Pergamon

Chaos, Solitons & Fractals, Vol. 8, No. 9, pp. 1517–1522, 1997

0960-0779/97 $17.00 + 0.00

PII: S0960-0779(96)00151-8

Experimental Evidence of Locally Intermingled Basins of Attraction in Coupled Chua's Circuits

T. KAPITANIAK

Division of Dynamics, Technical University of Łódź, Stefanowskiego 1/15, 90-924 Łódź, Poland

L. O. CHUA and G.-Q. ZHONG

Department of Electrical Engineering and Computer Science, University of California, Berkeley, CA 94720, USA

Abstract—We show experimentally that two coupled chaotic systems initially operating on two different simultaneously co-existing attractors can be synchronized. Synchronization is achieved as one of the systems switches its evolution to the attractor of the other one. The final attractor of the synchronized state strongly depends on the actual position of trajectories on their attractors at the moment when coupling is introduced. Coupling introduced in such systems can lead to the locally intermingled basins of attraction of coexisting attractors. Even if the initial location of trajectories on attractors A_1 and A_2 is known with infinite precision, we are unable to determine, on the basis of any finite calculation, in which basin this location lies and finally we cannot be sure on which attractor the evolution will synchronize. We investigate this uncertainty in chaos synchronization in numerical and experimental studies of two coupled Chua's circuits. © 1997 Elsevier Science Ltd

1. INTRODUCTION

Chaos synchronization procedures [1–8] require the introduction of some kind of coupling between two chaotic systems. One of the synchronization procedures is based on the mutual coupling of two chaotic systems $\dot{\mathbf{x}} = f(\mathbf{x})$ and $\dot{\mathbf{y}} = f(\mathbf{y})$, where $\mathbf{x},\mathbf{y} \in \mathbf{R}^n$, with $n \geq 3$, by one-to-one negative feedback mechanism

$$\text{(a)}\ \dot{\mathbf{x}} = f(\mathbf{x}) + \mathbf{d}(\mathbf{y} - \mathbf{x}), \quad \text{(b)}\ \dot{\mathbf{y}} = f(\mathbf{y}) + \mathbf{d}(\mathbf{x} - \mathbf{y}), \tag{1}$$

where $\mathbf{d} = [d,d,...,d]^T \in \mathbf{R}^n$ is a coupling vector. System (1) can be rewritten as

$$\dot{\mathbf{z}} = g(\mathbf{z}), \tag{2}$$

where $\mathbf{z} = [\mathbf{x},\mathbf{y}]^T \in \mathbf{R}^{2n}$. The manifold defined by the synchronized state $\mathbf{x} = \mathbf{y}$ is an invariant n-dimensional manifold of the system (2); i.e. any trajectory initiated in this manifold remains there for all the time. This manifold is called the synchronization manifold. In this paper, we consider for simplicity $n = 3$ although the phenomena described are characteristic for any $n \geq 3$.

The problem of synchronization of chaotic 3-dimensional systems can be understood as a problem of stability of 3-dimensional chaotic attractor of system (1)(a) or (b) in 6-dimensional phase space of coupled system (2). Let A be a chaotic attractor. The basin of attraction $\beta(A)$ is the set of points whose ω-limit set is contained in A. In Milnor's definition [9] of an attractor, the basin of attraction need not include the neighborhood of the attractor. Attractor A is an asymptotically stable attractor if it is Lyapunov stable (i.e. $\beta(A)$ has positive Lebesque measure) and $\beta(A)$ contains a neighborhood of A. Recently, it has been shown that for certain types of system the basin of attraction of attractor A can be riddled [6, 10–14]. A riddle basin has positive Lebesgue measure but does not contain any neighborhood of the attractor; i.e. for any point $\mathbf{x}_0$ in the riddled basin of an attractor, a ball in the phase space of arbitrarily small radius r has a nonzero fraction of its volume in some other attractor's basin. The basin of the other attractor may or may not be riddled by the first basin. If the second basin is also riddled by the first one, we call such basins intermingled. Riddled basins have been observed numerically and experimentally in a few physical systems [6, 11–14].

Most of the work on the chaos synchronization problem has been associated with identical systems operating on some chaotic attractor. If the trajectory of one system is on the attractor A_1 and the trajectory of the other one is on the co-existing attractor A_2 to achieve synchronization, one of the trajectories, say one on the attractor A_1, has to be perturbed in such a way that it goes to the basin of attraction $\beta(A_2)$ of the other attractor A_2. Recently a new class of basins of attraction, namely locally intermingled basins, was shown [17, 18] to occur in the system like (2). Locally intermingled basins of attraction of the attractors A_1 and A_2 are not intermingled in whole 6-dimensional phase space of the system (2) (in the 6-dimensional phase space the basins $\beta(A_1)$ and $\beta(A_2)$ have only fractal boundary), but are intermingled in the lower 3-dimensional manifolds on which attractors A_1 and A_2 are located. As coupling in system (3) is introduced when both subsystems (1)(a) and (2)(b) are either on A_1 or A_2, all initial conditions for 6-dimensional system (2) are located on the 3-dimensional manifold where basins $\beta(A_1)$ and $\beta(A_2)$ are intermingled. Such basins of attraction which are intermingled on some lower-dimensional submanifold of the phase space but which are not intermingled in the whole phase space we call locally intermingled.

In what follows, we investigate the hyperchaotic attractors in a chain of coupled identical Chua's circuits, as shown in Fig. 1. The state equations for the circuit of Fig. 1 are

$$C_1 \frac{dv_{C_1}^{(1)}}{dt} = G(v_{C_2}^{(1)} - v_{C_1}^{(1)}) - f(v_{C_1}^{(1)}) + dH(t - t_0)(v_{C_1}^{(2)} - v_{C_1}^{(1)}),$$

$$C_2 \frac{dv_{C_2}^{(1)}}{dt} = G(v_{C_1}^{(1)} - v_{C_2}^{(1)}) + i_l^{(1)} + dH(t - t_0)(v_{C_2}^{(2)} - v_{C_2}^{(1)}),$$

$$L \frac{di_l^{(1)}}{dt} = v_{C_2}^{(1)}.$$

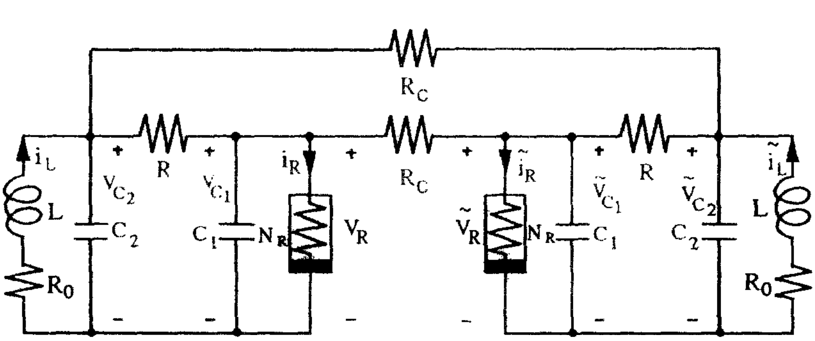

Fig. 1. Two coupled Chua's circuits

$$C_1\frac{dv_{C_1}^{(2)}}{dt} = G(v_{C_2}^{(2)} - v_{C_1}^{(2)}) - f(v_{C_1}^{(2)}) + dH(t-t_0)(v_{C_2}^{(1)} - v_{C_2}^{(2)}),$$

$$C_2\frac{dv_{C_2}^{(2)}}{dt} = G(v_{C_1}^{(2)} - v_{C_2}^{(2)}) + i_L^{(2)} + dH(t-t_0)(v_{C_2}^{(1)} - v_{C_2}^{(2)}), \tag{3}$$

$$L\frac{di_L^{(2)}}{dt} = v_{C_2}^{(2)}.$$

Each Chua's circuit [15, 16] contains three linear energy-storage elements (an inductor and two capacitors), a linear resistor, and a single nonlinear resistor N_R, namely Chua's diode [15], with a three-segment linear characteristic defined by

$$f(v_R) = m_0 v_R + \frac{1}{2}(m_1 - m_0)[|v_R + B_p| - |v_R - B_p|], \tag{4}$$

where the slopes in the inner and outer regions are m_0 and m_1, respectively, and $\pm B_p$ denotes the breakpoints. Each Chua's circuit is coupled to the next one in such a way that the difference between the signals $v_{C_{1,2}}^{(1,2)}$ and $v_{C_{1,2}}^{(2,1)}$, that is,

$$dH(t-t_0)(v_{C_{1,2}}^{(1,2)} - v_{C_{1,2}}^{(2,1)}),$$

is introduced into each first circuit as a negative feedback. We consider the stiffness $d > 0$ of the perturbation as a control parameter. $H(t-t_0)$ is a Heaviside function: $H(t-t_0) = 1$ for $t >= t_0$ and $H(t-t_0) = 0$ for $t < t_0$. In our experiments, we took $C_1 = 10\,\text{nF}$, $B_p = 1$ V, $C_2 = 99.34\,\text{nF}$, $m_1 = -0.76\,\text{mS}$, $m_0 = -0.41$ mS, $L = 18.46\,\text{mH}$, $R = 1.80\,\text{K}\Omega$.

In what follows, we assume that for $d = 0$ both chaotic systems evolve on different co-existing spiral type chaotic attractors. The first system (the first three of equation (3)) is assumed to evolve on the attractor A_1, shown in Fig. 2(a) and the second one (the last three of equation (3)) to evolve on the attractor A_2 shown in Fig. 2(b).

Both chaotic systems evolve on different attractors when coupling is introduced for $t = t_0$. (In our experiments, we took $d = 1$, i.e. $Rc = 20\,\text{k}\Omega$.) For $t > t_0$, after a transition period the evolution of both systems is either synchronized on one of the co-existing attractors A_1 or

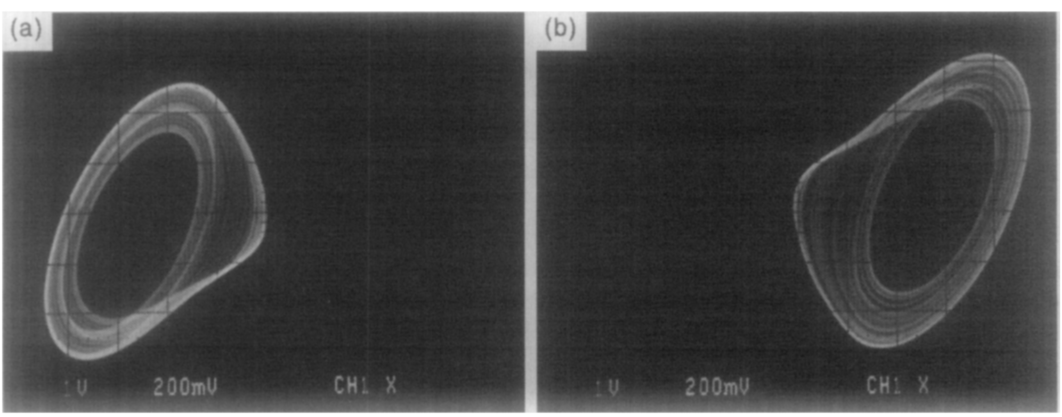

Fig. 2. Two co-existing attractors of uncoupled eqn (3): $C_1 = 10\,\text{nF}$, $B_p = 1$ V, $C_2 = 99.34\,\text{nF}$, $m_1 = -0.76\,\text{mS}$, $m_0 = -0.41$ mS, $L = 18.46\,\text{mH}$, $R = 1.80\,\text{k}\Omega$, $R_c = 0$. (a) A_1, (b) A_2, with horizontal axis: v_{C_2} 200 mv/div, vertical axis: v_{C_2} 1 v/div.

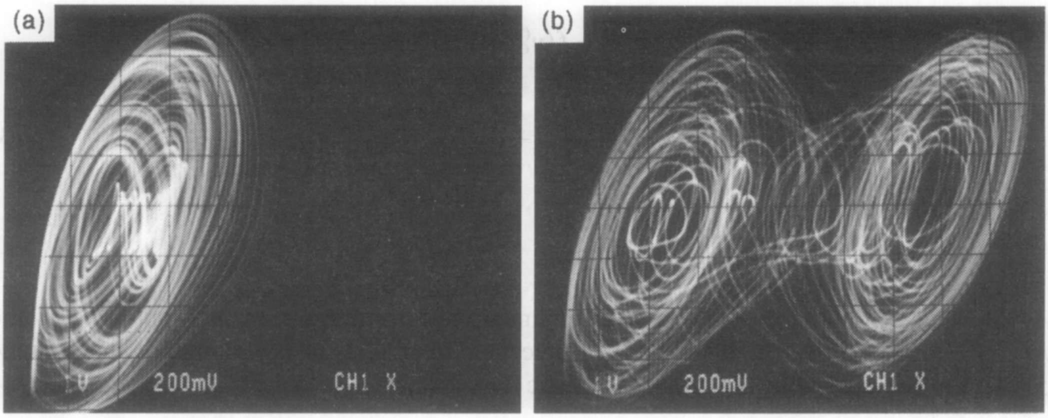

Fig. 3. Evolution of coupled eqn (3): $C_1 = 10$ nF, $B_p = 1$ V, $C_2 = 99.34$ nF, $m_1 = -0.76$ mS, $m_0 = -0.41$ mS, $L = 18.46$ mH, $R = 1.80$ kΩ, $R = 20$ kΩ (initially both systems evolve on attractors shown in Fig. 2). Horizontal axis: $v = 200$ mV/div, vertical axis $v_c = 1$ V/div.

A_2. Such evolutions shown in different projections are presented in Fig. 3(a) and (b). During the transitional period, the evolution of the first system was switched from attractor A_1 to A_2 then synchronized with the evolution of the second system on the attractor A_2 as shown in Fig. 3(a). A symmetrical situation with the synchronization on the attractor A_1 is shown in Fig. 3(b).

The switch between attractor A_1 and A_2 (or A_2 and A_1) of the evolution of one of the systems is possible as the perturbation $d(\mathbf{X}_2 - \mathbf{X}_1)$ (or $d(\mathbf{X}_1 - \mathbf{X}_2)$), where $\mathbf{X}_1 = [x,y,z]^T$ and $\mathbf{X}_2 = [u,v,w]^T$ moves trajectory $\mathbf{X}_1(t)$ (or $\mathbf{X}_2(t)$) out of the basin of attraction $\beta(A_1)$ of attractor A_1 (or $\beta(A_2)$ of attractor A_2) to the basin of attraction $\beta(A_2)$ (or $\beta(A_1)$). Simultaneously, the perturbation $d(\mathbf{X}_1 - \mathbf{X}_2)$ (or $d(\mathbf{X}_2 - \mathbf{X}_1)$) cannot move the trajectory $\mathbf{X}_2(t)$ (or $\mathbf{X}_1(t)$) out of $\beta(A_2)$ (or $\beta(\mathbf{A}_1)$). Perturbed trajectory $\mathbf{X}_2(t)$ (or $\mathbf{X}_1(t)$) leaves attractor A_2 (or A_1) but evolves within its basin of attraction.

We observed that the final attractor of the synchronized states or unsynchronized state strongly depends on the value of t_0, the time when coupling is introduced, i.e. on the initial locations $\mathbf{X}_{1,2}(t_0)$ of trajectories on attractors A_1 and A_2. At the time $t = t_0$, chaotic trajectories $\mathbf{X}_{1,2}(t)$ are at the points $\mathbf{X}_1(t_0) \in A_1$ and $\mathbf{X}_2(t_0) \in A_2$ which strongly depend on initial conditions $\mathbf{X}_1(0)$ and $\mathbf{X}_2(0)$, characterizing trajectories of both systems. Introducing coupling at $t = t_0$ we are unable to predict on which attractor the synchronization occurs. $\mathbf{Z}(0) = [\mathbf{X}_1(0),\mathbf{X}_2(0)]^T$ can be considered as the initial conditions for the augmented $2n$-dimensional system (2). We performed our computations for 10^4 randomly chosen initial conditions $\mathbf{X}_1(0) = [1.0 \pm 0.1, 2.0 \pm 0.1, 0 \pm 0.1]^T$ and $\mathbf{X}_2(0) = [-2. \pm 0.1, 0 \pm 0.1, 0 \pm 0.1]^T$ and introduced coupling at $t_0 = 10^4$ when both systems are on their attractors. Our results show that both chaotic attractors A_1 and A_2 are equally probable as a place of an asymptotic regime. The basins of attractors A_1 and A_2 of the coupled system (3) (considered as a 6-dimensional system of the type (2)) on the chaotic attractors A_1 and A_2 of the two uncoupled systems, obtained from the first three and the last three equations of (3), are intermingled as has been shown in Fig. 4. The basins of attraction of attractors A_1 and A_2 are indicated in black and white, respectively. In the computations shown in Fig. 4, initial location of the trajectory $\mathbf{X}_2(t)$ on attractor A_2 has been fixed in the point $\mathbf{X}_2(t_0) = [-1.0705, 0.5684, 1.529]^T$ and the location of the trajectory $\mathbf{X}_1(t)$ on attractor A_1 has been varied. To get the experimental evidence that the basins of A_1 and A_2 are locally intermingled, we performed our experiments 1000 times for randomly chosen values of t_0. We observed 631 events of the synchronization on the attractor A_1 and 469 on the attractor

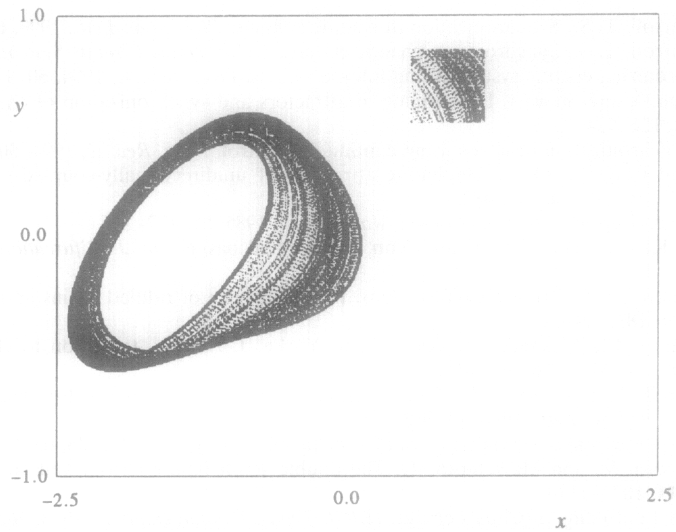

Fig. 4. Locally intermingled basins of attraction. The basins of attraction of attractors A_1, A_2 are indicated in black and white, respectively. In the computations shown here, initial location of the trajectory $\mathbf{X}_2(t)$ on attractor A_2 has been fixed at the point $\mathbf{X}_2(t_0) = [-1.0705, 0.5684, 1.529]^T$ and the location of the trajectory $\mathbf{X}_1(t)$ on attractor A_1 has been varied.

A_2. This test can be considered as a statistical evidence of locally intermingled basins of attraction.

Basins of attraction of the attractors A_1 and A_2 are not intermingled in whole 6-dimensional phase space of the system (3), but are intermingled in the lower 3-dimensional manifolds on which attractors A_1 and A_2 are located. All our numerical computations have been carried out using the software INSITE [19].

2. CONCLUSIONS

Coupling introduced in chaos synchronization schemes of quasi-hyperbolic systems which initially evolve on different co-existing attractors can lead to the locally intermingled basins of attraction of co-existing attractors. Even if the initial location of trajectories on the two attractors is known with infinite precision, we are unable to determine, on the basis of any finite calculation, in which basin this location lies and finally we cannot be sure on which attractor the evolution will synchronize. This type of uncertainty seems to be common for this class of dynamical system with invariant lower-dimensional manifold (synchronization manifold) and may have some practical implications. In the experimental systems, it is characterized by the lack of repetition of results as it is uncertain at which of co-existing attractors we shall obtain a synchronized regime.

Acknowledgement—This work was partially supported by the KBN (Poland) under the project No. 7TO7A 039 10.

REFERENCES

1. Fujisaka, H. and Yamada, T., Synchronization in chaotic systems. *Progr. Theoret. Phys.*, 1983, **69**, 32–47.
2. Afraimovich, V. S., Verichev, N. N. and Rabinovich, M. I., Stochastic synchronization of oscillations in dissipative systems. *Radiophys. Quant. Electron.*, 1986, **29**, 795–803.

3. Pecora, L. and Carroll, T. S., Synchronization in chaotic systems. *Phys. Rev. Lett.*, 1990, **64,** 821–824.
4. Pecora, L. and Carroll, T. S., Synchronizing chaotic circuits. *IEEE Trans. Circuits Systems*, 1991, **38,** 453–456.
5. Carroll, T., Synchronizing chaotic systems using filtered signals. *Phys. Rev. E*, 1994, **50,** 1563–1575.
6. Ashwin, P., Buescu, J. and Stewart, I., Bubbling of attractors and synchronization of chaotic oscillators. *Phys. Lett. A*, 1994, **193,** 126–134.
7. Kapitaniak, T., Synchronization of chaos using continuous control. *Phys. Rev. E*, 1994, **50,** 1642–1644.
8. Kapitaniak, T. and Chua, L. O., Hyperchaotic attractors of unidirectionally-coupled Chua's circuits. *Int. J. Bifurcation and Chaos*, 1994, **4,** 477–482.
9. Milnor, J., On the concept of attractor. *Comm. Math. Phys.*, 1985, **99,** 177–205.
10. Alexander, J. C., Kan, I., Yorke, J. A. and You, Z., Riddled basins. *Int. J. Bifurcation and Chaos*, 1992, **2,** 795–812.
11. Ott, E. and Sommerer, J. C., Blowout bifurcations: the occurrence of riddled basins and on-off intermittency. *Phys. Lett. A*, 1994, **188,** 39–47.
12. Ott, E., Sommerer, J. C., Alexander, I., Kan, I. and Yorke, J. A., The transition to chaotic attractors with riddled basins. *Phys. D*, 1994, **76,** 384–410.
13. Heagy, J. F., Carroll, T. and Pecora, L., Experimental and numerical evidence for riddled basins in coupled chaotic systems. *Phys. Rev. Lett.*, 1994, **73,** 3528–3531.
14. Kapitaniak, T., Experimental observation of riddled behavior. *J. Phys. A*, 1995, **28,** 63–66.
15. Chua, L. O., Komuro, M. and Matsumoto, T., The double scroll family parts I and II. *IEEE Trans. Circuits Systems*, 1986, **3311,** 1073–1118.
16. Chua, L. O., Global unfolding of Chua's circuit. *IEICE Trans. Fundamentals E*, 1993, **76A,** 704–734.
17. Kapitaniak, T. and Chua, L. O., Locally intermingled basin of attraction: uncertainty in chaos sysnchronization. *Int. J. Bifurcation and Chaos*, 1996, **6,** 357–366.
18. Kapitaniak, T., Uncertainty in coupled chaotic systems: locally intermingled basins of attraction. *Phys. Rev.* In press.
19. Parker, T. S. and Chua, L. O., *Practical Numerical Algorithms for Chaotic Systems*. Springer, New York, 1989.