topology and as such can be minimized but cannot be avoided. It is clear that the upper frequency of operation must be kept significantly below the $f_T$ of the op-amps used in order to allow sufficient high-frequency harmonics to be present within the output waveform so as not to degrade the performance excessively.

The second degradation mechanism identified is due to the fact that the diode feedback path becomes open-circuit around the zero-crossings, resulting in a missing segment in the output waveform for a time interval $t_d$. It has been confirmed that the input signal amplitude is small during $t_d$, and the op-amp operates in the linear region, with $t_d$ being inversely proportional to $\sqrt[3]{f_T}$ of the op-amp. From the results obtained, though high SR appears not to be directly important in providing high-speed precision rectification, it is generally the case that high SR op-amps do have a high $f_T$, and as such there is indirect correlation between high slew-rate and high-frequency PFWR performance.

## References


[1] F. J. Lidgey, K. Hayatleh, and C. Toumazou, "New current-mode precision rectifiers," in *Proc. IEEE ISCAS'93,* Chicago, IL, May 1993, pp. 1322–1325.

[2] K. Hayatleh, S. Porta, and F. J. Lidgey, "High frequency performance of current-mode PFWR," in *Proc. IEEE ISCAS'94,* London, UK, May–June 1994, vol. 5, pp. 401–404.


# Chaos in a Fractional Order Chua's System

Tom T. Hartley, Carl F. Lorenzo, and Helen Killory Qammer

***Abstract*—This brief studies the effects of fractional dynamics in chaotic systems. In particular, Chua's system is modified to include fractional order elements. By varying the total system order incrementally from 2.6 to 3.7, it is demonstrated that systems of "order" less than three can exhibit chaos as well as other nonlinear behavior. This effectively forces a clarification of the definition of order which can no longer be considered only by the total number of differentiations or by the highest power of the Laplace variable.**

## I. Introduction

It is well known that chaos cannot occur in continuous-time systems of order less than three. This assertion is based on the usual concepts of order, such as the number of states in a system, the highest power of the Laplace variable $s$ in the system, or the total number of separate differentiations or integrations in a system. Unfortunately, these concepts of order do not directly relate to systems having fractional order components. The purpose of this paper is to demonstrate that systems whose order is less than three, as defined in the usual way, can still display chaotic behavior. The first section below provides a brief review of fractional calculus. Useful approximations for these fractional operators follow in the next section. Finally, an example is given which demonstrates that systems whose order is less than three can display chaos. This is shown both experimentally via simulations, and predicted analytically using the describing function method.

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T. T. Hartley is with the Department of Electrical Engineering, The University of Akron, Akron, OH 44325 USA on leave with the System Dynamics Branch of NASA Lewis Research Center and The Ohio Aerospace Institute.

C. F. Lorenzo is with the System Dynamics Branch, NASA Lewis Research Center, Cleveland, OH 44135 USA.

H. Killory Qammar is with the Department of Chemical Engineering, The University of Akron, Akron, OH 44325 USA.

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## II. Review of Fractional Operators

The idea of fractional integrals and derivatives has been known since the development of the regular calculus, with the first reference probably being associated with Leibniz in 1695 [27]. Although not well known to most engineers, the fractional calculus has been considered by prominent mathematicians [8] as well as the "engineers" of the operational calculus [3], [13]. In fact many textbooks written before 1960 usually have some small section on fractional calculus [6], [9], [14], [26], [30], [32]. An outstanding historical survey can be found in [27] who also give what is unquestionably the most readable and complete mathematical presentation of the fractional calculus. Other bound discussions of the area are given by [24], [25], and [29]. Unfortunately, many of the results in the fractional calculus are given in the language of functional analysis and are not readily accessible to the general engineering community.

Many systems are known to display fractional order dynamics. Probably the first physical system to be widely recognized as demonstrating fractional behavior is the semi-infinite lossy (*RC*) line. Looking into the line, the current is equal to the half-derivative of the applied voltage, that is, the impedance is

$$V(s) = \frac{1}{\sqrt{s}} I(s).$$

Although studied by many, [13] considered this system extensively using the operational calculus. He states "there is a universe of mathematics lying in between the complete differentiations and integrations" and that "fractional (operators) push themselves forward sometimes, and are just as real as the others." Another equivalent system is the diffusion of heat into a semi-infinite solid. Here the temperature looking in from the boundary is equal to the half integral of the heat rate there. Other systems that are known to display fractional order dynamics are viscoelastic systems [1], [16]–[18], [31]; colored noise [23]; electrode-electrolyte polarization [15], [36]; dielectric polarization [35]; boundary layer effects in ducts [34]; and electromagnetic waves [13]. As many of these systems depend upon specific material and chemical properties, it is expected that a wide range of fractional order behaviors are possible using different materials.

Two commonly used definitions for the general fractional differintegral are the Grunwald definition and the Riemann–Liouville definition [27]. The Riemann–Liouville definition is given here;

$$\frac{d^q f}{dt^q} = \frac{1}{\Gamma(-q)} \int_o^t \frac{f(\tau)}{(t-\tau)^{q+1}}\, d\tau, q < 0.$$

Here $q$ can have noninteger values, and thus the name fractional differintegral. Notice that the definition is based on integration, and more importantly is a convolution integral for $q < 0$. When $q > 0$ then the usual integer $n$th derivative is taken of the fractional $(q-n)$th integral;

$$\frac{d^q f}{dt^q} = \frac{d^n}{dt^n}\left[\frac{d^{q-n} f}{dt^{q-n}}\right], q > 0 \text{ and } n \text{ an integer} > q.$$

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This appears so vastly different from the usual intuitive definition of derivative and integral that the reader must abandon the familiar concepts of slope and area and attempt to get some new insight (which still remains elusive).

Fortunately, the basic engineering tool for analyzing linear systems, the Laplace transform, is still applicable and works as one would expect;

$$L\left\{\frac{d^q f(t)}{dt^q}\right\} = s^q L\{f(t)\} - \sum_{k=0}^{n-1} s^k \left[\frac{d^{q-1-k} f(t)}{dt^{q-1-k}}\right]_{t=0},$$

for all $q$,

where $n$ is an integer such that $n-1 < q < n$. Upon considering the initial conditions to be zero, this formula reduces to the more expected and comforting form

$$L\left\{\frac{d^q f(t)}{dt^q}\right\} = s^q L\{f(t)\}.$$

## III. Approximation of Fractional Operators

The standard definitions of the fractional differintegral do not allow direct implementation of the operator in time-domain simulations of complicated systems with fractional elements. Thus, in order to effectively analyze such systems, it is necessary to develop approximations to the fractional operators using the standard integer order operators. In the work that follows, the approximations are effected in the Laplace $s$-variable. The resulting approximations provide sufficient accuracy for time domain hardware implementations.

Some work has been done in this area already, but it has not been highly organized. [27] and [28] give several discrete-time approximations based on numerical quadrature. In continuous-time, engineers have used network theory approximations [4], [5], [10], and [33]. More recently [7], [15], and [27] have developed other network theory approximations. Even more recently, a discrete-time fractional calculus has been developed similar to the theory of linear multistep methods for numerical integration [19]–[22].

The approximation approach taken here is that of [7]. Basically the idea is to approximate the system behavior in the frequency domain. This is done for a given $q$, by creating an approximation with Bode magnitude response roll off of 20 times $q$ dB/decade, and which will consequently have a phase shift of approximately 90 times $q$ degrees over the required frequency band. This approximation is created by choosing an initial breakpoint (the low frequency accuracy limit of the approximation), the allowable error in dB's, and the number of $s$-plane poles in the approximation. The high frequency limit of the usable bandwidth can be varied by changing the allowable error and the number of poles. Thus an approximation of any desired accuracy over any frequency band can be achieved. Table I gives approximations for $1/s^q$ with $q = 0.1$–$0.9$ in steps of 0.1. These were obtained by trial and error and are reasonably good from 0.01 rad/s to 100 rad/s. These approximations are used in the study that follows.

## IV. A Fractional Chua's System

Chua's system is well known and has been extensively studied. The particular form to be considered here was presented by [11] and used further for the study of [12]. This system is different from the usual Chua system in that the piecewise-linear nonlinearity is replaced by an appropriate cubic nonlinearity which yields very similar behavior. It is studied here in two different, but equivalent, system representations.

TABLE I

List of Integer Order Approximations to Fractional Operators. Each has an Error of Approximately 2 dB from $w = 10^{-2}$ to $10^2$ rad/s.

$$\frac{1}{s^{0.1}} \approx \frac{220.4s^4 + 5004s^3 + 5038s^2 + 234.5s + 0.4840}{s^5 + 359.8s^4 + 5742s^3 + 4247s^2 + 147.7s + 0.2099}$$

$$\frac{1}{s^{0.2}} \approx \frac{60.95s^4 + 816.9s^3 + 582.8s^2 + 23.24s + 0.04934}{s^5 + 134.0s^4 + 956.5s^3 + 383.5s^2 + 8.953s + 0.01821}$$

$$\frac{1}{s^{0.3}} \approx \frac{23.76s^4 + 224.9s^3 + 129.1s^2 + 4.733s + 0.01052}{s^5 + 64.51s^4 + 252.2s^3 + 63.61s^2 + 1.104s + 0.002267}$$

$$\frac{1}{s^{0.4}} \approx \frac{25.00s^4 + 558.5s^3 + 664.2s^2 + 44.15s + 0.1562}{s^5 + 125.6s^4 + 840.6s^3 + 317.2s^2 + 7.428s + 0.02343}$$

$$\frac{1}{s^{0.5}} \approx \frac{15.97s^4 + 593.2s^3 + 1080s^2 + 135.4s + 1}{s^5 + 134.3s^4 + 1072s^3 + 543.4s^2 + 20.10s + 0.1259}$$

$$\frac{1}{s^{0.6}} \approx \frac{8.579s^4 + 255.6s^3 + 405.3s^2 + 35.93s + 0.1696}{s^5 + 94.22s^4 + 472.9s^3 + 134.8s^2 + 2.639s + 0.009882}$$

$$\frac{1}{s^{0.7}} \approx \frac{5.406s^4 + 177.6s^3 + 209.6s^2 + 9.197s + 0.01450}{s^5 + 88.12s^4 + 279.2s^3 + 33.30s^2 + 1.927s + 0.0002276}$$

$$\frac{1}{s^{0.8}} \approx \frac{5.235s^3 + 1453s^2 + 5306s + 254.9}{s^4 + 658.1s^3 + 5700s^2 + 658.2s + 1}$$

$$\frac{1}{s^{0.9}} \approx \frac{1.766s^2 + 38.27s + 4.914}{s^3 + 36.15s^2 + 7.789s + 0.01000}$$

One representation is the usual state space form

$$\dot{x} = \alpha\left[y + \frac{x - 2x^3}{7}\right]$$
$$\dot{y} = x - y + z$$
$$\dot{z} = -\frac{100y}{7} = -\beta y$$

The other representation is a decoupled nonlinear feedback arrangement of Fig. 1, which is equivalent to the above state space representation when $q = 1$. In each case $\beta$ is defined to be 100/7 and $\alpha$ is allowed to vary. The state space configuration is used to verify chaos by computing Lyapunov exponents. The feedback configuration of Fig. 1 is used to perform a more thorough bifurcation study.

To study the effect of fractional derivatives on the dynamics of this system, the state space configuration is considered first. Here, the vector derivative is replaced by a vector fractional derivative as follows;

$$\frac{d^q x}{dt^q} = \alpha\left[y + \frac{x - 2x^3}{7}\right]$$
$$\frac{d^q y}{dt^q} = x - y + z$$
$$\frac{d^q z}{dt^q} = -\frac{100y}{7} = -\beta y$$

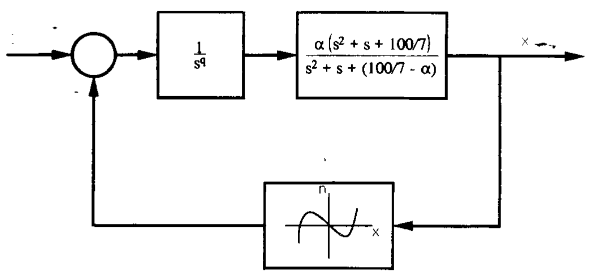



Fig. 1. The feedback configuration for Chua's system which allows easy change of order; $q = 1$ is the nominal Chua system.

Simulations were then performed using $q = 0.8$, 0.9, 1.0, and 1.1. The approximations from Table I are used for the simulations of the appropriate $q$th integrals. When $q$ is less than 1, then the approximations are used directly. It should further be noted that approximations used in the simulations for $1/s^q$, when $q > 1$, are obtained by using $1/s$ times the approximation for $1/s^{q-1}$ from Table I.

The results from this state space study verified that chaos could indeed occur in a system of mathematical order less than 3. This was determined by computing the Lyapunov exponents for each of the simulations with $q = 0.9$, 1.0, and 1.1, using the method of [2]. These results are given in Table II where the largest several exponents are given as a function of system order. In each case, the second exponent was near zero. The 2.7 order system approximation had an additional six negative exponents which were not listed. Also the 3.3 order system approximation was so large as to prohibit a timely calculation of any exponents but the first. Since the order of this system was greater than three anyway, these calculations were not pursued. In all cases, the one positive exponent clearly indicates that the system is behaving chaotically. The numerical simulations further indicated that the lower limit of the vector fractional derivative $q$ for this system to remain capable of generating chaos is between 0.8 and 0.9. The lowest value obtained for mathematical order to yield chaos was 2.7 using the $q = 0.9$ fractional vector derivative. No upper limit was obtained. Phase plane plots for these systems are given in Fig. 2.

The feedback configuration of Fig. 1 is now considered. To change the total system mathematical order, the separated $1/s$ in Fig. 1 is allowed to change powers, that is

$$\frac{1}{s} \to \frac{1}{s^q}.$$

A variety of simulations were performed on the resulting systems as discussed below. Here, the approximations from Table I are used to represent the fractional integral where again the approximations for $1/s^q$, when $q > 1$, are obtained by using $1/s$ times the approximation for $1/s^{q-1}$.

Bifurcation diagrams for several of these systems are given in Fig. 3. Here, a particular value of $q$ was chosen, and the parameter $\alpha$ was varied to obtain the particular bifurcation plot. These diagrams were generated by simulation using Euler's method and a simulation timestep of 0.001. These were verified by further reducing the timestep by an order of magnitude with little change in the overall bifurcation structure. To obtain these diagrams, the values of the output $x$-variable are plotted whenever its slope changes sign. Although it is felt that the bifurcation diagrams are reasonably correct and are sufficiently accurate for this particular study, more correct diagrams could possibly be obtained by using more accurate approximations of the fractional derivative than those given in Table I or a more accurate simulation. Observation of the bifurcation diagrams indicates behavior similar to that from the state space study. For the feedback configuration, decreasing the power of $s$, shifts the bifurcation diagram to the right as a function of $\alpha$, and conversely. The limits on the system mathematical order to have a chaotic response as measured from the bifurcation diagrams are approximately $2.5 < n < 3.8$. The overall behavior from the simulation studies is summarized in Fig. 4.

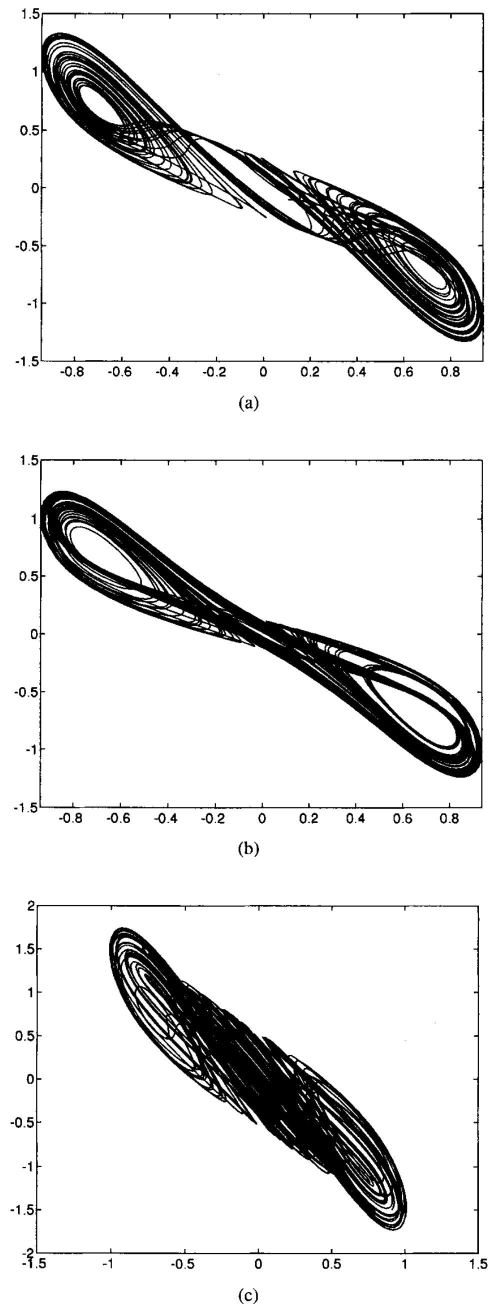
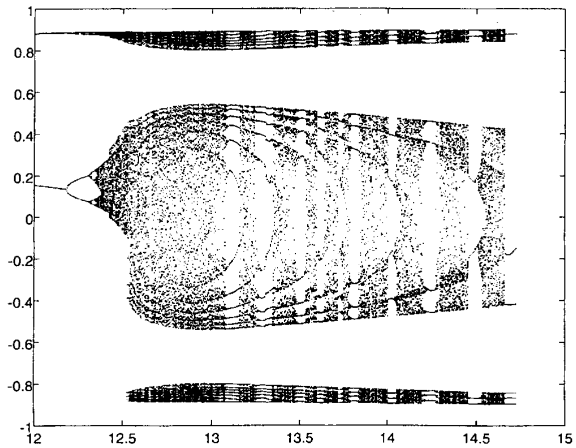


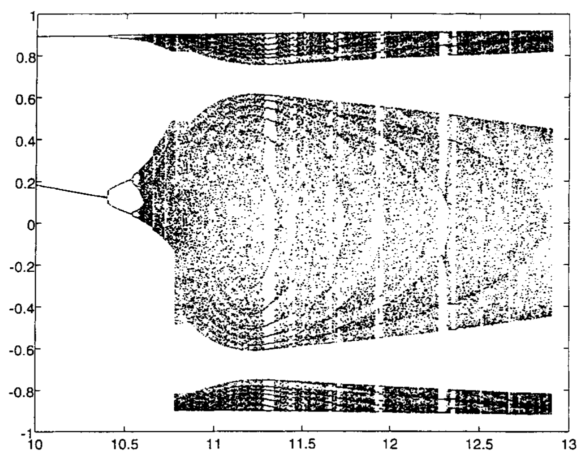

Fig. 2. Phase plane projections for the state space configuration of Chua's system: (a) total mathematical system order is 3.0, $x$ versus $z$, $\alpha = 9.5$; (b) total mathematical system order is 2.7, $x$ versus $z$, $\alpha = 12.75$; (c) total mathematical system order is 3.3, $x$ versus $z$, $\alpha = 7.0$.

TABLE II
LARGEST LYAPUNOV EXPONENTS FOUND IN THE STATE SPACE CONFIGURATION FOR $q$ = 0.9, 1.0, AND 1.1 WHICH GIVES A TOTAL SYSTEM MATHEMATICAL ORDER OF 2.7, 3.0, AND 3.3, RESPECTIVELY

| Mathematical System Order | Order of System Approximation | $\alpha$ - Used | Largest Exponent, $\lambda_1$ | $\lambda_2$ | $\lambda_3$ |
|---|---|---|---|---|---|
| 2.7 | 9 | 12.75 | 0.338 | -0.000201 | -0.132 |
| 3.0 | 3 | 9.50 | 0.248 | -0.00412 | -3.07 |
| 3.3 | 18 | 7.00 | 0.318 | * | * |

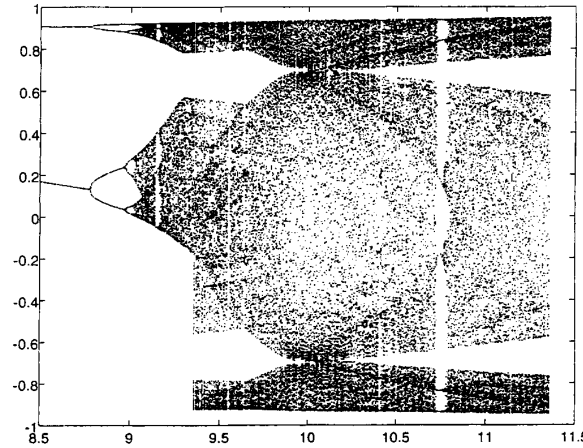



(a)

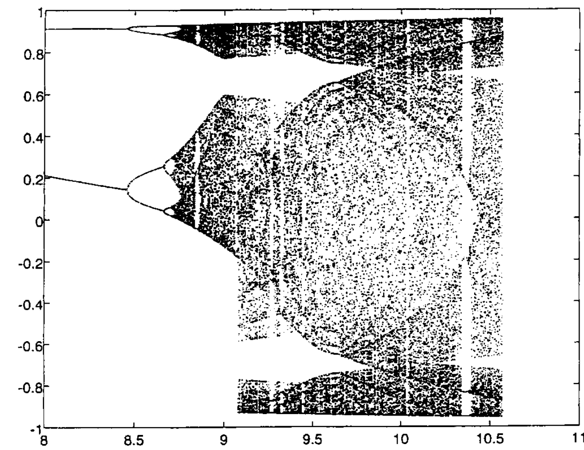



(b)

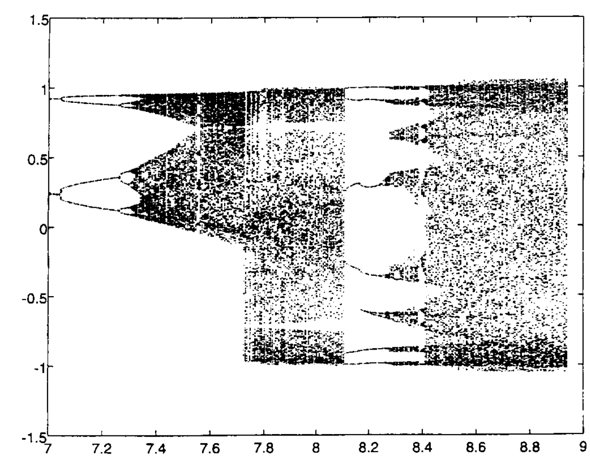
Fig. 3. Bifurcation diagram for the feedback configuration of Chua's system, max and min of $x$ versus $\alpha$: (a) fractional integral of order 0.7, total mathematical system order 2.7; (b) fractional integral of order 0.8, total mathematical system order 2.8; (c) fractional integral of order 0.9, total mathematical system order 2.9; (d) fractional integral of order 1.0, total mathematical system order 3.0; and (e) fractional integral of order 1.3, total mathematical system order 3.3.




(c)




(d)




(e)

Fig. 3. (*Continued.*)

An advantage to the feedback configuration is that it allows easy system analysis using describing functions, as discussed in [12]. Here the idea is that the frequency response of the linear block in the feedback configuration is plotted in the Nyquist plane, along with minus one over the appropriate describing function of the nonlinearity, as in Fig. 5. The fractional order integral in the loop is handled directly by taking the frequency response on the primary

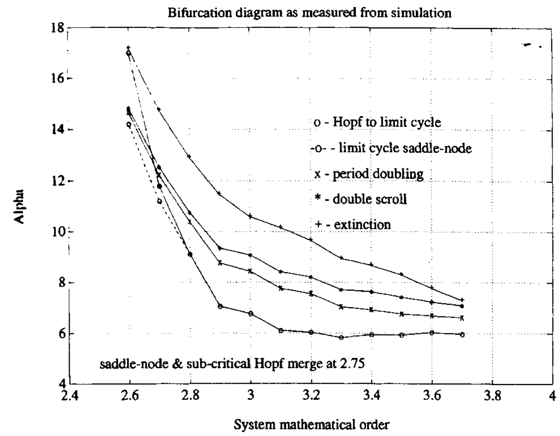



Fig. 4. Bifurcation diagram in the $\alpha$ versus mathematical system order plane based on simulation studies of the fractional Chua system.

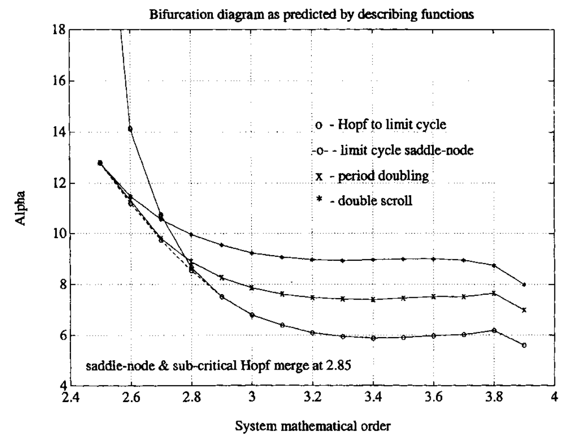



Fig. 6. Bifurcation diagram in the $\alpha$ versus mathematical system order plane based on describing function analysis of the fractional Chua system.

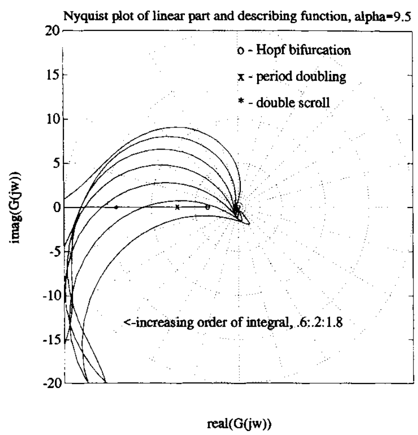



Fig. 5. Nyquist plane plot showing the frequency response of the linear part of Fig. 1 (set of curved lines) for various $q$ and $\alpha$ = 9.5; and the describing function of the nonlinearity (solid line on real-axis from -3.5 to -20 shown).

Riemann sheet, and essentially poses no complication or confusion in application of the describing function approach. In other words, the fact that fractional powers of $s$ are present does not require any frequency domain approximation as in the time-domain simulation, rather the fractional powers of $s$ can be used as is, in computing the frequency response of the linear block. In [12] it is shown that the important points from the nonlinearity of this system in the Nyquist plane are:

1) $\text{Re}[H(j\omega)] > -3.5$, $\text{Im}[H(j\omega)] = 0$, indicates two stable points at $x = \pm\sqrt{0.5}$;
2) $\text{Re}[H(j\omega)] \leq -3.5$, $\text{Im}[H(j\omega)] = 0$, indicates a Hopf bifurcation of the stable points of (a) into a limit cycle;
3) $\text{Re}[H(j\omega)] \leq -7$, $\text{Im}[H(j\omega)] = 0$, indicates that period doubling of the limit cycle of (b) occurs (this progresses into spiral chaos);
4) $\text{Re}[H(j\omega)] \leq -14$, $\text{Im}[H(j\omega)] = 0$, indicates merging of the spiral chaos into the double scroll behavior.

Extinction of the double scroll (meaning its disappearance) is not directly predicted using the describing function approach, but a reasonable approximate value is $\text{Re}[H(j\omega)] \leq -23$, $\text{Im}[H(j\omega)] = 0$. A diagram indicating the usage of the describing function is given in Fig. 5.

Using these results, and varying the power of the integrator in the loop allowed a theoretical prediction of the simulation results of Fig. 4. These theoretical results are given in Fig. 6. It should be noted that the qualitative features are very well predicted using the describing function approach, and that the quantitative results are reasonably close. Furthermore, for mathematical system order less than approximately 2.85, the describing function approach predicted the appearance of a stable and unstable limit cycle as $\alpha$ increased (via an apparent saddle-node bifurcation). These limit cycles coexist with each of the stable fixed points. Eventually, as $\alpha$ increased further, the unstable cycles merged with the stable fixed points via a subcritical Hopf bifurcation, leaving an unstable fixed point. This entire process basically became a supercritical Hopf bifurcation for mathematical order greater than 2.85. This was then verified in the simulations with this bifurcation structure occurring for mathematical system order less than approximately 2.75. In fact, for the mathematical order equal to 2.6, the simulation gave the points at $x = \pm\sqrt{0.5}$ to be stable and each coexisting with spiral chaos. It is a true testament to the utility of the describing function approach that it could predict the behavior of this system as accurately as it does.

## V. Discussion

This paper has introduced the idea of fractional derivatives from the dynamic systems viewpoint. It has been demonstrated that the usual idea of system order must be modified when fractional derivatives are present. The usual approach of calculating the mathematical system order by determining the highest derivative in the system does not work in this situation.

It has been further demonstrated that chaos, as well as the other usual nonlinear dynamic phenomena, can occur in systems with mathematical order less than three via Chua's system. This is surprising given the usual nonlinear system paradigms concerning chaos and order. It is not clear at this point whether the chaos in fractional order systems should be characterized differently than chaos in regular integer order systems.

It should be noticed that the describing function approach usually requires at least -180 degrees of phase shift in the linear part of the feedback loop to ever predict Hopf bifurcations, and consequently chaos, for memoryless nonlinearities. As the linear part can be a nonminimum phase transfer function, it is further conjectured that chaos can occur in systems with mathematical order less than three and probably less than one. Furthermore, the feedback configuration indicates that as long as the linear part of the loop has at least -180 degrees of phase shift, the possibility of chaos in the system depends primarily on the nonlinearity, and how its particular describing function behaves.

As has been demonstrated, the idea of fractional derivatives requires one to reconsider dynamic system concepts that are often taken for granted. Some of these concepts have been discussed in this paper. Some others that require much further consideration are the concept of Lyapunov exponents for fractional states, the use of fractional states in which to embed attractors, and the relationship between fractional order and fractal dimension.

## REFERENCES

[1] R. L. Bagley and R. A. Calico, "Fractional order state equations for the control of viscoelastically damped structures," *J. Guid., Cont. & Dyn.*, vol. 14, no. 2, pp. 304–311, Mar.–Apr. 1991.

[2] G. Benettin, L. Galgani, A. Giogilli, and J. Strelcyn, "Lyapunov characteristic exponents for smooth dynamical systems and for hamiltonian systems," *Meccanica*, vol. 15, pp. 9–30, 1980.

[3] V. Bush, *Operational Circuit Analysis.* New York: Wiley, 1929.

[4] G. E. Carlson and C. A. Halijak, "Simulation of the fractional derivative operator sqrt(s) and the fractional integral operator 1/sqrt(s)," in *Proc. of the Central States Simulation Council Meeting*, KSU, Manhattan, KS, May 1961, pp. 1–22.

[5] ——, "Approximation of fractional capacitors $(1/s)^{(1/n)}$ by a regular Newton process," *IEEE Trans. Circuit Theory*, pp. 210–213, June 1964.

[6] H. S. Carslaw and J. C. Jeager, *Operational Methods in Applied Mathematics.* Oxford: Oxford University Press, 1941, 2nd ed.

[7] A. Charef, H. H. Sun, Y. Y. Tsao, and B. Onaral, "Fractal system as represented by singularity function," *IEEE Trans. Automat. Contr.*, vol. 37, pp. 1465–1470, Sept. 1992.

[8] R. Courant and D. Hilbert, *Methods of Mathematical Physics.* New York: Intersci., 1953.

[9] S. Goldman, *Transformation Calculus and Electrical Transients.* Engelwood Cliffs, NJ: Prentice-Hall, 1949.

[10] C. A. Halijak, "An RC impedence approximant to $(1/s)^{1/2}$," *IEEE Trans. Circuit Theory*, pp. 494–495, Dec. 1964.

[11] T. T. Hartley, "The duffing double scroll", in *Proc. Amer. Cont. Conf.*, Pittsburgh, PA, June 1989, pp. 419–423.

[12] T. T. Hartley and F. Mossayebi, "Control of Chua's circuit," *J. Circuits, Syst., Comput.*, vol. 3, no. 1, pp. 173–194, Mar. 1993.

[13] O. Heaviside, *Electromagnetic Theory.* New York: Chelsea, 1893 and 1971.

[14] J. G. Holbrook, *Laplace Transforms for Electronic Engineers.* Oxford: Pergamon, 1966, 2nd ed.

[15] M. Ichise, Y. Nagayanagi, and T. Kojima, "An analog simulation of noninteger order transfer functions for analysis of electrode processes," *J. Electroanal. Chem.*, vol. 33, pp. 253–265, 1971.

[16] R. C. Koeller, "Application of fractional calculus to the theory of viscoelasticity," *J. Appl. Mech.*, vol. 51, pp. 299, June 1984.

[17] ——, "Polynomial operators, stieltjes convolution, and fractional calculus in hereditary mechanics," *Acta Mechanica*, vol. 58, pp. 251–264, 1986.

[18] J. C. Lopez-Marcos, "A difference scheme for a nonlinear partial integrodifferential equation," *SIAM J. Numer. Anal.* , vol. 27, no. 1, pp. 20–31, Feb. 1990.

[19] C. Lubich, "Fractional multistep methods for Abel-Volterra integral equations of the second kind," *Math. Comp.* , vol. 45, no. 172, pp. 463–469, Oct. 1985.

[20] ——, "Discretized fractional calculus," *SIAM J. Math. Anal.*, vol. 17, no. 3, May 1986.

[21] ——, "Convolution quadrature and discretized operational calculus, I," *Numer. Math.*, vol. 52, pp. 413–425, 1988.

[22] ——, "Convolution quadrature and discretized operational calculus, II," *Numer. Math.*, vol. 52, pp. 129–145, 1988.

[23] B. Mandelbrot, "Some noises with $1/f$ spectrum, a bridge between direct current and white noise," *IEEE Trans. Inform. Theory*, vol. IT-13, no. 2, Apr. 1967.

[24] A. C. McBride, *Fractional Calculus and Integral Transforms of Generalized Functions.* San Francisco, CA: Pitman, 1979.

[25] A. C. McBride and G. F. Roach, Eds. *Fractional Calculus* . Boston, MA: Pitman, 1985.

[26] J. Mikusinski, *Operational Calculus.* New York: Pergamon, 1959.

[27] K. B. Oldham and J. Spanier, *The Fractional Calculus* . San Diego, CA: Academic, 1974.

[28] R. Piche, "Noise-filtering properties of numerical methods for the inverse Abel transform," *IEEE Trans. Instrum. Measure.* , vol. 41, no. 4, pp. 517–522, Aug. 1992.

[29] B. Ross, Ed. *Fractional Calculus and Its Applications* . Berlin: Springer-Verlag, 1975.

[30] E. J. Scott, *Transform Calculus.* New York: Harper & Bros., 1955.

[31] S. B. Skaar, A. N. Michel, and R. K. Miller, "Stability of viscoelastic control systems," *IEEE Trans. Automat. Contr.*, vol. 33, pp. 348–357, Apr. 1988.

[32] B. J. Starkey, *Laplace Transforms for Electrical Engineers.* London: Iliffe & Sons, 1954.

[33] K. Steiglitz, "An RC impedence approximant to $s^{-1/2}$," *IEEE Trans. Circuit Theory*, vol. 11, pp. 160–161, Mar. 1964.

[34] N. Sugimoto, "Burgers equation with a fractional derivative; hereditary effects on nonlinear acoustic waves," *J. Fluid Mech.*, vol. 225, pp. 631–653, 1991.

[35] H. H. Sun, A. A. Abdelwahab, and B. Onaral, "Linear approximation of transfer function with a pole of fractional order," *IEEE Trans. Automat. Contr.*, vol. AC-29, no. 5, pp. 441–444, May 1984.

[36] H. H. Sun, B. Onaral, and Y. Tsao, "Application of the positive reality principle to metal electrode linear polarization phenomena," *IEEE Trans. Biomed. Eng.*, vol. BME-31, pp. 664–674, Oct. 1984.

[37] Y. Y. Tsao, B. Onaral, and H. H. Sun, "An algorithm for determining global parameters of minimum-phase systems with fractional power spectra," *IEEE Trans. Instrum. Measure.*, vol. 38, pp. 723–729, June 1989.