Exploring Chaos in Chua's Circuit via Unstable Periodic Orbits

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Abstract — Unstable periodic orbits inherently embedded in strange attractors provide a useful characterisation of chaotic behavior. In this paper we analyse unstable periodic orbits existing in chaotic orbits existing in Chua's circuit. Uncovering of such orbits is feasible and numerically tractable - specific algorithms and programs have been developed for this purpose. Unstable periodic orbits provide not only information about the attractor geometry and can be further used for calculation of metric descriptors such as dimension or Lyapunov exponents but can be treated as a basis for controlling chaos.

I. Introduction

Efforts of many research groups have been concentrated during the last decade on describing and understanding chaotic phenomena in deterministic systems. Existence of countable infinity of unstable periodic orbits as in the case of a Smale's horseshoe is considered as one of the main attributes of chaotic systems (e.g. chaos in the Shi'lnikov sense).

Two main appraoaches towards description of chaotic phenomena can be distinguished.

The first approach, often referred to in the literature as the *metric approach*, uses time-series data measured from the considered system to compute metric invariants such as fractal dimension, entropy, Lyapunov exponents or spectrum of singularities. All of these quantities represent averages over the attractor and thus require very large data sets (long time series), are difficult to compute, and the results are not always reliable.

The second approach, referred to as *topological*, describes spatial layout of system trajectories and provides information about geometry of the strange attractor. An interesting method belonging to this category, based on extraction of unstable periodic orbits embedded in the chaotic attractor has been elaborated in a series of papers [1], [2], [7], [8]. The basic idea of this approach is to find an approximation to the curvatures of any nonlinear multidimensional Poincaré map using a continuous polygonal surface made of hyperplanes in such a way that these hyperplanes are tangent to the graph of the map at the unstable periodic points and their slopes are determined by eigenvalues of the Jacobian matrices calculated at these points. One can obtain any needed accuracy of approximation as there exists a countable infinite number of unstable periodic orbits with growing periods and these orbits are dense on the asymptotic strange set — recovering more and more unstable cycles we obtain better approximations. The unstable periodic orbits existing within a particular type of chaotic atractor can be used also for calculating its metric characteristics e.g. the topological entropy and fractal dimension, universal constants such as the Feigenbaum constant and Lyapunov exponents [1], [2], [8], [10].

The main features of the characterisation in terms of unstable periodic orbits can be summarised as follows:

- Periodic orbits and their eigenvalues are topologically invariant — different representations of the same system (up to a smooth transformation of coordinates) must preserve their topological properties (a fixed point must remain a fixed point in any representation and the same applies to periodic orbits),
- Periodic orbits constitute a "skeleton" for the attractor — they determine its spatial layout,
- The eigenvalues of closed orbits are metric invariants — they describe the scaling between different pieces of the attractor.
- There exists a hierarchical ordering of unstable periodic orbits — short cycles give good approximations of the strange set.
- Periodic orbits are robust — they vary slowly with smooth parameter variations. The same applies to their eigenvalues.
- Unstable periodic orbits can be successfully extracted from experimental data - specific computational methods have been developed for this purpose and implemented in computer prrograms.

In this paper we analyse unstable periodic orbits embedded within different chaotic attractors existing in the canonical Chua's circuit.

II. Canonical Chua's Circuit

Canonical Chua's circuit proposed in [6] unifies all chaotic phenomena encountered so far in third order autonomous circuit in a single circuit consisting of two linear capacitors (C1, C2), one lineaar inductor (L), two linear resistors (R and G) and a single nonlinear (piecewise-linear) resistor called Chua's diode (Fig. 1). The dynamics of this circuit are described by:

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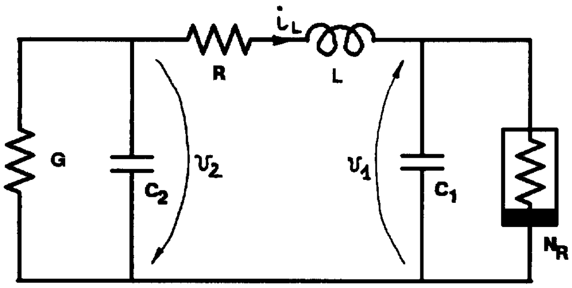

Fig. 1. The canonical Chua's circuit.

$$\begin{aligned}\frac{dv_1}{dt} &= \frac{1}{C_1}[-f(v_1)+i_L]\\ \frac{dv_2}{dt} &= \frac{1}{C_2}(-Gv_2+i_L)\\ \frac{di_L}{dt} &= -\frac{1}{L}(v_1+v_2+Ri_L)\end{aligned} \quad (1)$$

where the nonlinear resistor function is:
$i_R = f(v_R) = G_b v_R + \frac{1}{2}(G_b - G_a)(|v_R - 1| + |v_R + 1|)$,
G_a and G_b are the inner and outer slopes respectively of the nonlinear resistor

We use the following sets of circuit parameters (comp. Table 3 in [6]):
1. C1=1, C2=-0.6 , G=0.01 , GA=-0.445, GB=0.851 , L=-1.10 , R=-0.409 for which Rössler-type chaotic attractor has been observed.
2. C1=1, C2=-0.632, G=-0.0033, GA=-0.419, GB=0.839 , L=-1.02 , R=-0.330 for which the double scroll attractor exists.

In each case we analysed a computer generated data set of 500000 steady-state points (first 10000 points rejected to avoid transients) equally spaced in time (time step 0.1). Each trajectory was started with the initial condition $(0.1, 0, 0)$.

A modification of the Lathrop and Kostelich [10] technique has been used for recovering the unstable periodic orbits from an experimental time series. In this paper we adopt the following coding of unstable periodic orbits: all orbits are denoted by Π. The subscript 0 denotes orbits which do not wind around the origin. The superscript denotes the number of windings around the outer equilibria P^+ or P^-. Symmetric orbits winding around both of the equilibria P^+ and P^- have no subscripts. In the case of asymmetric orbits winding around P^+ and P^- the subscript denotes the number of windings around one of these points and the superscript denotes the number of windings around the other.

III. Periodic Orbits in the Rössler-type Attractor

Rössler-type chaotic orbits are the simplest of all chaotic trajectories found in the canonical Chua's circuit. It turns out that the unstable periodic orbits orbits embedded in this kind of attractor are also very simple and their complexity changes following a simple period-adding rule. For the particular choice of parameters the simplest orbit Π_0^2 winds twice around the equilibrium point P^- existing in the system. We found a single orbit with two windings, a single orbit with four windings Π_0^4, two distinct orbits with six windings Π_0^6, five orbits with ten windings Π_0^{10} etc. No orbits with eight windings were found. The number of orbits of a particular period length is specific to the analysed attractor and the particular set of parameter values. This is confirmed by experiments with Rössler-type attractors existing in Chua's circuit for different parameter settings — in all tested cases we found different structures of embedded unstable periodic orbits. One of the explanations of such a phenomenon could be the fact that regions of existence of chaos are separated by periodic windows and some of the bifurcating (destabilised) orbits from the windows are later found within the chaotic attractor (and thus the structure of characteristic unstable periodic orbits changes). It is also possible to explain the absence of any period-1 orbits - an orbit of this kind exists in the state space but lies far from the observed attractor.

IV. Unstable Periodic Orbits in the Double Scroll

The double scroll chaotic attractor is probably the most interesting of all the attractors encountered in Chua's circuit and persists for a wide range of parameter values. Typical attractor of this kind encountered for the parameter choice as given above is shown in Fig. 2a. During the bifurcation analysis it has been confirmed that the double scroll is born via the merging of two coexisting Rössler-type attractors when varying the bifurcation parameter value. Three types of unstable periodic orbits could be distinguished among the orbits detected within the double scroll atractor:

1. Pairs of simple (similar to the ones detected in the Rössler-type attractor which appear before the birth of the double scroll) asymmetric periodic orbits in two linear domains symmetrically placed about the origin (Π_0^1, Π_0^2, Π_0^3, Π_0^4 - Fig. 2b-f respectively ; Fig. 2d-e show two distinct kinds of period-3 Π_0^3 orbits).
2. Symmetric orbits passing through three linear domains, similar to the ones found in periodic windows and often similar in shape to heteroclinic orbits [9], [11]. Typical orbits of this kind are Π^2 (Fig. 2g), Π^4 (Fig. 2h) and Π^3 (Fig. 2i). The orbit Π^4 of Fig. 2h is a "period doubled" version of the orbit Π^2 shown in Fig. 2g. Possibly there exist also more complicated, longer period (4,8 etc.) orbits of this kind (these orbits were not detected durnig our experiment as in our search procedure we limited the maximum length of the searched orbit to 5000 iterates to reduce the computation time).
3. Asymmetric periodic orbits in three linear domains. These orbits are characterised by unequal number of windings around the equilibria P^+ and P^- (eg. Π_3^5 - Fig. 2j; Π_4^3 - Fig. 2k; Π_6^3 - Fig. 2l).

Comparing the unstable periodic orbits found in the case of the Rössler-type attractor with those detected in the double scroll it is interesting to note that more simple orbits were found in the double scroll than in the Rössler-type attractor — we have no explanation why the double scroll contains also "fractional period orbits" ie. ones with periods smaller than (1/2 etc.) or between (3/2 etc.) those found in the Rössler-type attractor analysed in the previous section. This seems to contradict the observation that the double scroll is born via a merging of coexisting Rössler spiral attractors. One possible explanation could be given on

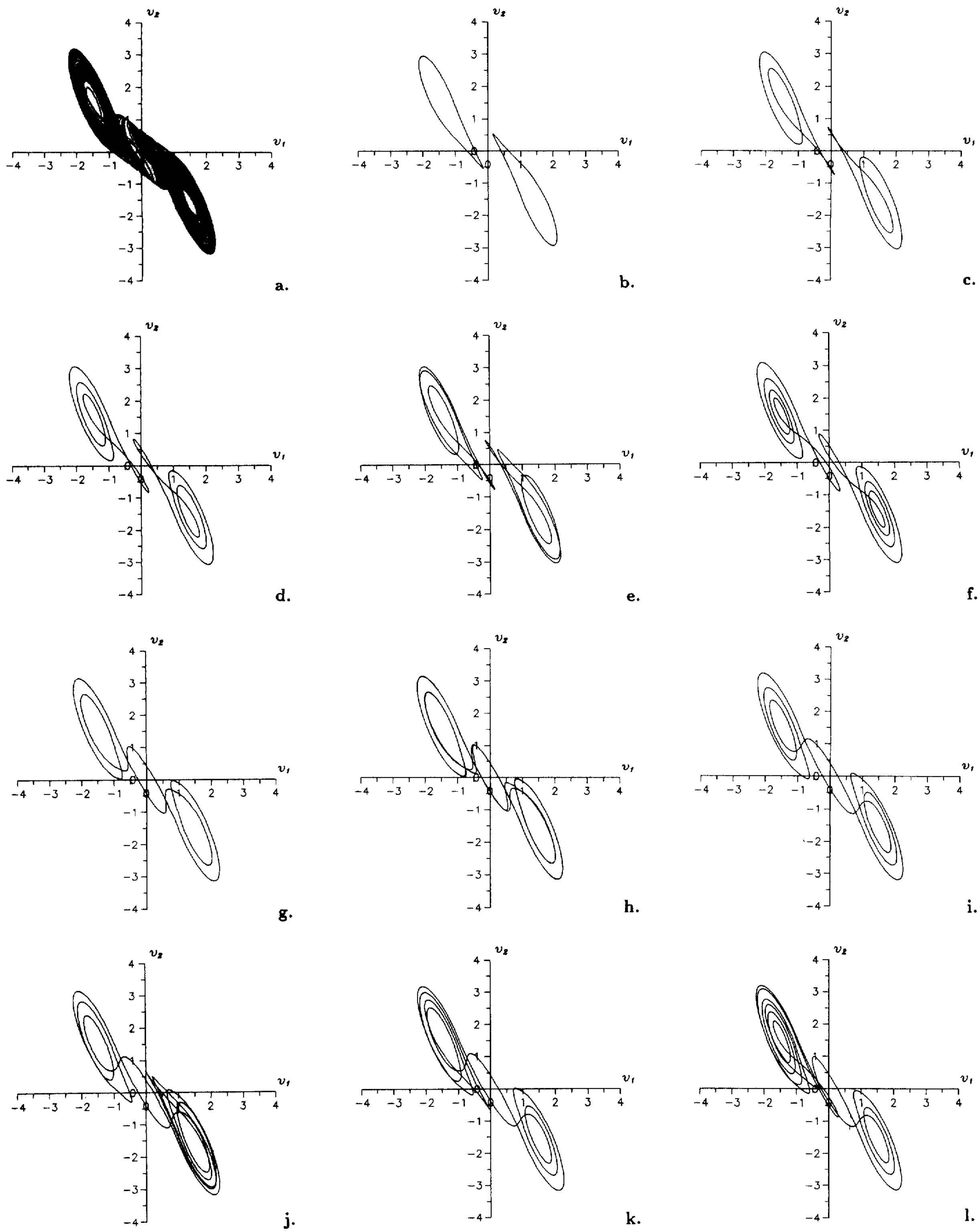

Fig.2. Double scroll chaotic attractor encountered in Chua's circuit (a); some of the unstable periodic orbits embedded within this attractor (b-l) (see text for description and comments).

the basis of analysis of the bifurcation diagrams. Among the unstable periodic orbits found within a particular kind of chaotic attractor there are not only the orbits which lost stability in the period doubling bifurcations (of period 2^n times the basic period of period-one orbit) but also unstable orbits born via other types of bifurcations e.g. intermittency or saddle-node bifurcations in periodic windows.

Our observation supports the claim that the skeleton of periodic orbits is specific to the particular type of attractor observed and some of the simple (two domain) orbits characterising the double scroll cannot exist in the Rössler attractor.

V. Conclusions

Using a simple software package we were able to uncover, from discrete time series of state variables, the hierarchy of unstable periodic orbits embedded within typical chaotic attractors encountered in the canonical Chua's circuit. This hierarchy (ie. the lengths and the number of orbits of distinct types) is specific to the particular sets of circuit parameters and gives a characterisation of the attractor which exists for this choice of parameters — it is possible to distinguish between the attractors by looking at their respective hierarchies of unstable periodic orbits.

There are several numerical problems that should be mentioned. Firstly, fixing arbitrarily the accuracy (admissible error ε) at the beginning of the search procedure for periodic orbits we overlook the possibility that there may exist orbits of high period which pass many times through the assumed neighborhood. Secondly, it turns out that it is quite difficult to identify similar but different orbits reliably. The criterion for distinguishing between similar orbits fails in many situations. It is very difficult to tell on the basis of time series analysis alone how many distinct orbits of a given length are emebedded within the attractor. This is due to the fact that we always operate on a finite data set and in fact there is no practical indication how large the threshold (admissible error) in this case should be chosen. Usually this is achieved by trial and error and requires some skill. In some cases this problem could be overcome by additional analysis of the Poincaré map if its analytic form is accessible. It may also be possible to develop a description of unstable periodic orbits in the canonical Chua's circuit in terms of symbolic dynamics or knot theory.

The uncovered unstable periodic orbits could be used not only for the characterisation of attractor geometry. One of the main research problems in the last two years has been the control of chaotic dynamical systems [12], [13], [14]. One of the approaches to controlling chaotic sytems is the so-called OGY method (Ott-Grebogi-Yorke).

The key development in this approach is control in terms of stabilisation of any arbitrarily chosen unstable periodic regime existing in the chaotic state. It has been demonstrated that this type of control can be achieved by an infinitesimal change of the system parameters. The results presented in this paper can be considered as part of a larger project the goal of which is to control chaos in Chua's circuit.

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