IV. Conclusion

The various models associated with the Chua's circuit have been recently found helpful in demonstrating and explaining the various facets of chaos [3]. Attempts are made in the literature to discover new members of the large family of Chua's circuits and to find their close relatives [8], [9]. An example for such a recently discovered relative is described in [10], where a nonautonomous related circuit is discussed. (The latter circuit can also be regarded as a relative of the RL diode circuit). The present communication deals with the well known Colpitts oscillator, which is shown to be topologically similar to Chua's circuit. It is also shown here that when the nonlinearity of the active device in the Colpitts oscillator is modified to be purely odd, then the circuit exhibits chaotic phenomena closely related to those exhibited by the classical [3] Chua's circuit. Hence, yet another relative of Chua's circuit has been discovered.

A reviewer has pointed out that recent works of Chua *et al.* [8], [9] have established mathematically the exactly detailed relationship between Chua's oscillator and relatively many other 3-D systems. The works prove that such 3-D systems are topologically conjugate to Chua's oscillator (or in circuit terms, they are equivalent to Chua's oscillator [11]). A Chua's oscillator is obtained by adding a resistor in series with the inductor in Chua's circuit [11]. The classical Chua's oscillator [11] is, therefore, conjugate to the circuit in Fig. 1(b). Hence, by demonstrating that there exists a robust relationship between the Colpitts oscillator in Fig. 1(a) and the Chua's oscillator [11], one can show that the two systems of Fig. 1 are not simply loose relatives, but they are even strongly related and can be regarded as being conjugate one to the other. Reference [9], which is strongly related to [8], cites an example due to Arneodo *et al.* [12] of a 3-D system that is strictly proved in [9] as being conjugate to the Chua's oscillator. It is interesting that the latter example ((10) in [9]) is in fact the same equation as the one that represents the presently discussed Colpitts oscillator (3). Hence, we can conclude that the two member systems of Fig. 1 are not merely related, but they are even conjugate one to the other. Therefore, due to the helpful remarks of the reviewer, we are assured that the answer to the question posed in the title of the present communication is strongly yes.

Acknowledgment

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References

[1] G. Sarafian and B. Z. Kaplan, "A new approach to the modeling of the dynamics of RF VCO's and some of its practical implication," *IEEE Trans. Circuits Syst.*, vol. 40, no. 12, pp. 895–901, Dec. 1993.
[2] L. O. Chua and G. Lin, "Intermittency in a piecewise-linear circuit," *IEEE Trans. Circuits Syst.*, vol. 38, no. 5, pp. 510–520, May 1991.
[3] ——, "Canonical realization of Chua's circuit family," *IEEE Trans. Circuit Syst.*, vol. 37, no. 7, pp. 885–902, July 1990.
[4] G. Sarafian, "Practical problems in the dynamics of voltage controlled oscillators," Thesis submitted in partial fulfillment of the requirements for M.Sc. Degree, Ben-Gurion University of the Negev, Beer-Sheva Israel, Faculty of Engineering Sciences, Department of Electrical and Computer Engineering (in Hebrew), Nov. 1992.
[5] E. J. Altman, "Normal form analysis of the Chua's circuit with application for trajectory recognition," *IEEE Trans. Circuits Syst.*, vol. 40, no. 10, pp. 675–682, Oct. 1993.
[6] X. Zhou and A. S. Daryoush, "An injection locked push-pull oscillator at Ku-band," *IEEE Microwave and Guided Wave Lett.*, vol. 3, no. 8, pp. 244–246, Aug. 1993.
[7] R. N. Madan, Ed., *Chua's Circuit: A Paradigm for Chaos.* Singapore: World Scientific, 1993.
[8] L. O. Chua, "Global unfolding of Chua's circuit," *IEICE Trans. Fundamentals*, vol. E76-A, pp. 704–734, May 1993.
[9] L. O. Chua, C. W. Wu, A. Huang, and G. Q. Zhong, "A universal circuit for studying and generating chaos—Part II: Strange attractors," *IEEE Trans. Circuits Syst.*, vol. 40, no. 10, pp. 745–761, Oct. 1993.
[10] K. Murali, M. Lakshmanan, and L. O. Chua, "The simplest dissipative nonautonomous chaotic circuit," *IEEE Trans. Circuits Syst.*, vol. 41, no. 6, pp. 462–463, June 1994.
[11] L. O. Chua, C. W. Wu, A. Huang, and G. Q. Zhong, "A universal circuit for studying and generating chaos-Part I: Routes to chaos," *IEEE Trans. Circuits Syst.*, vol. 40, no. 10, pp. 732–744, Oct. 1993.
[12] A. Arneodo, P.Coullet, and E. A. Spiegel, "Chaos in finite macroscopic systems," *Phys. Lett.*, vol. 92A, no. 8, pp. 369–373, 1982.

On the Relationship Between the Chaotic Colpitts Oscillator and Chua's Oscillator

Michael Peter Kennedy

***Abstract*— In this letter, we show that the two-region third-order piecewise-linear dynamics of the chaotic Colpitts oscillator may be mapped to a Chua's oscillator with an asymmetric nonlinearity.**

I. Introduction

It has recently been shown that the dynamics of a chaotic Colpitts oscillator (shown in Fig. 1(a)) can be captured by a third-order autonomous circuit model containing just one nonlinear element—a two-segment piecewise-linear resistor (Fig. 1(b)), [1].

The circuit is described by a system of three autonomous state equations

$$\begin{aligned} C_1 \frac{V_{CE}}{dt} &= I_L - I_C \\ C_2 \frac{V_{BE}}{dt} &= -\frac{V_{EE} + V_{BE}}{R_{EE}} - I_L - I_B \\ L \frac{I_L}{dt} &= V_{CC} - V_{CE} + V_{BE} - I_L R_L. \end{aligned}$$

We model the transistor as a two-segment piecewise-linear voltage-controlled resistor N_R and a linear current-controlled current source. Thus

$$\begin{aligned} I_B &= \begin{cases} 0 & \text{if} \quad V_{BE} \le V_{TH} \\ \frac{V_{BE} - V_{TH}}{R_{ON}} & \text{if} \quad V_{BE} > V_{TH} \end{cases} \\ I_C &= \beta_F I_B \end{aligned}$$

where V_{TH} is the threshold voltage ($\approx$ 0.75 V), R_{ON} is the small-signal on-resistance of the base-emitter junction, and β_F is the forward current gain of the device.

Fig. 2 shows a chaotic attractor in this two-region piecewise-linear oscillator.

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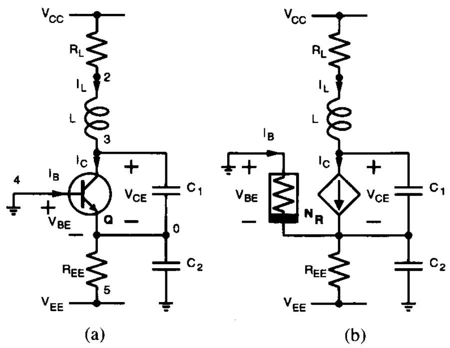

Fig. 1. (a) BJT Colpitts oscillator and (b) its equivalent circuit.

In this letter, we address the question: what is the connection between this two-region third-order piecewise-linear dynamical system and Chua's oscillator [2] (shown in Fig. 3)? We will show that the chaotic Colpitts oscillator may be mapped to a Chua's oscillator with the same eigenvalue pattern if the Chua diode [3] is allowed to have an asymmetric driving-point characteristic.

II. Chua's Oscillator

Chua *et al.* have shown that almost every continuous third-order odd-symmetric three-region piecewise-linear vector field $\mathbf{F}'$ may be mapped onto a Chua's oscillator (whose vector field $\mathbf{F}$ is topologically conjugate to $\mathbf{F}'$) by means of the following algorithm [2]:

1) Calculate the eigenvalues (μ_1', μ_2', μ_3') and (ν_1', ν_2', ν_3') associated with the linear and affine regions, respectively, of the vector field $\mathbf{F}'$ of the circuit or system whose *attractor* is to be *reproduced* (up to linear conjugacy) by Chua's oscillator.
2) Find a set of circuit parameters $\{C_1, C_2, L, R, R_0, G_a, G_b, E\}$ so that the resulting eigenvalues μ_j and ν_j for Chua's oscillator satisfy $\mu_j = \mu_j'$ and $\nu_j = \nu_j'$, $j = 1, 2, 3$.

III. Piecewise-linear Structure of the Chaotic Colpitts Oscillator

Cutoff region ($V_{BE} \le V_{TH}$):

In the cutoff region, the chaotic Colpitts oscillator shown in Fig. 1(b) is described by

$$\begin{aligned}
\frac{dV_{CE}}{dt} &= \frac{1}{C_1} I_L \\
\frac{dV_{BE}}{dt} &= -\frac{1}{R_{EE}C_2} V_{BE} - \frac{1}{C_2} I_L - \frac{V_{EE}}{R_{EE}C_2} \\
\frac{dI_L}{dt} &= -\frac{1}{L} V_{CE} + \frac{1}{L} V_{BE} - \frac{R_L}{L} I_L + \frac{V_{CC}}{L}.
\end{aligned}$$

The equilibrium point in this region is given by

$$\begin{bmatrix} V_{CE} \\ V_{BE} \\ I_L \end{bmatrix} = \begin{bmatrix} V_{CC} - V_{EE} \\ -V_{EE} \\ 0 \end{bmatrix}.$$

If $-V_{EE} > V_{TH}$, then this equilibrium point lies *outside* the cutoff region and is called a *virtual* equilibrium point of the system [4]; in fact, it lies in the forward active region.

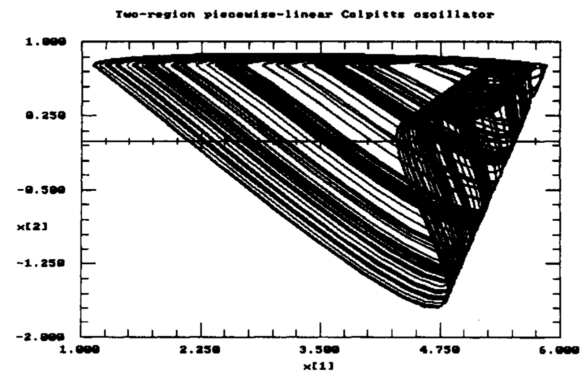

Fig. 2. INSITE simulation of the piecewise-linear BJT Colpitts oscillator model. Model parameters are: V_{CC} = 5 V, R_L = 35 Ω, L = 98.5 μH, C_1 = 54 nF, C_2=54 nF, R_{EE} = 400 Ω, and V_{EE} = -5 V. The BJT is specified by three parameters: V_{TH} = 0.75 V, R_{ON} = 100Ω, and β_F = 200. Horizontal axis: x[1] = V_{CE} (V); vertical axis: x[2] = V_{BE} (V).

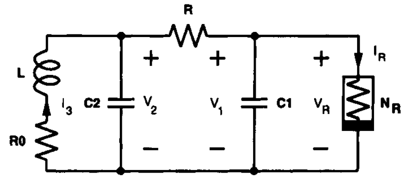

Fig. 3. Chua's oscillator consists of a linear inductor L with series resistance R_0, a linear resistor R, two linear capacitors C_1 and C_2, and a nonlinear resistor N_R.

Forward active region ($V_{BE} > V_{TH}$):

In the BJT's forward active region of operation, the circuit is described by

$$\begin{aligned}
\frac{dV_{CE}}{dt} &= -\frac{\beta_F}{R_{ON}C_1} V_{BE} + \frac{1}{C_1} I_L + \frac{\beta_F V_{TH}}{R_{ON}C_1} \\
\frac{dV_{BE}}{dt} &= -\left(\frac{1}{R_{EE}C_2} + \frac{1}{R_{ON}C_2}\right) V_{BE} - \frac{1}{C_2} I_L \\
&\quad - \frac{V_{EE}}{R_{EE}C_2} + \frac{V_{TH}}{R_{ON}C_2} \\
\frac{dI_L}{dt} &= -\frac{1}{L} V_{CE} + \frac{1}{L} V_{BE} - \frac{R_L}{L} I_L + \frac{V_{CC}}{L}.
\end{aligned}$$

This region has an equilibrium point defined by

$$\begin{bmatrix} V_{CE} \\ V_{BE} \\ I_L \end{bmatrix} = \begin{bmatrix} V_{CC} + \frac{-V_{EE}R_{ON} + (1+\beta_F)R_{EE}V_{TH}}{R'} + \frac{\beta_F R_L (V_{EE} + V_{TH})}{R'} \\ \frac{-V_{EE}R_{ON} + (1+\beta_F)R_{EE}V_{TH}}{R'} \\ -\frac{\beta_F (V_{EE} + V_{TH})}{R'} \end{bmatrix}$$

where $R' = R_{ON} + (1 + \beta_F)R_{EE}$. This equilibrium point lies in the forward active region.

Thus, the chaotic Colpitts oscillator [1] has a true equilibrium point in its forward active region and a virtual equilibrium point associated with the cutoff region, also lying in the forward active region.

Chua's oscillator always possesses an equilibrium point at the origin (in the so-called D_0 region) because the driving-point characteristic of its nonlinear resistor goes through the origin. In addition, it has a symmetrically-placed pair of true or virtual equilibrium points associated with the outer regions [4].

Therefore, we identify the forward active region with the D_0 region of Chua's oscillator and the cutoff region with the outer regions (D_{-1} and D_1) and attempt to map the dynamics of the chaotic Colpitts oscillator to Chua's oscillator by following the algorithm presented in [2].

IV. From Eigenvalues to Circuit Parameters

Step 1:

Let $\{p_1, p_2, p_3, q_1, q_2, q_3\}$ be the equivalent eigenvalue parameters defined by

$$\left.\begin{aligned} p_1 &= \mu_1' + \mu_2' + \mu_3' \\ p_2 &= \mu_1\mu_2' + \mu_2'\mu_3' + \mu_3'\mu_1' \\ p_3 &= \mu_1'\mu_2'\mu_3' \\ \\ q_1 &= \nu_1' + \nu_2' + \nu_3' \\ q_2 &= \nu_1'\nu_2' + \nu_2'\nu_3' + \nu_3'\nu_1' \\ q_3 &= \nu_1'\nu_2'\nu_3' \end{aligned}\right\} \tag{1}$$

where (μ_1', μ_2', μ_3') and (ν_1', ν_2', ν_3') denote the eigenvalues associated with the vector fields in the D_0 and D_1 regions, respectively.

The eigenvalues of the forward active region are the roots of the characteristic polynomial

$$\lambda^3 + \left(\frac{G_a}{C_2} + \frac{R_L}{L}\right)\lambda^2 + \left(\frac{G_a R_L}{LC_2} + \frac{1}{LC_1} + \frac{1}{LC_2}\right)\lambda + \frac{G_a'}{LC_1C_2}$$

where $G_a = 1/R_{EE} + 1/R_{ON}$ and $G_a' = 1/R_{EE} + (1+\beta_F)/R_{ON}$. We label these roots μ_1', μ_2', and μ_3'.

The eigenvalues of the cutoff region in the Colpitts oscillator are defined as the roots of the characteristic polynomial

$$\lambda^3 + \left(\frac{G_b}{C_2} + \frac{R_L}{L}\right)\lambda^2 + \left(\frac{G_b R_L}{LC_2} + \frac{1}{LC_1} + \frac{1}{LC_2}\right)\lambda + \frac{G_b'}{LC_1C_2}$$

where $G_b = G_b' = 1/R_{EE}$. We label these eigenvalues ν_1', ν_2', and ν_3'.

Step 2:

Define

$$\begin{aligned} k_1 &= -p_3 + \left(\tfrac{p_3-q_3}{p_1-q_1}\right)\left(p_1 - \tfrac{p_2-q_2}{p_1-q_1}\right) \\ k_2 &= p_2 - \left(\tfrac{p_3-q_3}{p_1-q_1}\right) - \left(\tfrac{p_2-q_2}{p_1-q_1}\right)\left(p_1 - \tfrac{p_2-q_2}{p_1-q_1}\right) \\ k_3 &= -\left(\tfrac{p_2-q_2}{p_1-q_1}\right) - \tfrac{k_1}{k_2} \\ k_4 &= -k_1k_3 + k_2\left(\tfrac{p_3-q_3}{p_1-q_1}\right). \end{aligned}$$

Step 3:

The corresponding normalized circuit parameters are given by

$$\left.\begin{aligned} C_1 &= 1 \\ C_2 &= -\tfrac{k_2}{k_3^2} \\ L &= -\tfrac{k_3^2}{k_4} \\ R &= -\tfrac{k_3}{k_2} \\ R_0 &= -\tfrac{k_1k_3^2}{k_2k_4} \\ G_a &= -p_1 + \left(\tfrac{p_2-q_2}{p_1-q_1}\right) + \tfrac{k_2}{k_3} \\ G_b &= -q_1 + \left(\tfrac{p_2-q_2}{p_1-q_1}\right) + \tfrac{k_2}{k_3} \end{aligned}\right\}. \tag{2}$$

This equation is a corrected version of (23) in [2].

Step 4:

The breakpoints $\pm E$ of the piecewise-linear Chua diode can be chosen arbitrarily since the choice of E does not affect either the eigenvalues or the dynamics; it simply *scales* the circuit voltages.

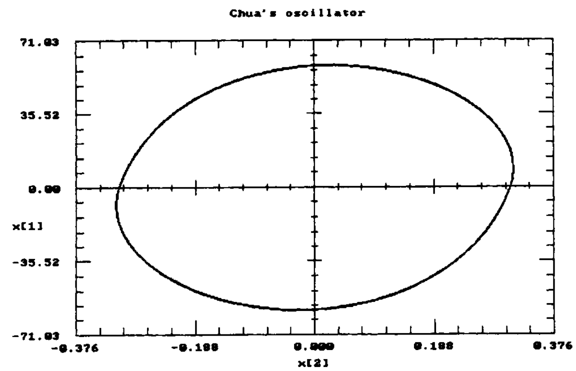

Fig. 4. Periodic attractor in Chua's oscillator with parameters given by (3) and E = 1 V. Horizontal axis: x[2] = V_2 (V); vertical axis: x[1] = V_1 (V).

Step 5:

Finally, one may transform the time and current scales by means of frequency and impedance scaling factors k_f and k_Z

$$\begin{aligned} C_1 &\rightarrow \tfrac{1}{k_Z k_f}C_1 \\ C_2 &\rightarrow \tfrac{1}{k_Z k_f}C_2 \\ L &\rightarrow \tfrac{k_Z}{k_f}L \\ R &\rightarrow k_Z R \\ R_0 &\rightarrow k_Z R_0 \\ G_a &\rightarrow \tfrac{1}{k_Z}G_a \\ G_b &\rightarrow \tfrac{1}{k_Z}G_b. \end{aligned}$$

V. Simulation Results

Throughout the following discussion, we consider a single set of model parameters V_{CC} = 5 V, R_L = 35 Ω, L = 98.5 μH, C_1 = 54 nF, C_2=54 nF, R_{EE} = 400 Ω, and V_{EE} = -5 V. The BJT is specified by three parameters: V_{TH} = 0.75 V, R_{ON} = 100Ω, and β_F = 200.

The eigenvalues in the forward active and cutoff regions are (to ten significant figures)

$$\mu_1' = -2042294.350$$
$$\mu_2' = 727741.4596 + j1703284.985$$
$$\mu_3' = 727741.4596 - j1703284.985$$

$$\nu_1' = -22674.29324$$
$$\nu_2' = -189475.9761 + j589887.5958$$
$$\nu_3' = -189475.9761 - j589887.5958.$$

With $k_f = 0.01$ and $k_Z = 10^{12}$, the corresponding parameters for Chua's oscillator (to ten significant figures) are

$$\left.\begin{aligned} C_1 &= 100.0000000\text{pF} \\ C_2 &= 39.23427816\text{nF} \\ L &= 6.747352901\text{mH} \\ R &= 8.253831974\text{k}\Omega \\ R_L &= 3.139471850\Omega \\ G_a &= -120.9243651\mu\text{S} \\ G_b &= -121.1095502\mu\text{S} \end{aligned}\right\}. \tag{3}$$

Fig. 4 shows the attractor of Chua's oscillator for this parameter set with E = 1 V. This *periodic* attractor, which lies in *all three* regions of the state space, is qualitatively different from the original *chaotic* Colpitts attractor, which was confined to just *two* regions.

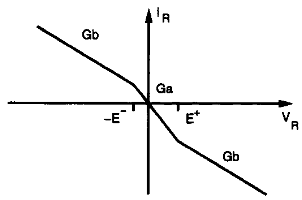

Fig. 5. Three-segment piecewise-linear driving-point characteristic of the asymmetric Chua diode. The inner and outer regions have slopes G_a and G_b, respectively. The breakpoints are at $-E^-$ and E^+.

The problem may be removed by breaking the symmetry of the nonlinearity to ensure that the trajectory is confined to the inner and just one of outer regions. In particular, we can move the left breakpoint to the right so that the steady-state trajectory never enters the D_1 region. The modified Chua diode nonlinearity, defined explicitly by

$$\begin{aligned} I_R &= f(V_R) \\ &= \frac{1}{2}(G_a - G_b)\left(E^+ - E^-\right) + G_b V_R \\ &\quad + \frac{1}{2}(G_a - G_b)\left(|V_R + E^-| - |V_R - E^+|\right) \end{aligned}$$

is shown in Fig. 5. A piecewise-linear voltage-controlled nonlinear resistor of this type may be implemented using the synthesis procedure described in [5]. We refer to Chua's oscillator with an asymmetric nonlinearity as an *asymmetric Chua's oscillator.*

With $E^- = 0.02$ V and $E^+ = 1$ V, the resulting attractor in the asymmetric Chua's oscillator is confined to the D_{-1} and D_0 regions and is chaotic (see Fig. 6).

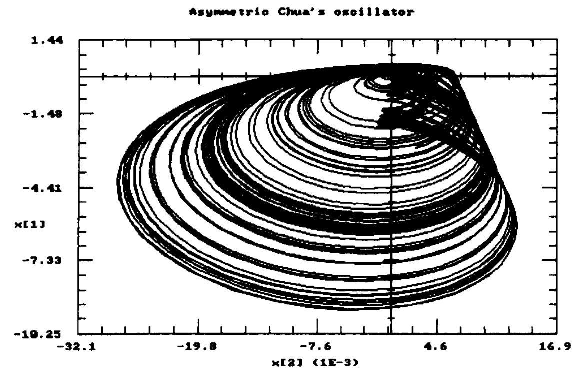

Fig. 6. Chaotic attractor in the asymmetric Chua's oscillator with parameters given by (3), $E^- = 0.02$ V, and $E^+ = 1$ V. Horizontal axis: x[2] = V_2 (V); vertical axis: x[1] = V_1 (V).

VI. Concluding Remarks

The dynamics of the continuous two-region piecewise-linear Colpitts oscillator may be mapped onto an equivalent asymmetric Chua's oscillator by choosing the breakpoints of the asymmetric Chua diode such that the resulting attractor is confined to two regions of the state space. The dynamics of the circuits are equivalent in the sense that the vector field of the Colpitts oscillator is linearly conjugate to that of the asymmetric Chua's oscillator.

References

[1] M. P. Kennedy, "Chaos in the Colpitts oscillator," *IEEE Trans. Circuits Syst.-I*, vol. 41, no. 11, pp. 771–774, Nov. 1994.

[2] L. O. Chua, C. W. Wu, A. Huang, and G.-Q. Zhong, "A universal circuit for studying and generating chaos—part I: Routes to chaos," *IEEE Trans. CAS I: Fundamental Theory and Applications*, vol. CAS-40, no. 10, pp. 732–744, Oct. 1993.

[3] M. P. Kennedy, "Robust op amp realization of Chua's circuit," *Frequenz*, vol 46, no. 3–4, pp. 66–80, Mar.–Apr. 1992.

[4] ——, "Three steps to chaos part II: A Chua's circuit primer," *IEEE Trans. Circuits Syst.-I*, vol. 40, no. 10, pp. 657–674, Oct. 1993.

[5] ——, "Synthesis of continuous piecewise-linear resistors for Chua's circuit family using operational amplifiers, diodes, and linear resistors," *Int. J. Circuit Theory Appl.*, vol. 21, no. 5, pp. 551–558, 1993.