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CONTROL BIFURCATION STRUCTURE OF RETURN MAP CONTROL IN CHUA'S CIRCUIT

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We demonstrate that *return map control* and *adaptive tracking* can be used together to locate, stabilize, and track unstable periodic orbits (UPOs). Through bifurcation studies as a function of some control parameters of return map control, we observe the *control bifurcation* (CB) phenomenon which exhibits another *route to chaos.* Nearby an UPO there are a lot of *driven periodic orbits* (DPOs) along the CB route. DPOs are not embedded in the original chaotic attractor, but they are generated artificially by driving the system slightly in a direction with feedback control. Based on the CB phenomenon, our adaptive tracking algorithm searches for the location and the exact control condition of the UPO by minimizing feedback perturbations. We discuss the *universality* of the CB phenomenon and the possibility of *immediate control* which does not require much prior analysis of the system.

1. Introduction

The basic concept of controlling chaos is that a lot of unstable periodic orbits (UPOs) are embedded in a chaotic attractor and they can be stabilized by applying small time-dependent parameter perturbation. Recently, extensive theoretical and experimental researches have been made in this field that the concept of controlling chaos is considered as quite common. Using these control methods, we cannot only control a chaotic system to periodic motion, but also design new chaotic systems in which we utilize chaotic behaviors in a positive sense.

OGY [Ott *et al.*, 1990] proposed a feedback control method which is able to stabilize UPOs by making small time-dependent perturbation on some available system parameters. Although OGY method have provided an important breakthrough of the theoretical aspect of controlling chaos, it is a little complicated because we have to reconstruct an attractor and establish a Poincaré map from an experimental time series. Furthermore it requires much prior analysis of the system dynamics and its responses on the change of system parameters. A number of modifications of OGY have been introduced depending on the systems under study.

Hunt [1991] proposed occasional proportional feedback (OPF) method as an 1D simplification of OGY. In this approach, an experimental time series is sampled with some sampling period T, and the feedback control proportional to the difference between the set point and the sampled data is applied to the system if the sampled data is within the control window. It was proven to be very effective especially for high-frequency systems such as chaotic lasers [Roy *et al.*, 1992; 1994]. In this approach, however, the sampling period must be nearly matched with the system's natural period for successful control.

Peng and Petrov [1991, 1992] simplified OGY in another way. They established an 1D return map from an experimental time series and targeted the fixed point of an unstable periodic orbit in it. They demonstrated their method using the chemical Belousov–Zhabotinsky (BZ) reaction system. Compared to OPF, it has the advantage of using the natural period of the system.

For the tracking of UPO, Triandaf and Schwartz [1993] used a *predictor-corrector scheme* as an experimental continuation method. Using the current and the past control conditions, a new control condition for a slightly moved system is predicted. Applying the predicted control, system responses are analyzed and the control condition is corrected for the error to be brought to zero. For example, the fixed point X_F of an UPO is adjusted until the mean of the parameter perturbation is brought to zero.

Petrov *et al.* [1994] proposed another tracking procedure. Based on return map control, they applied *mild convergent* and *mild divergent* controls alternately to the system, analyzed the system responses, and calculated a new control condition.

In this paper we use return map control and adaptive tracking together to locate, stabilize, and track UPO. Through bifurcation studies as a function of some control parameters of return map control, we study the change of the controlled periodic orbit when some control parameters of return map control are moved away from the exact control condition to UPO. We observe the *control bifurcation* (CB) phenomenon which suggests the existence of another *route to chaos* induced by feedback control. We show that there are a lot of *driven periodic orbits* (DPOs) nearby an UPO along the CB route. Our adaptive tracking method is based on the CB phenomenon.

We first briefly review return map control, the CB phenomenon, and our adaptive tracking method. The experimental results applied to the chaotic Chua's circuit are continued. We further discuss the universality of the CB phenomenon and the possibility of immediate control which does not require much prior analysis of the system.

2. Control and Tracking Algorithm

2.1. *Control algorithm*

We use return map control with the concept of control window. From a measured time series we collect all the local maximum states X_n and construct a return map from them. In the local vicinity of the fixed point of an UPO, a *local linear approximation* can be made. Once an orbit X_n occasionally comes within the local linear region, we apply feedback perturbation proportional to the difference between the measured state and the targeted fixed point to some accessible system parameters or state variables.

The dynamics in the local vicinity of a fixed point on the return map can be approximated by

$$X_{n+1} = \lambda(X_n - X_F) + X_F \,, \tag{1}$$

where X_n is the nth local maximum state, X_F is the targeted fixed point, and λ is the Floquet multiplier. If the width of the control window is set to X_w and X_n comes into the control window ($X_F - X_w < X_n < X_F + X_w$), the feedback perturbation to stabilize the UPO X_F can be calculated by [Petrov *et al.*, 1994]

$$\delta p = k(X_n - X_F) \,, \tag{2}$$

where the proportionality constant k is given by

$$k = \frac{\lambda}{(\lambda - 1)\dfrac{\partial X_F}{\partial p}} \,. \tag{3}$$

The feedback perturbation δp is applied only during a kick time d. The kick time d is an another control parameter and it should be less than the natural period of the system.

The concept of the control window represents the range of linear approximation of the dynamics. If it is too narrow, the length of the chaotic transient will increase and the dynamics will be sensitive to external noise reducing the stability of the controlled periodic orbit. If it is too wide, the approximated linear control given by Eqs. (1)–(3) will fail. In this study we use a relatively wide control window for the global control of the period 1 UPO. We also try to show that our adaptive tracking method can be used to find the exact location and the control condition of UPO over a wide control window. Furthermore we are interested in the change of controlled orbits when the control condition moves away from the exact control condition to UPO.

To stabilize higher periodic UPOs, one usually has to construct a *delayed return map* of X_{n+m} versus X_n where m is the period of UPO. But the introduction of the control window makes it easier with the same delay 1 return map. If we set a narrow control window and only consider the states which come into it, some high periodic UPOs can be selected and stabilized depending on the selection of a fixed point X_F and the width of a control window X_w.

The feedback perturbation δp is considered as the *control output* of the feedback controller. Depending on the application of the control output, feedback control can be classified as the *parameter perturbation* and the *state perturbation.* In the next section, we will show that both of them work well under the same theoretical basis.

2.2. *Control bifurcation*

Bifurcation is one of the important characteristics of chaotic systems which represents the hierarchical coexistence of order and chaos in a chaotic system. The *system bifurcation* (SB) as a function of some system parameters is the property of the system itself. Now we consider the case of feedback controlled chaotic systems and the bifurcation as a function of some control parameters of feedback control.

We are interested in the change of controlled orbits when the control condition moves away from the exact control condition to UPO. The control parameters in return map control are the fixed point X_F, the proportionality constant k, the width of the control window X_w, and the kick time d. Assume that we have the exact control condition to UPO, i.e., $X_F = X_{F_0}$ and $k = k_0$, for some fixed $X_w = X_{w_0}$ and $d = d_0$. What happens if we fix $X_F = X_{F_0}$ and move k around k_0, or fix $k = k_0$ and move X_F around X_{F_0}? Through bifurcation studies as a function of some control parameters, we observe the *control bifurcation* (CB) diagrams qualitatively similar to SB. We will present only several results in the following section, but CB seems to be a *universal phenomenon* which is widely observed in many chaotic systems and in many feedback control methods.

2.3. *Adaptive tracking of unstable periodic orbit*

The CB route suggests a new possibility for the control and the tracking of chaotic systems. CB shows that there is a bifurcation phenomenon as a function of some control parameters and UPO is located in the middle of CB. There are *two directions* in CB, one to order and the other to chaos. If we check the fluctuation of feedback perturbations along the CB route, we observe that UPO is the point of zero-mean of the perturbations along the X_F-mode CB and the point of minimum deviation of the perturbations along the k-mode CB. Based on these facts, we can start control immediately, if we know the approximate location of the fixed point and the *direction to order.*

Firstly we get an approximate fixed point of UPO from the analysis of the original return map. To find the direction to order, we apply some *small test feedback controls* to the system along the two possible directions and analyze the system responses. If the dynamical range of the system *shrinks* along one direction, it is the direction to order. For the adaptive tracking of UPO we apply feedback control with some starting control conditions, collect all the data which come into the control window of the return map, and analyze the running average and running deviation of the data. We adjust the fixed point X_F to the running average and adjust the proportionality constant k for the running deviation to be brought to zero using the following equations,

$$X_F(n+1) = X_F(n) + \alpha(X^* - X_F(n)), \quad (4)$$

$$k(n+1) = k(n) + \mathrm{sgn}(k(n))\beta\delta^*, \quad (5)$$

where X^* is the running average and δ^* is the running deviation. α and β are some stiffness constants of the adjustment and $\mathrm{sgn}(k(n))$ is the predetermined direction to order. In this way we can locate and stabilize UPO automatically.

When the system condition is moving slowly, tracking of UPO can be made using the same method. The change of system parameter causes the change of feedback perturbation and our adaptive tracking procedure chooses a new control condition for the feedback perturbation to be brought to zero.

3. Results

3.1. *Locating and stabilizing unstable periodic orbit*

We demonstrate our control and tracking method numerically with the well known *Chua's circuit* [Shil'nikov, 1994]. It is a simple electronic circuit which displays a variety of dynamical behaviors, from chaos to order. Its simplified dynamical equations are

$$\begin{aligned} \dot{x} &= A(y - x - f(x)), \\ \dot{y} &= x - y + z, \\ \dot{z} &= -By, \end{aligned} \quad (6)$$

where $f(x) = bx + \frac{1}{2}(a-b)(|x+1| - |x-1|)$ is the piecewise-linear negative resistance, and A, B, a, and b are system parameters.

In this paper we use a *standard system parameter set* ($A = 8.3$, $B = 14.87$, $a = -1.27$, and $b = -0.68$) which provides a double-scroll chaotic attractor. To start return map control, we can construct a return map from any one of the three time series. In the point of the control output of the feedback controller, we can apply feedback perturbations to any one of either the system parameters or the state variables. As a *standard control condition* we construct the return map from the time series z and perturb the parameter A (denote as $z \to A$). Furthermore, we set the width of the control window X_w to 1.5 and the kick time d to 1.0. Note that the natural period of the system is about 2.1 for the standard system parameter set.

To find the location of period m UPO we collect all the local maximum values of z and construct the return map of X_{n+m} versus X_n. The intersection of the collected data with the $X_{n+m} = X_n$ bisectrix is the fixed point X_{F_1}. The fixed point obtained in this way has some experimental uncertainty. If we apply our adaptive tracking method starting from X_{F_1}, we can get the exact location of UPO and stabilize it.

The experimentally obtained fixed point of the period 1 UPO (UPO_1) is $X_{F_1} = 4.2$. Applying a small test feedback control $k_1 = -0.1$, we observe the shrinkage of dynamical range which represents that the direction to order is negative in this case. Starting from $X_{F_1} = 4.2$ and $k_1 = -0.1$, we apply the adaptive tracking given by Eqs. (4) and (5). Figure 1 shows the process of automatic search and stabilization of UPO_1. Note that the fixed point X_F is adjusted continually and the control signal converges rapidly to zero. The experimentally tracked exact location of the fixed point and its control condition are $X_{F_0} = 4.2385$ and $k_0 = -0.2$.

This is a simple experimental control procedure and does not require much information on the system dynamics. With an approximate location of the fixed point and the direction to order, we can start control immediately for any experimental system.

3.2. *Control bifurcation*

If the control parameters, X_F and k, are moved away from X_{F_0} and k_0, a *control bifurcation* (CB) appears. Typical CB diagrams are shown in Fig. 2. Figure 2(a) is the k-mode CB obtained by moving k from 0 to -0.4 for a given $X_F = X_{F_0}$. The direction to order is negative in k. As k moves from zero control ($k = 0$) to $k_{b_1} = -0.184$, a *reverse bifurcation* appears converging to UPO_1. k_{b_1} is the first bifurcation point between UPO_1 and period 2 orbits. It represents the least control condition required to stabilize UPO_1. When k moves further over some critical value, $k_c = -0.38$, control to UPO_1 becomes unstable and the system switches to another periodic orbits. For a wide range of k ($k_c < k < k_{b_1}$), UPO_1 can be stabilized. Figure 2(b) is the X_F-mode CB obtained by moving X_F from 4.2 to 4.6 for a given $k = k_0$. Another CB diagram appears and the direction to order is negative in X_F. Note that the characteristic period 7 double-scroll orbit appears in Fig. 2(a) and a period 7 single-scroll orbit appears in Fig. 2(b).

To get more systematic information about the CB structure of return map control, we search for the distribution of bifurcation points in the control

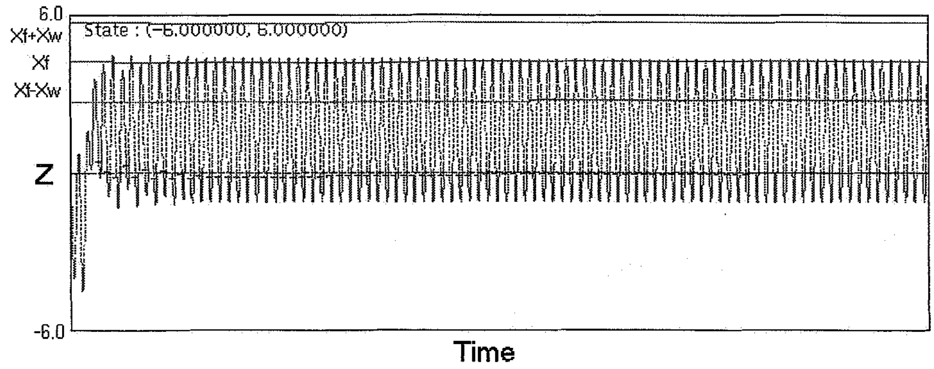


Fig. 1. Automatic search and stabilization of the period 1 UPO (UPO_1) by using return map control and adaptive tracking. Horizontal axis is time in arbitrary unit and vertical axis is the state variable z. Small horizontal signals around the $z = 0$ axis are the control signals. Note that the fixed point X_F is adjusted continually and the control signal converges rapidly to zero.

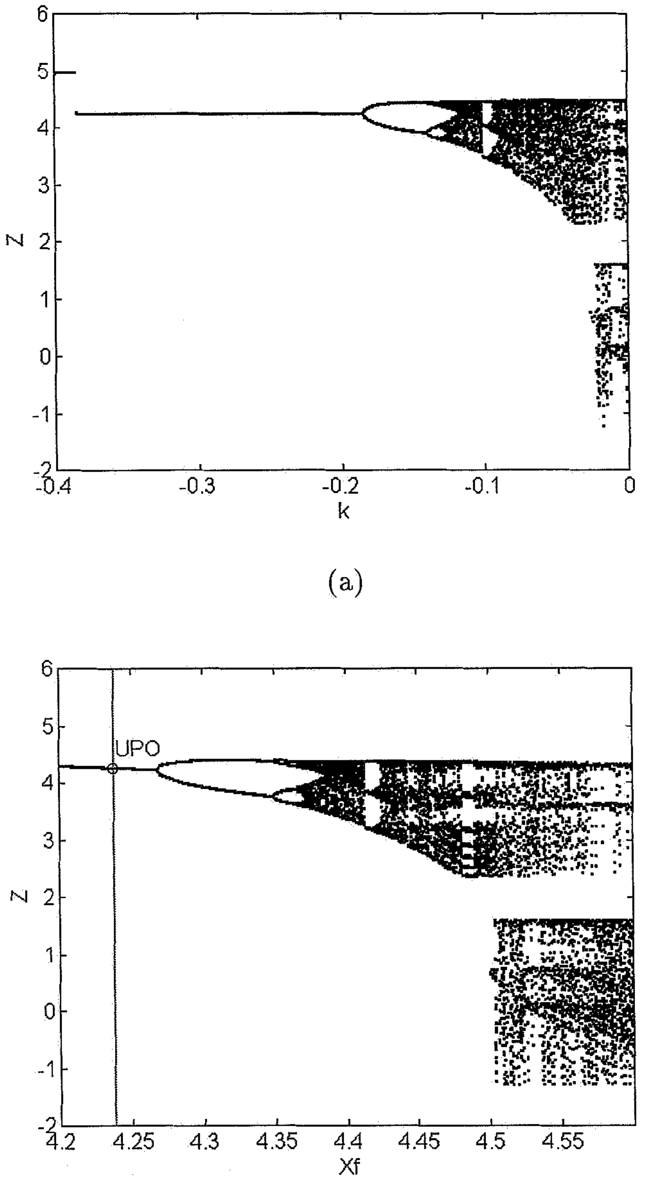


Fig. 2. Typical control bifurcation diagrams. (a) k-mode CB with $X_F = X_{F_0}$, (b) X_F-mode CB with $k = k_0$.

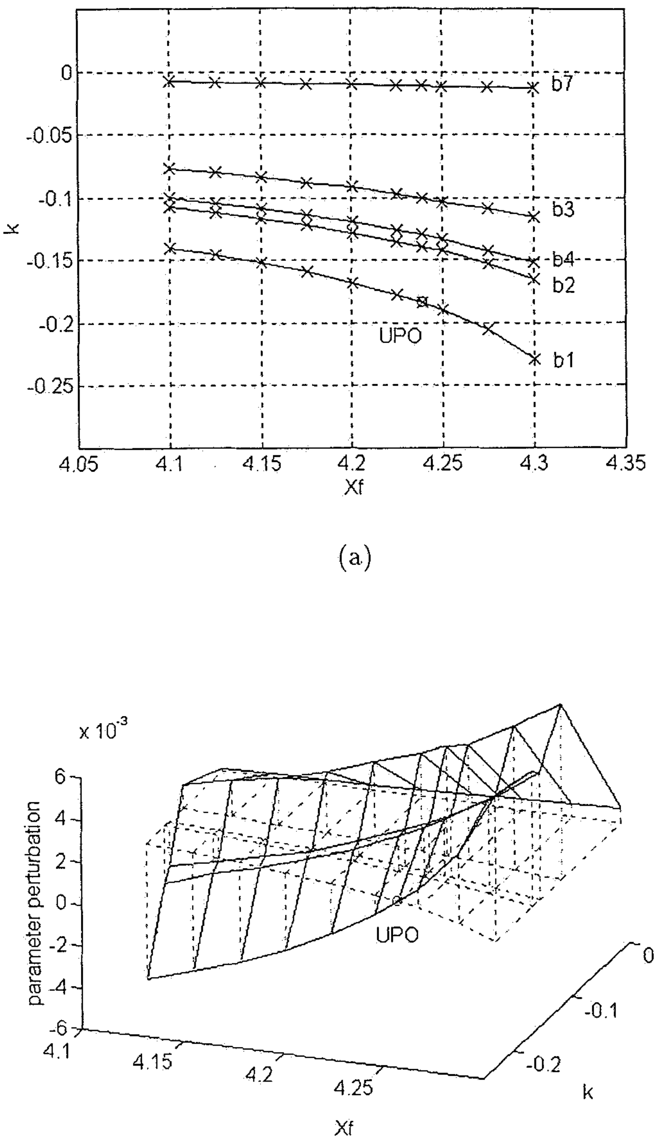


Fig. 3. Control bifurcation structure of return map control in (X_F, k) plane. (a) Distribution of bifurcation points in (X_F, k) plane, (b) 3D view of the mean of the feedback perturbation at the bifurcation points.

parameter space (X_F, k). Figure 3(a) shows the distribution of bifurcation points around UPO_1. b_1 is the bifurcation points from period 1 to period 2 orbit, b_2 from period 2 to period 4 orbit, and b_4 from period 4 to period 8 orbit, respectively. b_3 and b_7 are the locations of the periodic window of period 3 and period 7, respectively. A unified and continuous CB structure seems to exist in the control parameter space. Figure 3(b) shows the 3D view of the mean of the feedback perturbations at the bifurcation points. Note that the only UPO is $X_F = X_{F_0}$ for $k_c < k < k_{b_1}$ and the feedback perturbations have zero mean only at UPO. The possibility of adaptive tracking comes from this fact. Indeed, the adaptive tracking given by Eqs. (4) and (5) represents the process of automatic search for UPO along the CB route.

What are the periodic orbits appearing in the CB diagrams? All the periodic orbits except UPO are some kind of *driven periodic orbits* (DPOs) which are generated artificially by driving the chaotic system in a direction with feedback control. To stabilize UPO, only small and zero-mean feedback perturbation is required. On the other hand, to stabilize DPOs, we should drive the system with quite a lot and non-zero-mean feedback perturbations. Figure 4 shows that the period 2 DPO generated by $X_F = X_{F_0}$ and $k = -0.17$ is not embedded in the original chaotic attractor while UPO_1 is embedded in it. If we want to control the chaotic attractor to slightly different periodic orbits rather than UPO and a small system modification is allowed, DPOs in the CB route can be good candidates.

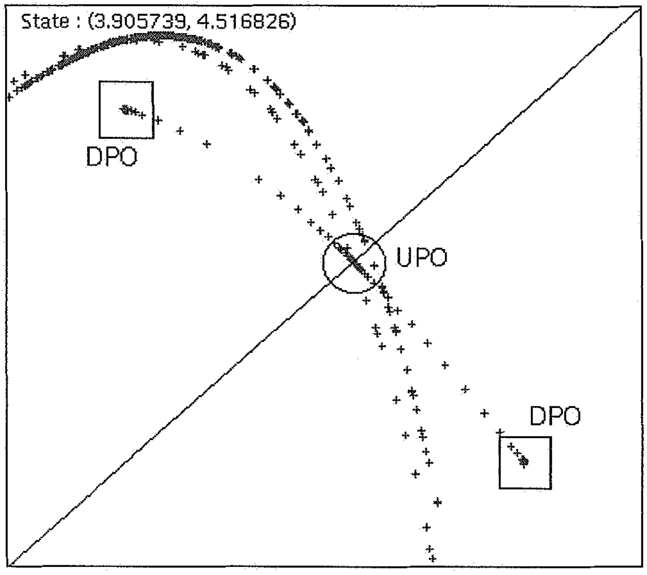


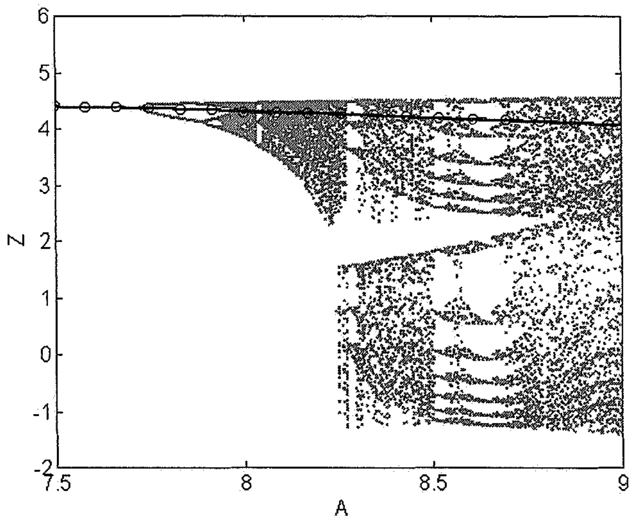
Fig. 4. DPO, UPO and uncontrolled chaotic attractor in return map. DPO is not embedded in the original chaotic attractor while UPO is embedded in it.

Table 1. k values of the first bifurcation points to the period 1 UPO for all combinations of control input and control output. They are all searched and stabilized by using adaptive tracking.

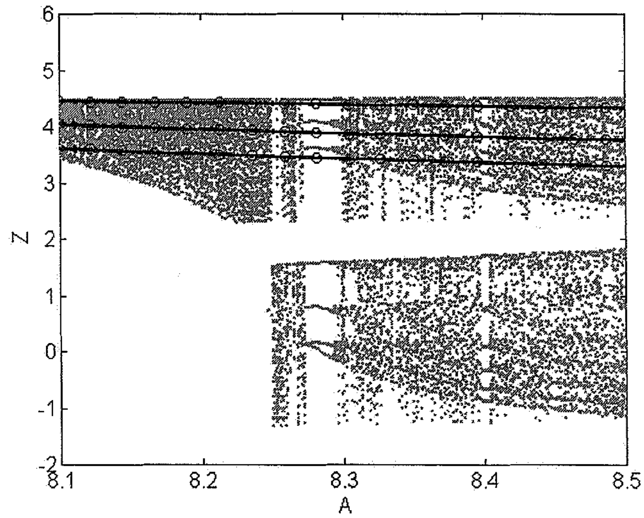
In / Out	x	y	z
	$X_{F_0} = 2.7771$	$X_{F_0} = 0.5473$	$X_{F_0} = 4.2385$
x	+0.937	−3.170	−0.454
y	+(*)	−8.100	−0.805
z	+1.410	−6.540	−0.833
A	−0.364	−4.000	−0.184
B	+0.384	+5.600	+0.165
a	+0.099	+0.490	+0.053
b	+0.326	+1.470	+0.214

(*) +1.563 with $d = 1.8$.

If we try the return map control and the adaptive tracking for other combinations of control input and control output including both parameter perturbation and state perturbation, we get similar results. Table 1 shows the k values of the *first bifurcation points* for all combinations. The existence of the first bifurcation points implies the existence of the CB phenomena and the controllability to UPO. Note that all combinations have



(a)



(b)

Fig. 5. Tracking UPO using our adaptive tracking method. Circled solid line represents the tracked UPO. (a) Tracking the period 1 UPO (UPO_1) when A moves from 7.5 to 9.0, (b) Tracking the period 3 UPO (UPO_3) when A moves from 8.1 to 8.5.

characteristic directions to order and the same UPO can be stabilized by applying feedback perturbation to any one of the system parameters or the state variables. Both the parameter perturbation and the state perturbation methods work well under the same theoretical basis.

3.3. *Tracking unstable periodic orbit*

Figure 5 shows the results of tracking UPO using our adaptive tracking method when the system parameter A moves slowly. Figure 5(a) represents the tracking of UPO_1 when A moves slowly from 7.5 to 9.0. As A increases, the system goes to more chaotic regime and the control to UPO_1 becomes unstable. The adaptive tracking procedure given by Eqs. (4) and (5) detects the change of the system responses under the parameter change and updates the control conditions, X_F and k, until the mean and the deviation of feedback perturbations converge to zero. Figure 5(b) represents the tracking of the period 3 UPO (UPO_3) when A moves slowly from 8.1 to 8.5. We target the topmost fixed point among the 3 possible fixed points and select a narrow control window of $X_w = 0.2$. Depending on the selection of a fixed point X_F and the width of a control window X_w, some high periodic UPOs can be selected, stabilized, and tracked. But a narrow control window can make the control process be sensitive to external noise.

4. Conclusions

We demonstrated that return map control and adaptive tracking method can be used together not only to track UPO, but also to locate and stabilize UPO. Our adaptive tracking method is based on the CB phenomenon. This is an experimental method which does not require much prior analysis of the dynamics, so it is easily applicable to a wide range of experimental systems.

CB generated by feedback control provides another *route to chaos* and it seems to be a *universal phenomenon.* In this paper we only presented the case of Chua's circuit and the case of return map control, but it is widely observed in many other chaotic systems and in many other feedback control methods. We are mainly interested in the control to UPO, but DPOs existing nearby UPO along the CB route are also available. DPOs are generated by driving the system slightly in a direction with nonzero-mean feedback perturbation and they are not embedded in the original chaotic attractor. If we need slightly different periodic orbits rather than UPO and a small system modification is allowed, DPOs can be good candidates.

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