Controlling Chua's Global Unfolding Circuit Family

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***Abstract*—The self-unified and conventional feedback control strategy developed by the present author and his colleagues for the control of chaotic dynamical systems is extended from the Chua's circuit to its global unfolding, driving the circuit dynamics from chaotic attractors to any desirable trajectories such as unstable limit cycles. A simple and realistic sufficient condition for such controllability of the global unfolding is derived. A similar sufficient condition for the controllability of the whole family of nonlinear dynamical systems that are topologically equivalent to the Chua global unfolding is also established.**

I. Introduction

An interesting and challenging research subject recently arising in the field of nonlinear dynamical systems is the control of chaos, namely, the investigation of bringing order into chaos. The state-of-the-art development of this stimulating and promising research can be seen from, for example, the survey and tutorial paper [1]. Our specific interest in this new research direction has been to study the design of conventional feedback controllers that can drive the phase-space trajectory of a nonlinear dynamical system from one region to another, particularly from a chaotic orbit of the system to one of its unstable limit cycles, which, as is well known, is rather difficult. The feedback control approach developed by the present author and his colleagues, which has been summarized in [1], provides a conventional and self-unified strategy. This technique has been successfully applied to the control of chaos, for both continuous-time and discrete-time nonlinear chaotic systems such as Chua's circuit [2], [3], Duffing's oscillator [4], and the Lozi and Henon systems [5], [6]. A general nonlinear controller design principle for this purpose was outlined in [7], using the Duffing system as an example.

The objective of this paper is to study the control of the Chua circuit family and its global unfolding. The idea and technique for the control of Chua's circuit developed in [2], [3] will be further generalized and extended to the Chua global unfolding circuit family, and as well to all the nonlinear dynamical systems that are topologically equivalent to this global unfolding.

To facilitate our discussion, let us first briefly review the structure of the Chua global unfolding circuit family. This whole circuit family is built on the Chua's circuit [8], with an additional linear resistor placed therein, as shown in Fig. 1. In the figure, L is an inductor, C_1, C_2 are two capacitors, g is a piecewise-linear resistor, and R and R_0 are two linear resistors where R_0 is the new element added to the canonical circuit, as described in [9]. The dynamics of this global unfolding circuit family is described by

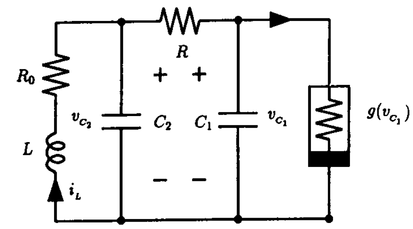

Fig. 1. The Chua global unfolding circuit family.

$$\begin{cases} C_1\dot{v}_{C_1} = G(v_{C_2} - v_{C_1}) - g(v_{C_1}) \\ C_2\dot{v}_{C_2} = G(v_{C_1} - v_{C_2}) + i_L \\ L\dot{i}_L = -v_{C_2} - R_0 i_L, \end{cases} \tag{1}$$

where i_L is the current through the inductor L, v_{C_1} and v_{C_2} the voltages across C_1 and C_2, respectively, $G = 1/R$, and

$$\begin{aligned} g(v_{C_1}) &= g(v_{C_1}; m_0, m_1) \\ &= m_0 v_{C_1} + \frac{1}{2}(m_1 - m_0)(|v_{C_1} + 1| - |v_{C_1} - 1|) \end{aligned}$$

with $m_0 < 0$ and $m_1 < 0$ being some appropriately chosen constants [8], [9].

By using the following transformation:

$$\begin{aligned} &x(\tau) = v_{C_1}(t), \quad y(\tau) = v_{C_2}(t), \\ &z(\tau) = \frac{1}{G}\, i_L(t), \quad \text{with} \quad \tau = \frac{G}{C_2}\, t, \end{aligned} \tag{2}$$

the circuit equations (1) can be reformulated as the following dynamically equivalent state equations:

$$\begin{cases} \dot{x} = p(-x + y - f(x)) \\ \dot{y} = x - y + z \\ \dot{z} = -qy - hz, \end{cases} \tag{3}$$

where $p = C_2/C_1 > 0$, $q = C_2/LG^2 > 0$, and $h = C_2/LGR_0$ are the main bifurcation parameters of the circuit and the nonlinear term $f(x)$ is a three-segment piecewise-

Manuscript received April 25, 1993; revised May 20, 1993. This work was supported by the President's Research and Scholarship Fund, University of Houston. This paper was recommended by Guest Editor L. O. Chua.

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IEEE Log Number 9211763.

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linear function defined by

$$\begin{aligned} f(x) &= g(x;\, m_0',\, m_1') \\ &= m_0' x + \frac{1}{2}(m_1' - m_0')(|x+1| - |x-1|) \\ &= \begin{cases} m_0' x + m_1' - m_0' & x \geq 1 \\ m_1' x & |x| \leq 1 \\ m_0' x - m_1' + m_0' & x \leq -1, \end{cases} \end{aligned}$$

where $m_0' = m_0/G < 0$ and $m_1' = m_1/G < 0$.

II. Controlling the Chua Global Unfolding Circuit Family

There are different methods in introducing order into the chaotic dynamics of Chua's circuit (see, from example, [10], [11]). Among them, the conventional feedback control approach developed by us in [2], [3] has the advantages that the given system structure and parameters need not be changed or be manipulated by the user, since the (linear or nonlinear) controller, after being designed, can automatically direct the system trajectory to the target, and that the method has a routine procedure to follow. This technique will be generalized and extended in this section, to control the whole family of global unfolding of Chua's circuit.

Consider the global unfolding described by the system (3). Let $(\overline{x}, \overline{y}, \overline{z})$ be a target trajectory of the system, which can be any orbit, like a (stable or unstable) limit cycle (including equilibrium points), a chaotic trajectory, etc., of the system. Observe that $(\overline{x}, \overline{y}, \overline{z})$ is itself a solution of system (3), namely, the system equations hold if (x, y, z) are replaced by $(\overline{x}, \overline{y}, \overline{z})$, where the resulting system will be labeled (3′) below. By subtracting (3′) from (3) and then adding the linear feedback control of the form

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = -\begin{bmatrix} K_{11} & 0 & 0 \\ 0 & K_{22} & 0 \\ 0 & 0 & K_{33} \end{bmatrix} \begin{bmatrix} x - \overline{x} \\ y - \overline{y} \\ z - \overline{z} \end{bmatrix} \tag{4}$$

into the resulting system, with the new notation

$$X = x - \overline{x}, \quad Y = y - \overline{y}, \quad \text{and } Z = z - \overline{z} \tag{5}$$

we arrive at the following new system:

$$\begin{cases} \dot{X} = p(-X + Y - \tilde{f}(x, \overline{x})) - K_{11} X \\ \dot{Y} = X - Y + Z - K_{22} Y \\ \dot{Z} = -qY - q\frac{G}{R_0} Z - K_{33} Z, \end{cases} \tag{6}$$

where all notation are defined as above, and $\tilde{f}(x, \overline{x}) = f(x) - f(\overline{x})$ (see [2] for more details about the exact expression of the nine-segment piecewise linear function $\tilde{f}$).

By imitating the proof for the case of the simple canonical circuit given in [2], we obtain the following result on the controllability of the Chua global unfolding circuit family (3):

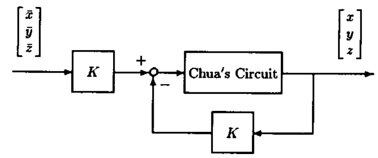

Fig. 2. Feedback control configuration for the Chua global unfolding circuit family.

Theorem 1: Let $(\overline{x}, \overline{y}, \overline{z})$ be any target trajectory of the Chua global unfolding circuit described by system (3). The chaotic trajectory (x, y, z) of the system can be driven to reach the target trajectory by the linear feedback control (4), if the constant feedback gains satisfy the following conditions:

$$K_{11} \geq -pm_1', \quad K_{22} \geq 0 \quad \text{and} \quad K_{33} \geq -C_2/LGR_0 \tag{7}$$

where the control can be applied to the chaotic trajectory at any time.

The closed-loop feedback control configuration of this system is shown in Fig. 2, from which one can see that the constant feedback gain matrix $K = \mathrm{diag}\,[K_{11}, K_{22}, K_{33}]$ is connected to the circuit from outside in a closed-loop manner and the circuit itself needs not to be modified.

We remark that if negative feedback control is preferred as usual, then the last inequality of (7) can be simply replaced by $K_{33} \geq 0$. In this case, and in the case that R_0 does not exist, the condition obtained above reduces to the one obtained in [2].

III. Controlling Equivalent Nonlinear Dynamical Systems

In this section, we further extend the above result for the controllability of the Chua global unfolding to a whole class of nonlinear systems that are topologically equivalent to the global unfolding.

Consider the so-called $\mathcal{C}/\epsilon_0$-family of nonlinear dynamical systems consisting of, roughly speaking, all the piecewise linear differential equations of the form

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \frac{1}{2}\{|x+1| - |x-1|\} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \tag{8}$$

where $\{a_{ij}, b_j\}_{i,j=1}^3$ are constants, such that

$$\Delta := \det \begin{bmatrix} 1 & 0 & 0 \\ a_{11} & a_{12} & a_{13} \\ \sum_{j=1}^3 a_{1j} a_{j1} & \sum_{j=1}^3 a_{1j} a_{j2} & \sum_{j=1}^3 a_{1j} a_{j3} \end{bmatrix} \neq 0,$$

(see [9] for the exact definition of $\mathcal{C}/\epsilon_0$ and for more detailed discussions).

We first introduce the following invertible linear transformation:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a_{11} & a_{12} & a_{13} \\ \sum_{j=1}^3 a_{1j}a_{j1} & \sum_{j=1}^3 a_{1j}a_{j2} & \sum_{j=1}^3 a_{1j}a_{j3} \end{bmatrix}^{-1} \times \begin{bmatrix} 1 & 0 & 0 \\ -\frac{G+m_0'}{C_1} & \frac{G}{C_1} & 0 \\ \left(\frac{G+m_0'}{C_1}\right) + \frac{G^2}{C_1C_2} & -\frac{G(G+m_0')}{C_1^2} - \frac{G^2}{C_1C_2} & \frac{G}{C_1C_2} \end{bmatrix} \times \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \tag{9}$$

where all notation are as before. Under this nonsingular linear transform, denoted by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = T \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix}$$

where

$$T_{11} = 1, \quad T_{12} = 0, \quad T_{13} = 0,$$

$$T_{21} = \left[a_{13}\left(\sum_{j=1}^3 a_{1j}a_{j1} - \left(\frac{G+m_0'}{C_1}\right)^2 - \frac{G^2}{C_1C_2} \right) - \left(a_{11} + \frac{G+m_0'}{C_1}\right) \sum_{j=1}^3 a_{1j}a_{j3} \right] / \Delta,$$

$$T_{22} = \frac{G}{C_1} \sum_{i=1}^3 a_{1j}a_{j3} / \Delta,$$

$$T_{23} = -a_{13}G/(C_1C_2\Delta),$$

$$T_{31} = \left[\left(a_{11} + \frac{G+m_0'}{C_1}\right) \sum_{j=1}^3 a_{1j}a_{j2} + a_{12} \cdot \left(\left(\frac{G+m_0'}{C_1}\right)^2 + \frac{G^2}{C_1C_2} - \sum_{j=1}^3 a_{1j}a_{j1} \right) \right] / \Delta,$$

$$T_{32} = -\left[\frac{G}{C_1} \sum_{j=1}^3 a_{1j}a_{j2} + a_{12} \left(\frac{G(G+m_0')}{C_1^2} + \frac{G^2}{C_1C_2} \right) \right] / \Delta,$$

$$T_{33} = a_{12}G/(C_1C_2\Delta),$$

with

$$\Delta = a_{12}\sum_{j=1}^3 a_{1j}a_{j3} - a_{13}\sum_{j=1}^3 a_{1j}a_{j2} \neq 0,$$

the nonlinear system (8) (with the state vector $[x\, y\, z]^T$) is transformed to the Chua global unfolding system (3) (with a new state vector $[\xi\, \eta\, \zeta]^T$), namely, to the global unfolding:

$$\begin{cases} \dot{\xi} = \frac{C_2}{C_1}(-\xi + \eta - \hat{f}(\xi)) \\ \dot{\eta} = \xi - \eta + \zeta \\ \dot{\zeta} = -\frac{C_2}{LG^2}\eta - \frac{C_2}{LGR_0}\zeta, \end{cases} \tag{10}$$

where

$$\hat{f}(\xi) = \frac{m_0}{G}\xi + \frac{1}{2}\frac{m_1 - m_0}{G}(|\xi + 1| - |\xi - 1|)$$

in which all circuit parameters C_1, C_2, G, etc. can be calculated by using the inverse transform T^{-1}, which is equivalent to using the "equivalent Chua's circuit algorithm" described in [9]. More precisely, the algorithm works as follows: Start with the given nonlinear dynamical system (8). First, calculate the eigenvalues (μ_1, μ_2, μ_3) and (ν_1, ν_2, ν_3) associated with the linear and affine vector fields, respectively, of the system. Then, set

$$\begin{aligned} p_1 &= \mu_1 + \mu_2 + \mu_3, & q_1 &= \nu_1 + \nu_2 + \nu_3 \\ p_2 &= \mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1, & q_2 &= \nu_1\nu_2 + \nu_2\nu_3 + \nu_3\nu_1, \\ p_3 &= \mu_1\mu_2\mu_3, & q_3 &= \nu_1\nu_2\nu_3. \end{aligned}$$

Next, set

$$\begin{aligned} k_1 &= -p_3 + \frac{q_3 - p_3}{q_1 - p_1}\left(p_1 + \frac{p_2 - q_2}{q_1 - p_1}\right), \\ k_2 &= p_2 - \frac{q_3 - p_3}{q_1 - p_1} + \frac{p_2 - q_2}{q_1 - p_1}\left(p_1 + \frac{p_2 - q_2}{q_1 - p_1}\right), \\ k_3 &= \frac{p_2 - q_2}{q_1 - p_1} - \frac{k_1}{k_2}, \\ k_4 &= -k_1k_3 + k_2\frac{p_3 - q_3}{p_1 - q_1}. \end{aligned}$$

Finally, calculate

$$\begin{aligned} C_1 &= 1, \\ C_2 &= -k_2/k_3, \\ G &= 1/R = -k_2/k_3, \\ L &= -k_3^2/k_4, \\ R_0 &= -k_1k_3^2/k_2k_4, \\ m_0 &= Gm_0' = \frac{k_2}{k_3}\left(p_1 + \frac{p_q - q_2}{p_1 - q_1}\right) - \frac{k_2^2}{k_3^2}, \\ m_1 &= Gm_1' = \frac{k_2}{k_3}\left(q_1 + \frac{p_2 - q_2}{p_1 - q_1}\right) - \frac{k_2^2}{k_3^2}. \end{aligned} \tag{11}$$

The reader is referred to [9] for more details about this equivalence of the two systems.

Now, we return to the control problem of the given nonlinear dynamical system (8). Let ($\overline{x}$, $\overline{y}$, $\overline{z}$) be any phase-space trajectory that we are targeting (e.g., unstable limit cycles) of the nonlinear dynamical system (8). We first transform system (8) to its equivalent Chua's global unfolding (10), and then derive the sufficient controllability conditions for system (10). After all, the inverse linear transform produces the desired controllability conditions for the original system (8). This is the basic procedure. Yet its algebraic manipulations are somewhat tedious, and are hence omitted. The final result is summarized in the following theorem:

Theorem 2: The chaotic trajectory (x, y, z) of the nonlinear dynamical system (8) of family $\mathcal{C}/\epsilon_0$ can be driven to reach any target trajectory ($\overline{x}$, $\overline{y}$, $\overline{z}$) of the system by a linear

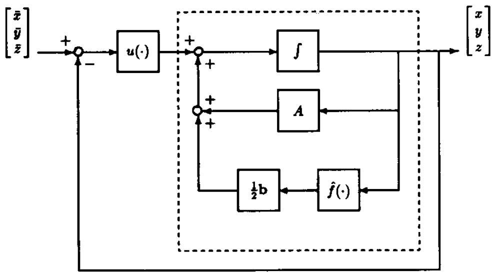

Fig. 3. Feedback control configuration for equivalent nonlinear dynamical systems.

feedback control input of the form

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = -\begin{bmatrix} K_{11} & 0 & 0 \\ 0 & K_{22} & 0 \\ 0 & 0 & K_{33} \end{bmatrix} \begin{bmatrix} x-\overline{x} \\ y-\overline{y} \\ z-\overline{z} \end{bmatrix}, \tag{12}$$

which is to be added to the right-hand side of the given system (8), if the three constant feedback gains of (11) satisfy the following conditions:

$$K_{11} \geq -\frac{C_2}{G} m_1,$$

$$K_{22} \frac{G\left(C_2 \sum_{i=1}^{3} a_{1i}a_{i3} - a_{13}\right)}{C_1C_2\left(a_{12}\sum_{i=1}^{3} a_{1i}a_{i3} - a_{13}\sum_{i=1}^{3} a_{1i}a_{i2}\right)} \geq 0,$$

$$\left(K_{33}+\frac{G}{R_0}\right) \frac{Ga_{12}}{C_1C_2\left(a_{12}\sum_{i=1}^{3} a_{1i}a_{i3} - a_{13}\sum_{i=1}^{3} a_{1i}a_{i2}\right)} \geq 0 \tag{13}$$

where C_1, C_2. $G.R_0$ and m_1 are given by (11). Moreover, the control can be applied to the chaotic trajectory at any time.

The closed-loop feedback control configuration of the system is shown in Fig. 3, where $A = [a_{ij}]_{3\times 3}$ and $\boldsymbol{b} = [b_1\, b_2\, b_3]^T$ are given in (8), and $\hat{f}$ is defined in (10).

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