LOCALIZATION PROBLEM FOR LIMIT CYCLES OF CHUA CIRCUIT

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Abstract.
A general method for estimation of domains with limit cycles and finding surfaces with the traces of all cycles is proposed. Corresponding estimations of domains with cycles for piecewise linear systems and Chua system are indicated.

1 INTRODUCTION

The many properties of nonlinear system

$$\dot{x} = f(x), \quad x \in \mathbf{R}^n, \quad f(x) \in C^\infty(\mathbf{R}^n) \tag{1.1}$$

depend on the existence of periodic oscillations in system (1.1).

Any T-periodic solution $x(t)$, $x(t+T) = x(t)$, to the nonlinear system (1.1) specifies the mapping $\mathbf{R} \to \mathbf{R}^n$, $t \to x(t)$, and image is called a *limit cycle* (or a cycle) of system (1.1).

In present time there is no analytical algorithm for finding limit cycles of dynamic systems. Usually they find limit cycles by means numerical methods. For using these methods we need to know two sets. They are a set of initial states and a set containing all limit cycles.

In the present paper we consider localization problem of the periodic oscillations for dynamic systems (1.1) in the following settings: 1) For system (1.1) find a surface S such that any limit cycle of system (1.1) either intersects or touches S. This surface is called Poincare universal intersection and it is a set of initial states for numerical integration; 2) For system (1.1) find a set Ω such that all limit cycles of system (1.1) are in Ω .

2 LOCALIZATION METHOD

For $\varphi \in C^1(\mathbf{R}^n)$ denote by $L_f\varphi$ derivative of the function φ with respect to system (1.1), i.e.

$$L_f\varphi(x) = \sum_{i=1}^{n} f_i(x)\partial\varphi(x)/\partial x_i,$$

where $(f_1(x), \ldots, f_n(x))^T = f(x)$.

Theorem 2.1 *For any function $\varphi \in C^1(\mathbf{R}^n)$, any cycle of system (1.1) contains at least two points of the set [1-2]*

$$S_\varphi = \{x : L_f\varphi(x) = 0\}. \tag{2.1}$$

For the function $\varphi \in C^1(\mathbf{R}^n)$ we put

$$\begin{aligned} \varphi_{sup} &= \sup\{\varphi(x) : x \in S_\varphi\}, \\ \varphi_{inf} &= \inf\{\varphi(x) : x \in S_\varphi\}. \end{aligned} \tag{2.2}$$

Theorem 2.2 *For any function $\varphi \in C^1(\mathbf{R}^n)$, all limit cycles of system (1.1) belong to the set [1-2]*

$$\Omega_\varphi = \{x : \varphi_{inf} \le \varphi(x) \le \varphi_{sup}\}. \tag{2.3}$$

Collorary 2.1 *All limit cycles of system (1.1) belong to the set [1-2]*

$$\Omega = \{\cap\Omega_\varphi, \varphi \in C^1(\mathbf{R}^n)\}. \tag{2.4}$$

3 PIECEWISE LINEAR SYSTEM

Let system (1.1) be a piecewise linear system of the form

$$\dot{x} = \begin{cases} Ax + b, & x_1 \le -l, \\ A_0x, & |x_1| < l \\ Ax + c, & x_1 \ge l, \end{cases} \tag{3.1}$$

where x, b, $c \in \mathbf{R}^n$, and matrices A, $A_0 \in M_n(\mathbf{R})$.

Any real matrix A can be transformed by a transformation $T^{-1}AT = A'$ into Jordan canonical form

$A' = \text{diag}\,(J_1(\lambda_1), J_2(\lambda_2), \ldots, J_m(\lambda_m))$, where λ_k, $k = \overline{1,m}$, are the roots of the characteristic equation of matrix A, and $J_s(\lambda_s)$, $s = \overline{1,m}$ are canonical Jordan blocks. Then in the new variables $z = T^{-1}x$ system $\dot{x} = Ax + c$ has the form $\dot{z} = A'z + T^{-1}c$.

If the condition $\text{Re}\lambda_k \neq 0$ is fulfilled for any root λ_k of the characteristic equation of matrix A, then there exists a function

$$\Phi(z) = \sum_{j=1}^{n} d_j z_j^2, \qquad (3.2)$$

where $d_j \neq 0$, $|d_j| > q_j$, such that the surface S_Φ is an ellipsoid El_c. For this function and the system $\dot{x} = Ax + b$ the corresponding surface S_Φ is also an ellipsoid El_c. For function (3.2) and the system $\dot{x} = A_0 x$ the corresponding surface $S_\Phi = \{z | z^T W z = 0\}$ is a cone $C_\Phi \neq \emptyset$. Therefore, for function (3.2) and piecewise linear system (3.1) we get

$$\begin{aligned} S_\Phi = & (El_b \cap \{z \colon x_1(z) \leq -l\}) \\ & \cup (C_\Phi \cap \{z \colon |x_1(z)| < l\}) \\ & \cup (El_c \cap \{z \colon x_1(z) \geq l\}). \end{aligned} \qquad (3.3)$$

Theorem 3.1 *For piecewise linear system (3.1), if 1) the system (3.1) is a continuous one; 2) $Re\lambda_k \neq 0$ for any root λ_k of the characteristic equation for matrix A; 3) the intersection of n-dimensional ellipsoid El_c and the hyperplane $x_1(z) = l$ is an $(n-1)$-dimensional ellipsoid; then surface (3.3) is a compact set.*

Theorem 3.2 *For continuous piecewise linear system (3.1), if the conditions of Theorem 3.1 are fulfilled, then there exists a compact set*

$$\Omega = \Omega_\Phi \cap \Omega_{\Phi'} \qquad (3.4)$$

containing all limit cycles of system (3.1), where Φ and Φ' are different functions of the form (3.2).

4 CHUA CIRCUIT

The Chua's circuit is a rather simple electronic oscillator (in the simplest case it consists of only four linear elements and one nonlinear element). So the Chua's circuit is a very suitable subject for study by means of both laboratory experiments and computer simulations because it admits an adequate modelling via the language of differential equations.

Consider the simplest case Chua's equations written in the following dimensionless form [3]:

$$\left\{ \begin{array}{l} \dot{x} = \alpha(y - h(x)), \\ \dot{y} = x - y + z, \\ \dot{z} = -by, \end{array} \right. \qquad (4.1)$$

where $h(x) = m_1 x + 0.5(m_0 + m_1)(|x-1| - |x+1|)$, $a > 0$, $b > 0$, $m_0 > 0$, $m_1 > 0$. Let $\bar{x} = (x, y, z)^T \in \mathbf{R}^3$. System Chua is a example of continuous piecewise linear systems. The system (4.1) is considered to be a Chua system if its equilibrium points are not asymptotically stable. Find all these points and research their characters.

The system (4.1) has three equilibrium points:

1) $O_1((m_0 + m_1)/m_1, 0, -(m_0 + m_1)/m_1)$ in the domain $\{x \geq 1\}$;

2) $O_3(0, 0, 0)$ in the domain $\{|x| < 1\}$;

3) $O_2(-(m_0 + m_1)/m_1, 0, (m_0 + m_1)/m_1)$ in the domain $\{x \leq -1\}$.

In the domain $\{|x| < 1\}$ the point O_3 is unstable. In the domain $\{|x| \geq 1\}$ system (4.1) has the following Jacoby matrix:

$$A = \begin{pmatrix} -am_1 & a & 0 \\ 1 & -1 & 1 \\ 0 & -b & 0 \end{pmatrix},$$

and the characteristic equation for A is

$$\lambda^3 + (1 + am_1)\lambda^2 + (b - a + am_1)\lambda + abm_1 = 0. \quad (4.2)$$

The points O_1 and O_2 are not asymptotically stable if and only if

$$(1 + am_1)(am_1 + b - a) - abm_1 \leq 0. \qquad (4.3)$$

Under this condition system (4.1) is a Chua system, and in the further consideration we suppose that it is fulfilled.

Theorem 4.1 *Characteristic equation (4.2) for the matrix A of system (4.1) has no roots with null real part iff $(1 + am_1)(am_1 + b - a) - abm_1 \neq 0$.*

Proof. In the domain $\{|x| \geq 1\}$ characteristic equation (4.2) has no null roots due to the fact that $abm_1 > 0$. Let $\lambda = \mu i$ be an imaginary root of (4.2), $\mu \in \mathbf{R}^1$, $\mu \neq 0$. If we submit $\lambda = \mu i$ into equation (4.2), then we will have

$$[-(1 + am_1)\mu^2 + abm_1] + i[-\mu^3 + \mu(am_1 + b - a)] = 0,$$

i.e.

$$\left\{ \begin{array}{rcl} abm_1 - (1 + am_1)\mu^2 & = & 0, \\ \mu(am_1 + b - a - \mu^2) & = & 0. \end{array} \right.$$

Since $\mu \neq 0$, we receive

$$\mu^2 = abm_1/(1 + am_1) = am_1 + b - a.$$

So equation (4.2) has an imaginary root under the conduction $(1+am_1)(am_1+b-a)-abm_1=0$. This completes the proof of Theorem 4.1. ▹

In the further consideration we insist that

$$(1+am_1)(am_1+b-a)-abm_1<0 \qquad (4.4)$$

in accordance with both (4.3) and the condition of Theorem 4.1. In this case it is possible to use all the results obtained for piecewise linear systems.

Theorem 4.2 *For Chua system (4.1) there exists a quadric quantic*

$$\varphi(\bar{x})=\alpha x^2+\beta y^2+\gamma z^2+2\lambda xy+2\mu xz+2\delta yz, \quad (4.5)$$

such that the set S_φ is a compact surface.

Proof. For function $\varphi(\bar{x})$ (4.5) the Lie derivation with respect to the system (4.1) is

$$\begin{aligned}L_f\varphi(\bar{x}) &= 2\{\lambda x^2-a\alpha h(x)x+(a\lambda-\beta-b\delta)y^2+\\ &+\delta z^2+(a\alpha+\beta-\lambda-b\mu)xy-a\lambda h(x)y+\\ &+(\lambda+\delta)xz-a\mu h(x)z+(a\mu+\beta-\delta-b\gamma)yz\}.\end{aligned}$$

Since $h(x)=m_1x+0.5(m_0+m_1)(|x-1|-|x+1|)$, in the domain $\{|x|\geq 1\}$

$$\begin{aligned}L_f\varphi(\bar{x}) &= 2\{(\lambda-am_1\alpha)x^2+(a\lambda-\beta-b\delta)y^2+\\ &+\delta z^2+(a\alpha+\beta-\lambda-b\mu-am_1\lambda)xy+\\ &+(\lambda+\delta-am_1\mu)xz+(a\mu+\beta-\delta-b\gamma)yz+\\ &+\operatorname{sign}(x)a(m_0+m_1)(\alpha x+\lambda y+\mu z)\}.\end{aligned}$$

Put

$$\begin{aligned}&c_1=\lambda-am_1\alpha, \quad c_2=a\lambda-\beta-b\delta,\\ &c_3=\delta, \quad c_4=a\alpha+\beta-\lambda-b\mu-am_1\lambda,\\ &c_5=\lambda+\delta-am_1\mu, \quad c_6=a\mu+\beta-\delta-b\gamma,\end{aligned}$$

and

$$C=\begin{pmatrix}2c_1 & c_4 & c_5\\ c_4 & 2c_2 & c_6\\ c_5 & c_6 & 2c_3\end{pmatrix}.$$

The quadric of $L_f\varphi(\bar{x})$ is a positive definite quadric quantic iff

$$\begin{aligned}&\Delta_1=c_1>0, \quad \Delta_2=4c_1c_2-c_4{}^2>0,\\ &\Delta_3=\det C>0.\end{aligned} \qquad (4.6)$$

Supposing

$$c_4=c_5=c_6=0,$$

then system (4.6) turns into the following system of conditions: $c_1>0$, $c_2>0$, $c_3>0$. In the original parameters of function (4.5) we get the equivalent system:

$$\begin{cases}\delta>0, \quad \lambda>m_1d_2\delta/(1-m_1d_1),\\ (d_1\lambda+d_2\delta)/a<\alpha<\lambda/(am_1),\\ \beta=(1+am_1)\lambda-a\alpha+b\mu,\\ \mu=(\lambda+\delta)/(am_1), \quad \gamma=(\beta+a\mu-\delta)/b,\end{cases} \qquad (4.7)$$

where $d_1=b/(am_1)+1+am_1-a$, $d_2=b(1+am_1)/(am_1)$. It is clear that this system is solvable and its solutions can be founded by the following way. First, in accordance with (4.7) we can choose δ, λ, α and then calculate μ, β and γ. Under conditions (4.7) the surface S_φ is a part of the ellipsoid

$$\frac{(x+p_1)^2}{R^2/c_1}+\frac{(y+p_2)^2}{R^2/c_2}+\frac{(z+p_3)^2}{R^2/c_3}=1 \qquad (4.8)$$

in the domain $\{x\geq 1\}$, where

$$\begin{aligned}&p_1=a(m_0+m_1)\alpha/(2c_1),\\ &p_2=a(m_0+m_1)\lambda/(2c_2),\\ &p_3=a(m_0+m_1)\mu/(2c_3),\\ &R^2=c_1p_1{}^2+c_2p_2{}^2+c_3p_3{}^2,\end{aligned} \qquad (4.9)$$

and in the domain $\{x\leq -1\}$ the surface S_φ is a part of the ellipsoid

$$\frac{(x-p_1)^2}{R^2/c_1}+\frac{(y-p_2)^2}{R^2/c_2}+\frac{(z-p_3)^2}{R^2/c_3}=1. \qquad (4.10)$$

The intersection of the plane $x=1$ and ellipsoid (4.8) is the ellipse

$$c_2(y+p_2)^2+c_3(z+p_3)^2=R_1{}^2, \quad x=1, \quad (4.11)$$

and the intersection of the plane $x=-1$ and ellipsoid (4.10) is the ellipse

$$c_2(y-p_2)^2+c_3(z-p_3)^2=R_1{}^2, \quad x=-1, \quad (4.12)$$

where

$$R_1{}^2=c_2p_2{}^2+c_3p_3{}^2-(1+2p_1)c_1. \qquad (4.13)$$

Therefore, the intersection of the surface S_φ and the planes $x=1$ and $x=-1$ are the ellipses (4.11) and (4.12) respectively, if

$$R_1{}^2\geq 0. \qquad (4.14)$$

This condition is an essential for the coeficients of the function $\varphi(\bar{x})$, however, it is always possible to take δ, λ, α that satisfy (4.7) and (4.14) as well. In fact, let δ and λ be already taken in accordance with (4.7). Since

$$\begin{aligned}R_1^2 &= a(m_0+m_1)^2\lambda^2/(4(\alpha-(d_1\lambda+d_2\delta)/a))\\ &-\lambda-am_0\alpha+(m_0+m_1)^2(\lambda+\delta)^2/(4\delta m_1{}^2),\end{aligned}$$

then $\lim\limits_{\alpha\to\alpha_0}R_1{}^2(\alpha)=+\infty$, where $\alpha_0=(d_1\lambda+d_2\delta)/a$. Hence, there exists α in some neighbourhood of α_0 such that $R_1{}^2(\alpha)\geq 0$.

Since $h(x)=-m_0x$ in the domain $\{|x|<1\}$,

$$\begin{aligned}L_f\varphi(\bar{x}) &= 2\{(\lambda+am_0\alpha)x^2+(a\lambda-\beta-b\delta)y^2+\\ &+\delta z^2+xy(a\alpha+\beta-\lambda-b\mu+am_0\lambda)+\\ &+xz(\lambda+\delta+am_0\mu)+yz(a\mu+\beta-\delta-b\gamma)\},\end{aligned}$$

and if we recall (4.9), in this domain we obviously obtain S_φ given as

$$\begin{aligned} c_1(1+2p_1)x^2 + c_2y^2 + c_3z^2 + \\ + 2c_2p_2xy + 2c_3p_3xz = 0, \quad |x| < 1. \end{aligned} \tag{4.15}$$

Since the intersection of S_φ and the planes $x = 1$ and $x = -1$ are the ellipses (4.11) and (4.12) (due to ${R_1}^2 \geq 0$), in the domain $\{|x| \geq 1\}$ the surface S_φ is a compact set. Now, by Theorem 3.1 S_φ is a compact set in the whole phase space. This completes the proof of Theorem 4.2. ▹

Thus, any limit cycle of system (4.1) contains at least two points of the surface S_φ given as

$$\begin{cases} \dfrac{(x+p_1)^2}{R^2/c_1} + \dfrac{(y+p_2)^2}{R^2/c_2} + \\ \qquad + \dfrac{(z+p_3)^2}{R^2/c_3} = 1, \quad x \geq 1, \\ \dfrac{(x-p_1)^2}{R^2/c_1} + \dfrac{(y-p_2)^2}{R^2/c_2} + \\ \qquad + \dfrac{(z-p_3)^2}{R^2/c_3} = 1, \quad x \leq -1, \\ (1+2p_1)c_1x^2 + c_2y^2 + c_3z^2 + \\ \qquad + 2p_2c_2xy + 2p_3c_3xz = 0, \quad |x| < 1, \end{cases} \tag{4.16}$$

where $c_1, c_2, c_3, p_1, p_2, p_3$ and R^2 are given by (4.9). All the limit cycles belong to the corresponding set Ω_φ.

Theorem 4.3 *For system (4.1) there exists the compact set Ω such that all limit cycles of the system belong to Ω.*

Proof. Since for the system Chua Theorem's 3.1 conditions are valid, by Theorem 3.2 there exists a compact set Ω as intersection (3.4). ▹

Example. Consider an example of Chua system, where

$$a = 9, \quad b = 100/7, \quad m_0 = 1/7, \quad m_1 = 2/7. \tag{4.17}$$

The system with these parameters' values is characterized with chaotic motions [3]. Equilibrium points $O_1\,(3/2, 0, -3/2)$, $O_2\,(-3/2, 0, 3/2)$, $O_3(0,0,0)$ are unstable. Since condition (4.4) holds it's possible to use obtained results. With reference to (4.7) we choose the function

$$\begin{aligned} \varphi(\bar{x}) = 3x^2 + 133.71y^2 + 13.7z^2 + \\ + 34xy + 14xz + 2yz, \end{aligned} \tag{4.18}$$

and the surface S_φ according to (4.16) is

$$\begin{aligned} & (x+0.27)^2/43.17 + (y+6.56)^2/80.17 + \\ & \quad + (z+13.5)^2/400.84 = 1, \quad x \geq 1, \\ & (x-0.27)^2/43.17 + (y-6.56)^2/80.17 + \\ & \quad + (z-13.5)^2/400.84 = 1, \quad x \leq 1, \\ & 20.86x^2 + 5y^2 + z^2 + \\ & \quad + 65.57xy + 27xz = 0, \quad |x| < 1. \end{aligned} \tag{4.19}$$

To define the set Ω_φ we need to solve a conditional extremum problem for the function (4.18) under condition (4.19). Using the numerical methods we obtain the localization set:

$$\Omega_\varphi = \{\, \bar{x} \mid -6.15 \leq \varphi(\bar{x}) \leq 34338.32 \,\}.$$

For function $\varphi'(\bar{x}) = 5x^2 + 115.71y^2 + 12.44z^2 + 34xy + 14xz + 2yz$, in the same way, we obtain the new localization set

$$\Omega_{\varphi'} = \{\, \bar{x} \mid -7.30 \leq \varphi'(\bar{x}) \leq 30279.66 \,\},$$

It is proved that the cones $\varphi(\bar{x}) = 0$ and $\varphi'(\bar{x}) = 0$ have the only common point $\bar{x} = \bar{0}$. So, by Theorem 3.2 the set $\Omega = \Omega_\varphi \cap \Omega_{\varphi'}$ is a compact set containing all limit cycles of the Chua system with parameters' value (4.17).

5 CONCLUSION

The suggested method can be efficiently used for solving the localization problem for periodic orbits of ordinary differential systems in various fields of science and technology. Here we use it for piecewise linear systems, the Chua's circuit in radiophysics and nonlinear electronics. We have shown that the method is very useful and efficient and it can give new interesting results.

REFERENCES

[1] A. Krishchenko, "Localization of Limit Cycles", *Differentsial'nye Uravneniya*, vol.31, No. 11, pp.1858-1865, 1995 (Russian).

[2] A. Krishchenko, "Estimations of Domain with Cycles", *Computers and Mathematics with Applications*, vol.34, No. 2-4, pp.325-332, 1997.

[3] L. Shil'nikov. "Chua's circuit: rigorous results and future problems", *International Journal of Bifurcation and Chaos*, No.3, Vol.4, pp.489-519, 1994.