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# EXPERIMENTAL IMPROVEMENT OF CHAOTIC SYNCHRONIZATION DUE TO MULTIPLICATIVE TIME-CORRELATED GAUSSIAN NOISE

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The effect of time-correlated zero-mean Gaussian noise on chaotic synchronization is analyzed experimentally in small-size arrays of Chua's circuits. Depending on the correlation time, an improvement of the synchronization is found for different values of the noise amplitude and coupling diffusion between circuits.

Recently there has been considerable interest in stochastic resonance, i.e. the enhanced response of a system to an external signal induced by noise [Wiesenfeld & Moss, 1995; Gammaitoni *et al.*, 1998; Luchinsky *et al.*, 1998], a phenomenon in which noise has a *creative* role. Moreover, the influence of noise has also been studied within the context of arrays of cells [Lindner *et al.*, 1996; Braiman *et al.*, 1995a, 1995b; Gailey *et al.*, 1997; Shuai *et al.*, 1998]. Here, the coupling strength and the noise intensity play an important role for *array enhanced stochastic resonance* [Lindner *et al.*, 1995]. The applications of noise to biological systems or in engineering problems could be of special relevance. On the other hand, the behavior of uncoupled chaotic systems under the influence of external noise has been the subject of recent work [Maritan & Banavar, 1994; Pikovsky *et al.*, 1994; Herzel & Freund, 1995; Malescio, 1996; Gade & Basu, 1996; Longa *et al.*, 1996; Shinbrot *et al.*, 1993; Sánchez *et al.*, 1997, 1999]. The main idea behind these papers is that uncoupled chaotic systems cannot be synchronized by means of an identical noise signal (Gaussian noise of zero mean), except for a noise with some nonzero bias.

In this Letter, the role of a time-correlated Gaussian noise on diffusively chaotic coupled cells is analyzed. The dynamical noise used in this Letter is a Gaussian noise of zero mean of the Ornstein–Uhlenbeck type [Sancho *et al.*, 1982], characterized by a correlation function

$$\langle \xi(t)\xi(t')\rangle = \frac{\alpha}{\tau}\exp\left(\frac{-|t-t'|}{\tau}\right) \tag{1}$$

where $\tau$ is the correlation time and $\alpha$ is the noise amplitude. In the limit $\tau \to 0$ the white-noise limit is recovered. We emphasize here the case of *global noise*, where the noise is identical at each site, as opposed to the case of incoherent or *local noise*, where the noise is uncorrelated from site to site.

Experiments have been performed with $N$ resistively coupled Chua's circuits ($N = 3$) [Madan, 1993; Chua, 1995] in the chaotic regime, according

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to the design introduced by [Sánchez *et al.*, 1997, 1999]. Each circuit, $j = 1, 2, 3$, in the array is defined by the following evolution equations,

$$\begin{aligned} C_1 \frac{dV_{1,j}}{dt} &= \frac{1}{R}(V_{2,j} - V_{1,j}) - h(V_{1,j}) \\ &\quad + \frac{1}{R_c}(V_{1,j+1} + V_{1,j-1} - 2V_{1,j}) \\ C_2 \frac{dV_{2,j}}{dt} &= \frac{1}{R}(V_{1,j} - V_{2,j}) + i_{L,j} \\ L \frac{di_{L,j}}{dt} &= -V_{2,j} - r_0 i_{L,j} \end{aligned} \tag{2}$$

where $V_1$, $V_2$, and $i_L$, the voltages across $C_1$ and $C_2$ and the current through $L$, respectively, are the three variables that describe the dynamical system, resulting from a straightforward application of Kirchhoff's law. The parameters have the following meaning: $C_1$ and $C_2$ are the two capacitances, $L$ the inductance, $R$ the resistance that couples the two capacitors and $r_0$ the inner resistance of the inductor. Circuits were connected through capacitor $C_1$ by resistances $R_c$, leading to a diffusion term in the potential differences [Chua, 1995], with a coupling coefficient $D \propto 1/R_c$. Circuits at the boundaries are only connected with one neighbor.

The three-segment piecewise-linear characteristic of the nonlinear resistor (Chua's diode) is defined by,

$$\begin{aligned} h(V_1) = G_b V_1 + \frac{1}{2}(G_a - G_b)[|V_1 + B_p| \\ - |V_1 - B_p|] \end{aligned} \tag{3}$$

where $G_a$ and $G_b$ are the slopes of the inner and outer regions of $h(V_1)$, respectively, and $B_p = 1\ V$ defines the location of the breaking points of the three-slope nonlinear characteristic $h(V_1)$.

An experimental setup of three identical Chua's circuits driven by noise has been built. Their components are defined by $(C_1, C_2, L, r_0, R) = (10\,\text{nF}, 100\text{ nF}, 10\text{ mH}, 20\ \Omega, 1.1\text{ k}\Omega)$. The slopes of the nonlinear characteristic $h(V)$, Eq. (3), are defined by $G_a = -8/7000$ and $G_b = -5/7000$. Figure 1 shows the schematic diagram of the experimental setup. The circuits were sampled with

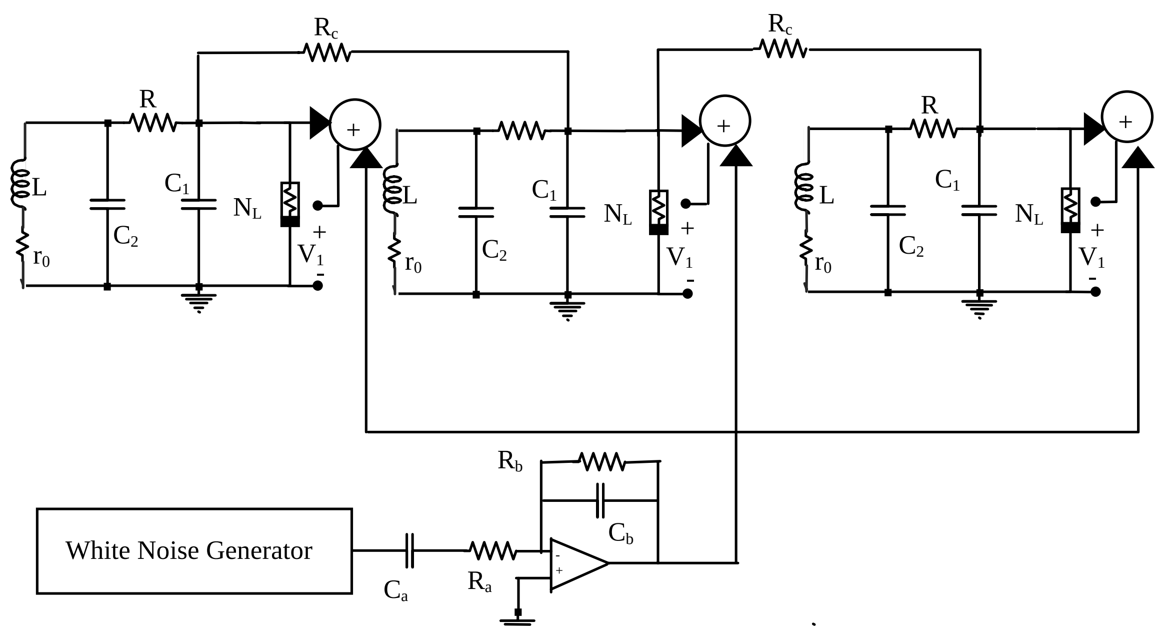


Fig. 1. Schematic diagram of the experimental setup used to introduce noise in a multiplicative way in an array of diffusively coupled Chua's circuits. Noise is added to voltage $V_1$ and is used to drive the nonlinear element of the circuit [see Eq. (4)]. The noise is buffered from all VCCS's ensuring thereby no interaction between the circuits except that due to the coupling resistances $R_c$. The white Gaussian noise generated by the function generator is transformed by the high-cut filter in a Gaussian noise of zero mean of the Ornstein–Uhlenbeck type [see Eq. (1)] with a correlation time $\tau = R_b C_b$, which is then added to the signal from capacitor $C_1$ (also buffered). The low-frequency cutoff of the filter, determined by $R_a C_a$, is fixed at $\approx 1$ Hz, and the high-frequency cutoff is adjustable by tuning $R_b$ ($C_b = 1\mu$F). The output noise finally passes through a variable-gain operational amplifier (not shown) before being applied to the circuits.

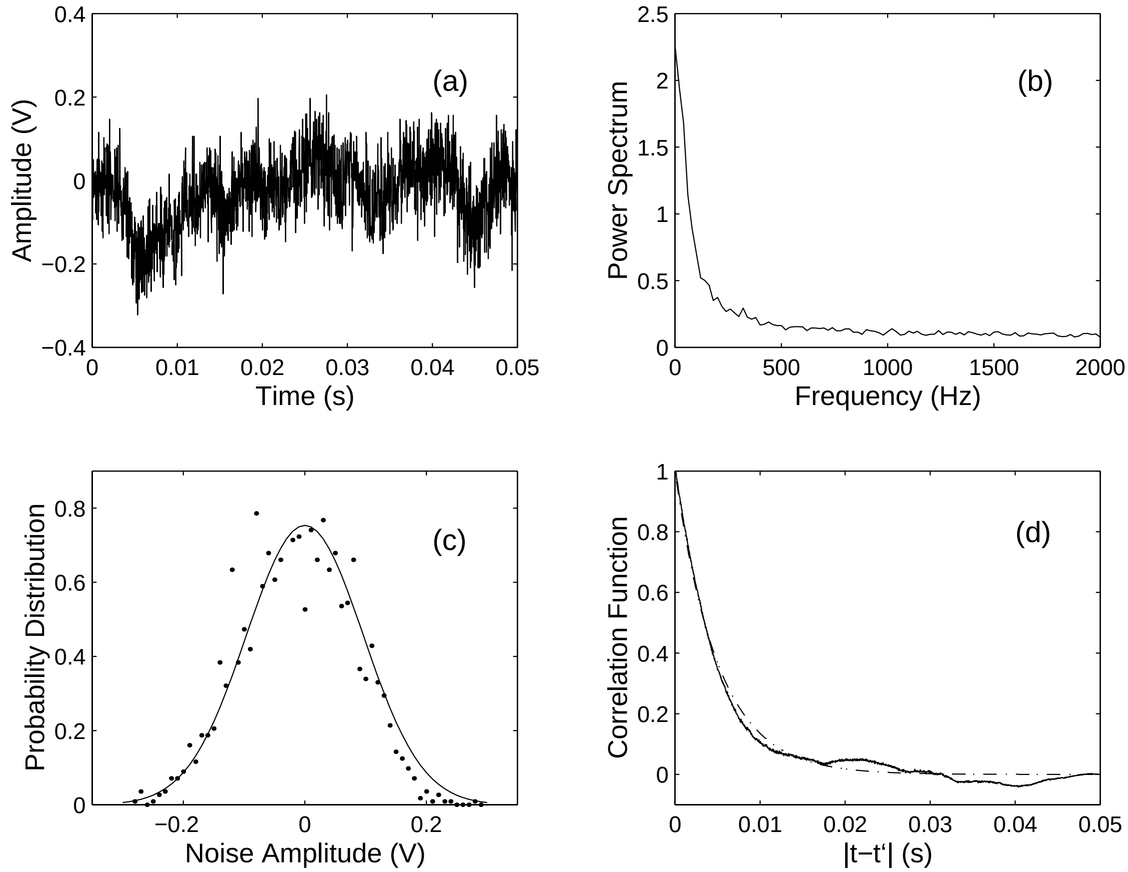


Fig. 2. Characterization of a time-correlated Gaussian noise obtained after a white zero-mean Gaussian noise is passed through a single-pole filter. (a) Temporal evolution of the color noise with amplitude 500 mV (peak-to-peak) and correlation time $\tau = 5$ ms. (b) Power spectrum of the noise signal. (c) Probability function distribution. The line shows a fitting of the experimental results to a Gaussian curve. (d) Correlation function (continuous line). The dashed-line corresponds to the theoretical curve given by Eq. (1).

a digital oscilloscope (Hewlett-Packard 54825A) with a maximum sampling rate of $4 \times 10^9$ samples per second, 1.5 GHz bandwith, and a record length of 32000 points, connected to a PC for data processing.

The external noise has been introduced multiplicatively using a recently introduced circuit [Sánchez *et al.*, 1997, 1999] that enables to drive the nonlinear element by using the voltage from an external source. The nonlinear element is driven, in general, by the voltage coming from an external source, not necessarily the voltage coming from capacitor $C_1$, as may happen in the case of a standard Chua's circuit. Thus, it is a voltage controlled current source (VCCS) with a characteristic defined by Eq. (3). This yields the following evolution equation for the voltage across the capacitor $C_1$,

$$C_1 \dot{V_1} = \frac{V_2 - V_1}{R} - h(V_1 + \xi(t)) \tag{4}$$

where it is easy to see that the noise term yields a multiplicative contribution.

The time-correlated noise, $\xi(t)$, [Eq. (1)], is obtained electronically by passing the output voltage of a white Gaussian noise generator, $\xi_w(t)$, through a single-pole active filter with a time constant $\tau = R_b C_b$, before being applied to the circuit as it is shown in Fig. 1 [McClintock & Moss, 1989; Luchinsky *et al.*, 1998]. The external white noise has been generated by using a function generator (Hewlett-Packard 33120A). Their characteristics, Gaussian distribution and zero mean in the absence of an offset, have been adequately checked.[1]

[1]As the bandwith ($\approx 10$ MHz) of the white noise from the HP generator is much higher than the characteristic frequencies of the Chua's circuit, then for our purposes, we can consider this ideal "white" noise.

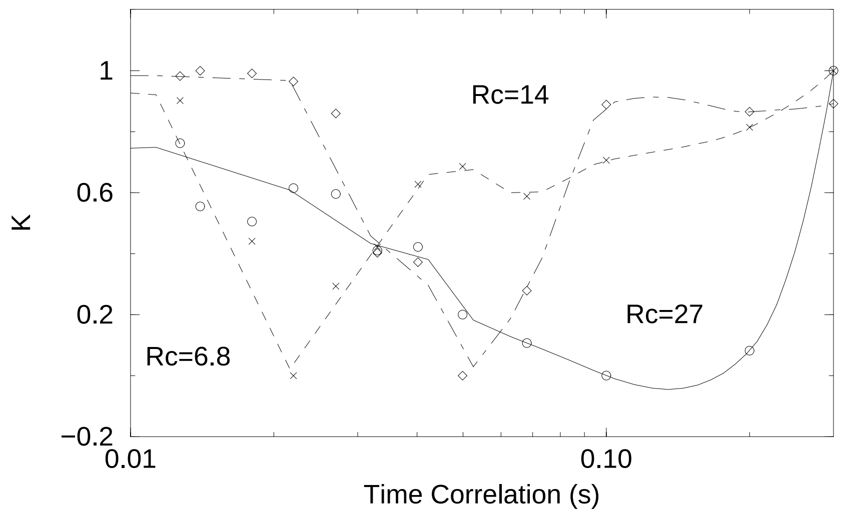


Fig. 3. Log-linear plot of $K$ as a function of the time correlation $\tau$ for three different values of the coupling resistance $R_c$. Since the length of the time series is determined by the record length of the oscilloscope and the time scale of the Chua's circuits, in order to improve the statistic, 50 realizations of the noise were carried out for each value of $\tau$. In doing so, we substitute the limiting value of $T \to \infty$, in Eq. (5), for $T \gg \tau$, in such a way that the most probable value of $K$ becomes identical to the ensemble average of $K$ when the number of experiments (realizations of noise) is large. The experimental data are shown as symbols, while lines represent an interpolation to the previous ones: crosses ($\times$) and a dashed-line for $R_c = 6.8\ \Omega$, rhombi ($\diamond$) and a dot–dashed line for $R_c = 14\ \Omega$, and circles ($\circ$) and a solid-line for $R_c = 27\ \Omega$. The obtained values of $K$ were scaled between 0 and 1 for a better representation of the phenomenon. The minimum and maximum values of $K$ for each coupling resistance are: 0.11 and 0.13 for $R_c = 6.8\ \Omega$, 0.34 and 0.37 for $R_c = 14\ \Omega$, and 1.39 and 1.51 for $R_c = 27\ \Omega$. Noise amplitude (peak-to-peak): 250 mV.

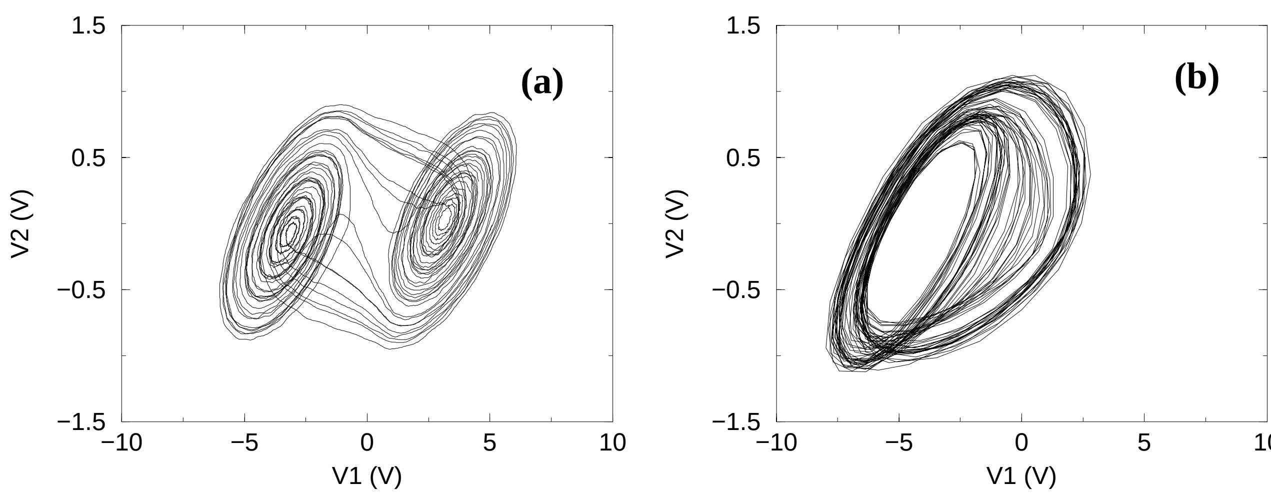


Fig. 4. Effect of a time-correlated noise on the double-scroll (chaotic) attractor (a) of the Chua's circuit for intermediate values of $\tau$. As $\tau$ increases, the attractor becomes smeared out as well as it becomes "periodically" asymmetric, finally losing the double-scroll appearance (b). The image presented in panel (b) was acquired after the oscilloscope was stopped at a given instant of time. Otherwise, other possible asymmetric shapes of the attractor could have been obtained. Noise amplitude (peak-to-peak): (a) 10 mV and (b) 400 mV. Correlation time; $\tau = 50$ ms. $R_c = 6.8\ \Omega$.

Figures 2(a)–2(d) show the physical properties of an experimental time-correlated noise before being added to the voltage $V_1$ in the Chua's circuit. The power spectrum of the noise (b) cannot be considered to be flat within the frequency range of interest, $\tau^{-1} \approx 200$ Hz. For small noise intensities, the obtained probability distribution does not fit perfectly to the Gaussian distribution as a consequence of the experimental noise in the sampling process [Fig. 2(c)]. The correlation function was also calculated experimentally (d) and compared with the theoretical function given in Eq. (1). Our realization of noise decays exponentially within a time scale almost equal to the expected theoretical value.

In order to characterize the *degree* of synchronization between cells of the array, we introduce the following time-averaged quantity,

$$K = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \left( \frac{1}{N-1} \sum_{j=2}^{N} [V_{1,j-1}^t - V_{1,j}^t]^2 \right) \quad (5)$$

This function is positive defined and it is equal to zero when all the cells in the array are globally synchronized. Since $K$ can serve as a measure of the array complexity, in this context it can be related to the Kolmogorov–Sinai entropy [Benettin *et al.*, 1976; Klimontovich, 1996].

The main effect of a colored Gaussian noise on an array of diffusively coupled chaotic cells is to improve the synchronization between units. Figure 3 shows the evolution of $K$ as a function of the correlation time $\tau$ for different values of the coupling resistance $R_c$. The mean value of $K$ increases with $R_c$ as expected, so a scaling factor was introduced in order to compare the different observed behaviors of $K$. In general, independently of the specific value of $R_c$ as $\tau$ increases, first $K$ decreases almost exponentially, reaches a minimum, and then rises smoothly until a constant saturation value is attained for $\tau \gg 1$. The minimum of $K(K_{\min})$ corresponds to an optimum choice of $\tau(\tau_{\min})$ to obtain the best synchronization.

For intermediate values of $\tau$, the time-correlated Gaussian noise periodically modulates $V_1$ when driving the nonlinear element. A resonance effect between the Chua's time scale and the noise correlation time, $\tau$, should be expected, since the power spectrum of the noise cannot be considered to be flat within the frequency range of interest, $\tau^{-1}$. This resonance effect could explain the improvement of chaotic synchronization observed in Fig. 3 for $K = K_{\min}$ [Lorenzo & Pérez-Muñuzuri, 1999]. Here, the double-scroll attractor becomes periodically asymmetric with increasing noise amplitude, as well as blurred [Figs. 4(a) and 4(b)], due to the slow dynamics of the noise.

The value of $\tau$ corresponding to $K_{\min}$ was found to increase with the coupling resistance as it is shown in Fig. 5. As $\tau \to \tau_{\min}$, a stronger interaction between the two characteristic time scales of both cell and noise should be expected, then improving the synchronization between circuits. For $\tau = \tau_{\min}$, in terms of frequency locking, the dynamics of the cell could be simplified to that of an oscillator forced periodically with a frequency equal to $\tau_{\min}^{-1}$. Then, for a chain of linearly-coupled oscillators, its dynamics can be described in terms of a plane-wave solution with a wave velocity proportional to $\sqrt{D}$. The wave dispersion relation is given by $\omega \propto \sqrt{D}/\lambda$, with $\omega$ the wave frequency and $\lambda$ the wavelength. For small-size arrays, it can be considered that $\lambda$ is fixed by the boundary conditions. In this case, the wave frequency increases with the coupling strength. Thus, in order to obtain

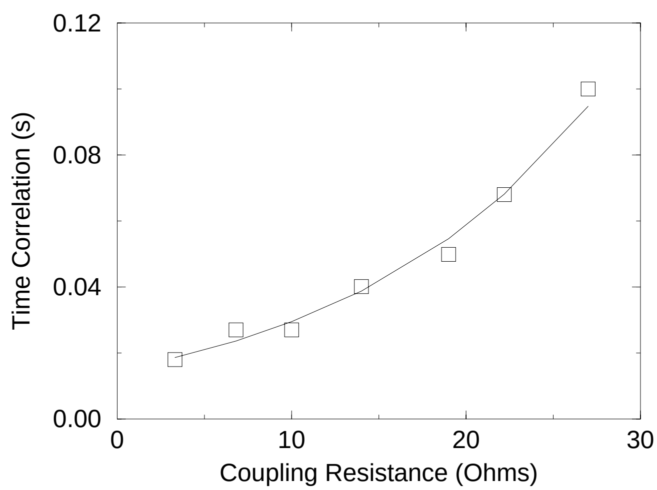


Fig. 5. Dependence of the time correlation value, corresponding to the minimum of $K$, with the coupling resistance, $R_c$. Noise amplitude (peak-to-peak): 250 mV.

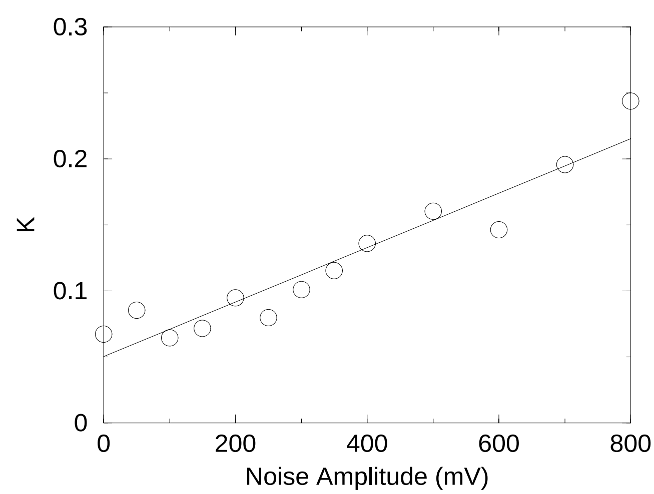


Fig. 6. Dependence of $K$ with the noise amplitude (peak-to-peak) of a time-correlated noise. Note the linear dependence observed for the range of used noise amplitudes. $R_c = 10\ \Omega$, $\tau = 0.01$ s.

locking between the internal oscillation frequency and the external periodic forcing, as the coupling resistance increases, the external forcing period (in our case this is related to the value of $\tau_{\min}$) should also increase. Obviously, the explanation above is a simplification of the problem, since the chaotic dynamics cannot be mapped in a simple way to that of an oscillator. Nevertheless, our aim is to stress the similarity between the classical frequency locking problem that occurs in a chain of oscillators forced periodically and the behavior of $K$ for $\tau \approx \tau_{\min}$. Here, the locking does not occur for a single value of the frequency, but for a range of frequencies that gives rise to a wide behavior of $K$ as a function of $\tau$ near the onset of resonance.

On the other hand, the two limits, $\tau \to 0$ and $\tau \to \infty$, in Fig. 3 deserve further comments. When $\tau \to 0$, the white Gaussian noise limit is recovered and circuits in the array do not become synchronized to each other independently of the variance of the noise [Sánchez *et al.*, 1997, 1999]. In this case, the structure of the unperturbed double-scroll attractor gets smeared out with the increasing noise amplitude, corresponding to a decreasing signal-to-noise ratio, while no evidence of synchronization behavior is observed. Similarly, when $\tau \to \infty$ the term $\xi(t)$ in Eq. (4) behaves as a constant value added to the voltage $V_1$. Noise affects the double-scroll dynamics that becomes asymmetric, while no synchronization is observed between cells within the array. In fact, for high enough noise amplitude, the main effect will be a biased signal that will induce a regularization in the system. This effect is analogous to that of some chaos suppression methods that achieve this result through perturbations in the system variables [Matías & Güémez, 1994, 1996].

Figure 6 shows the dependence of $K$ with the noise amplitude for a small value of $\tau$ near the white limit case. As expected, increasing the noise strength leads to a worse synchronization between circuits ($K$ increases).

The influence of the number of circuits in the array, $N$, on chaotic synchronization by time-correlated noise is also being studied [Lorenzo & Pérez-Muñuzuri, 1999]. Preliminar results show that the larger the array, the larger the needed fluctuations to improve chaotic synchronization; i.e. the variance of the noise must increase as $N$ increases.

We have observed a nonmonotonic dependence of the degree of synchronization measured in terms of $K$ as a function of the noise correlation time $\tau$ and the coupling between cells in a one-dimensional array. In fact, for values of $\tau$ of the order of the time scale of the chaotic attractor a stochastic resonance effect is found that could explain the observed minimum value of $K$. In other words, the effect of color noise to improve the chaotic synchronization between cells is more robust than that of white noise for a constant noise amplitude.

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