_{11} DT + P_{12} R - \beta T \\ Z_{13} &= P_{11} B + kC^T \Lambda \\ Z_{14} &= P_{11} DU + P_{12} S - \beta U \\ Z_{15} &= -P_{11} F \\ Z_{23} &= P_{21} B \\ Z_{24} &= P_{21} DU + P_{22} S + T^T U \\ Z_{25} &= -P_{21} F \end{aligned}$$

where $A_* = A - FH$.

The nonlinear H_∞ synchronization method aims then at minimizing the L_2-gain for the standard plant scheme [12]

$$\min_{F, P, \Lambda, \gamma} \gamma \;\text{ such that }\; Z(F, P, \Lambda, \gamma) < 0 \tag{6}$$

which leads to solving a non-convex optimization problem in order to find the controller F.

4. EXAMPLE: HYPERCHAOTIC SYSTEM WITH CHUA'S CIRCUITS

We consider the following system which consists of two unidirectionally coupled Chua circuits

$$\left\{\begin{array}{lcl} \dot{x}_1 & = & a\,[x_2 - h(x_1)] \\ \dot{x}_2 & = & x_1 - x_2 + x_3 \\ \dot{x}_3 & = & -b\,x_2 \\ \dot{x}_4 & = & a\,[x_5 - h(x_4)] + K(x_4 - x_1) \\ \dot{x}_5 & = & x_4 - x_5 + x_6 \\ \dot{x}_6 & = & -b\,x_5 \end{array}\right. \qquad (7)$$

with $h(x_i) = m_1 x_i + \frac{1}{2}(m_0 - m_1)(|x_i + c| - |x_i - c|)$ (i=1,4). For $m_0 = -1/7$, $m_1 = 2/7$, $a = 9$, $b = 14.286$, $c = 1$, $K = 0.01$ the system exhibits hyperchaotic behaviour with a double-double scroll attractor [8] (Fig.1). The system can be represented in Lur'e form by

$$A = \left[\begin{array}{ccc|ccc} -a\,m_1 & a & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & -b & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -a\,m_1 & a & 0 \\ 0 & -K & 0 & 1 & -1+K & 1 \\ 0 & 0 & 0 & 0 & -b & 0 \end{array}\right],$$

$$B = \left[\begin{array}{c|c} -a\,(m_0 - m_1) & 0 \\ 0 & 0 \\ 0 & 0 \\ \hline 0 & -a\,(m_0 - m_1) \\ 0 & 0 \\ 0 & 0 \end{array}\right],$$

$$C = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right] \qquad (8)$$

and $\sigma(\xi) = \frac{1}{2}(|\xi + c| - |\xi - c|)$ belonging to sector $[0, 1]$ with $n_h = 2$. For the synchronization scheme we define as outputs x_1, x_4, z_1, z_4 $(l = 2)$ with $H = [1\,0\,0\,0\,0\,0; 0\,0\,0\,1\,0\,0]$ (Fig.2) and for controlling the slave system $D = [1; 0; 0; 1; 0; 0]$. For $\mathcal{R}$ a first order Butterworth filter is chosen with cut-off frequency 10 Hz. The nonlinear H_∞ synchronization problem was solved by means of sequential quadratic programming (constr in Matlab). Instead of (6), it has been programmed as

$$\min_{F,Q,\Lambda,\gamma} \gamma \text{ such that } \lambda_{max}[Z(F, Q, \Lambda, \gamma)] + \delta < 0 \quad (9)$$

with $P = Q^T Q$ and $\delta = 0.01$. As starting point for the nonlinear optimization was taken $Q = I$, $\Lambda = 0.1I$, $\gamma = 100$ and F randomly chosen according to a Gaussian distribution with zero mean and standard deviation 0.1. On Fig.3 a locally minimizing solution is shown. The message signal $r = \text{sign}(\cos(0.5t))$ is binary valued. The message is successfully recovered from $\text{sign}(\beta^T e)$. Simulation results with zero mean white Gaussian noise with standard deviation 0.0001 are shown on Fig.4. The robustness with respect to the noise could be optimized further by penalizing ϵ with respect to r in the supply rate (4) as has been discussed in [12].

5. CONCLUSIONS

The method of nonlinear H_∞ synchronization has been applied to a hyperchaotic system consisting of two unidirectionally coupled Chua circuits. In this method the vector field of the master system is modulated by a filtered binary valued message signal. The message is recovered from a tracking error after representing the synchronization scheme in standard plant form. Channel noise is taken into account in the scheme. I/O properties are analyzed using the systemtheoretical concept of dissipativity with finite L_2-gain and a quadratic storage function. In this way dissipativity with a given L_2-gain is characterized by a matrix inequality. The static output feedback controller is obtained by minimizing this gain subject to the matrix inequality.

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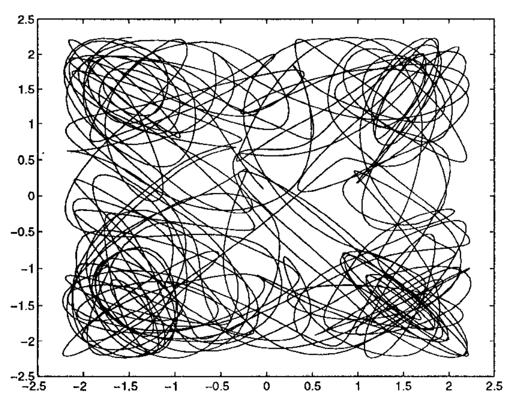

Fig.1. *Double-double scroll attractor according to Kapitaniak & Chua. Shown is (x_1, x_4) for this hyperchaotic system with 6 state variables.*

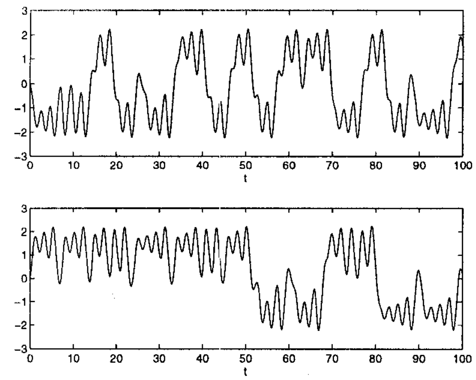

Fig.2. *Transmitted signals containing the message: (Top) $x_1(t)$; (Bottom) $x_4(t)$.*

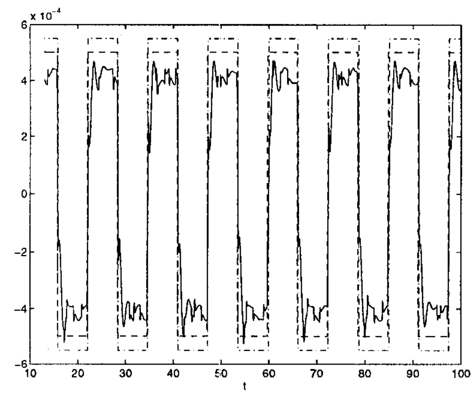

Fig.3. *Recovering the original message by nonlinear H_∞ synchronization: message signal (- -); $\beta^T e$ (-); recovered message* $\text{sign}(\beta^T e)$ *(-.).*

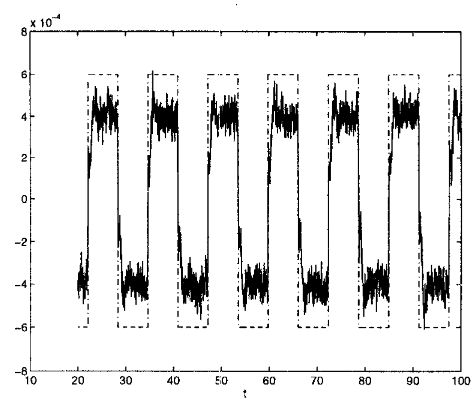

Fig.4. *Simulation for the resulting controller of Fig.3 with channel noise (zero mean white Gaussian noise with standard deviation 0.0001): $\beta^T e$ (-); recovered message* $\text{sign}(\beta^T e)$ *(- -).*