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Predicting Chaos Through an Harmonic Balance Technique: An Application to the Time-Delayed Chua's Circuit

Marco Gilli and Gian Mario Maggio

Abstract—The time-delayed Chua's circuit (TDCC) can be considered as a paradigm for studying chaos in circuits described by difference-differential equations. The dynamics of such circuits cannot be investigated by means of the standard time-domain techniques developed for finite dimensional systems. We show that through a spectral approach the occurrence of periodic limit cycles and of chaotic attractors can be easily predicted, without performing any simulations.

I. INTRODUCTION

In recent years much interest has been devoted to the study of the dynamics of circuits described by nonlinear equations with delay, which exhibit chaotic attractors; they have found interesting applications in secure communications (see [1]). The study of the

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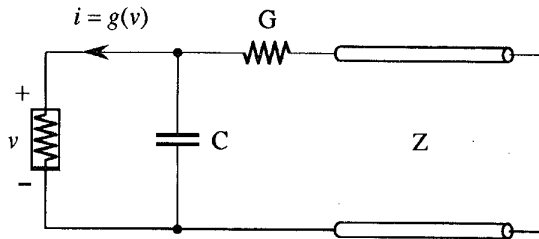


Fig. 1. The time-delayed Chua's circuit.

dynamic behavior of such circuits is very difficult because, owing to the presence of parasitics, they are governed by difference-differential equations (see [2]).

The TDCC in Fig. 1 and originally introduced in [3], can be considered an efficient tool for investigating the complex dynamic phenomena occurring in delayed systems. This circuit, in absence of the capacitor, is described by a difference equation which exhibits a chaotic behavior called period-adding phenomenon (see [3]). If the capacitor is not neglected the circuit is governed by a difference-differential equation of neutral type (see [2]): some partial results on its dynamics, mainly obtained through simulation, are reported in [4].

Owing to the infinite dimension of the TDCC state space, standard techniques suitable for finite-dimensional systems cannot be applied; moreover the simulation in all the parameter space is rather time consuming.

In [5] it was already shown that the spectral technique developed in [6] and [7] for nondelayed systems is particularly suitable for delayed cellular neural networks.

In this brief we apply such a technique to the study of the dynamics of the TDCC, which is described by a differential-difference equation of neutral type, i.e., by an equation more complex than those occurring in delayed cellular neural networks.

In particular we consider the double-scroll like attractor discovered in [4], and we show that its existence can be easily predicted, through the spectral technique, by simply solving a suitable set of transcendental equations and without performing any simulations.

We remark that the above spectral technique is an effective method for the investigation of the complex dynamics of delayed systems in the whole of the parameter space; moreover it allows to study the effects of the parasitics in circuits described by difference equations.

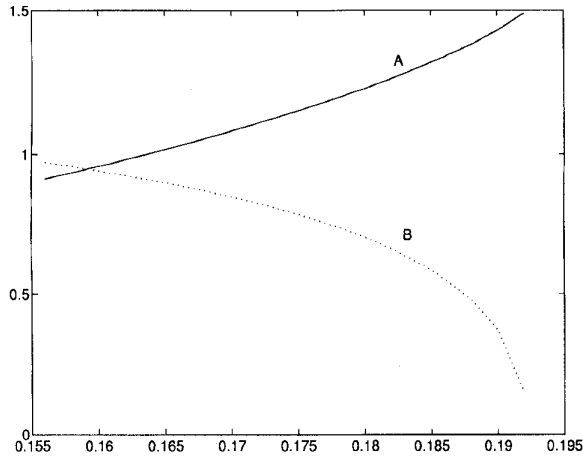
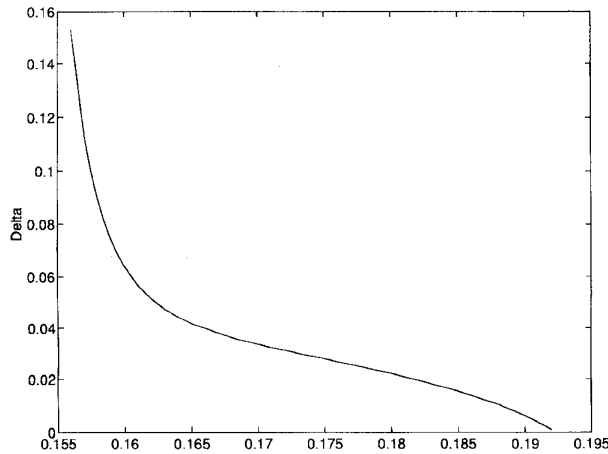
II. APPLICATION OF THE SPECTRAL TECHNIQUE TO THE TDCC

The TDCC, shown in Fig. 1, is derived from the classical Chua's circuit by substitution of the lumped LC resonator with an ideal transmission line (TL) [3]. The dynamics of the TDCC can be described by the following equation

$$\theta(1 + \zeta) \frac{d\tilde{v}(\tau + 1)}{d\tau} + \theta(1 - \zeta) \frac{d\tilde{v}(\tau)}{d\tau} + \tilde{v}(\tau + 1) + \tilde{v}(\tau) + (1 + \zeta)n(\tilde{v}(\tau + 1)) + (1 - \zeta)n(\tilde{v}(\tau)) = 0 \quad (1)$$

where τ is the time normalized with respect to the TL delay T ; the tilde indicates functions of the normalized time, $n(\cdot) = g(\cdot)/G$ represents the nonlinear characteristic of the Chua's diode, normalized with respect to G ; $\zeta = ZG$, $\theta = C/GT$, and Z is the TL characteristic impedance; G and C are the conductance and the capacitor respectively, shown in Fig. 1.

According to [7], in our analysis we assume that the normalized characteristic of the Chua's diode can be approximated by a cubic


 Fig. 2. Plots of A and B as a function of θ .

 Fig. 3. Plot of the distortion index as a function of θ .

function, i.e.,

$$n(y) = -\frac{m}{G}y + \frac{k}{G}y^3 \quad (2)$$

where m and k are suitable constants, that will be fixed below.

In particular we concentrate on the set of parameters which gives rise to the double-scroll like attractor discovered in [4], through simulation. Such a set of parameters, in terms of the variables introduced in (1) and (2) yields

$$\zeta = \frac{0.7}{\sqrt{T}}, \quad m = 4/5, \quad k = 2/45, \quad G = 0.7.$$

Note that, according to [7], the values of the parameters m and k have been chosen in such a way that the circuit described by the cubic characteristic (2) presents exactly the same equilibrium points of the TDCC considered in [4], which is described by a piecewise linear characteristic. Since in [4] a chaotic attractor has been found by varying the TL delay T in the range $[0.8, 1]$ here we expect to observe chaos, by varying the normalized parameter θ in the corresponding interval $\theta \in [0.155, 0.195]$.

We will investigate the occurrence of chaos by applying the harmonic balance (HB) technique developed in [7]: we remark that this is the only analytical tool available for predicting chaos in such a system.

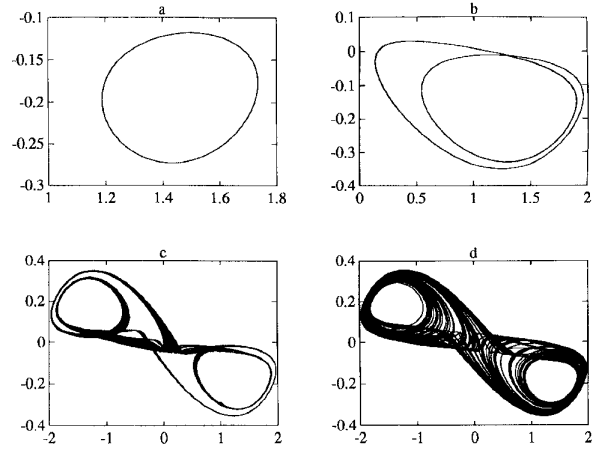


Fig. 4. Simulations of the TDCC: (a) $\theta = 0.19$: period-1 limit cycle. (b) $\theta = 0.165$: period-2 limit cycle. (c) $\theta = 0.160$ and (d) $\theta = 0.158$: double-scroll like attractor. The horizontal and the vertical axis refer to the normalized voltage \tilde{v} and to the voltage wave outcoming from the TL, respectively.

The algorithm proposed in [7] is based on the following three fundamental steps:

- 1) computation of the equilibrium points;
- 2) study of the existence of periodic solutions, called periodic limit cycles (PLC) through the describing function technique;
- 3) evaluation of the distortion index.

Then chaos is predicted if both the following conditions are satisfied:

- C1: there is an interaction between a stable limit cycle and an unstable equilibrium point;
- C2: the distortion index lies in a suitable range, corresponding to a medium filtering condition, i.e., to the occurrence of a noisy limit cycle (see [7]).

The coordinates of the equilibrium points turn out to be $\tilde{v} = 0$ and $\tilde{v} = \pm 1.5$, whereas their stability depends on θ .

Moreover it is easily seen that the nonlinear delayed system described by (1) admits of the Lur'e representation shown in Fig. 1 of [7]. In fact by taking the Laplace's transform of both the sides of (1) the transfer function of the linear part of the Lur'e system turns out to be

$$L(s) = \frac{(1 + \zeta)e^s + 1 - \zeta}{[(1 + \zeta)\theta s + 1]e^s + (1 - \zeta)\theta s + 1} \quad (3)$$

where it is worth noticing that the time delay is simply taken into account by the exponential term.

The describing function method (second step of the algorithm) is based upon considering the possible periodic solution (PLC)

$$\tilde{v}(\tau) \approx y_0(\tau) = A + B \sin(\omega_0 \tau) \quad (4)$$

and assuming that the nonlinearity output can be approximated in the form

$$n(y_0(\tau)) \approx N_0(A, B)A + N_1(A, B)B \sin(\omega_0 \tau). \quad (5)$$

The two real coefficients gains N_0 and N_1 are given by formulas (2) and (3), respectively, of [7].

Substituting (4) and (5) in (1) and equating the terms having the same frequency components, according to the HB principle, the following relations are obtained

$$1 + L(0)N_0(A, B) = 0 \quad (6)$$

$$1 + L(j\omega_0)N_1(A, B) = 0 \quad (7)$$

which represent the existence conditions for the PLC. Since $L(j\omega_0)$ is a complex number, (6) and (7) constitute three independent equations that can be solved with respect to A , B and ω_0 . Plots of A and B versus θ are shown in Fig. 2.

The third step for chaos prediction is the evaluation of the distortion index Δ , defined in (7) of [7] and reported as a function of θ in Fig. 3.

According to [7], chaos is expected in the parameter range such that the two conditions C1 and C2 reported above, are fulfilled. Condition C1 reduces to the inequality $B > A$; from Fig. 2 it is seen that C1 is satisfied if $\theta \leq 0.160$. Condition C2 can be expressed as $\delta^- < \Delta < \delta^+$, where δ^- is the upper bound corresponding to a true prediction, whereas δ^+ is the lower bound corresponding to an unreliable prediction (see p. 155 of [7]). By assuming $\delta^- = 0.03$ and $\delta^+ = 0.1$, the corresponding θ interval becomes approximately $0.157 \leq \theta \leq 0.175$. Then by intersecting the two conditions, we expect chaos in the range $0.157 \leq \theta \leq 0.160$. The time simulation of the circuit, shown in Fig. 4, confirms the prediction; in fact it is seen from Fig. 4 that at $\theta = 0.19$ the system exhibits a period-1 limit cycle; then a period doubling bifurcation occurs for $\theta = 0.165$ and eventually a double-scroll like attractor is observed for $\theta = 0.158$ and $\theta = 0.160$.

III. CONCLUSION

We have shown that the complex dynamics of the time-delayed Chua's circuit can be investigated through the spectral approach developed in [7], based upon the describing function technique and on the evaluation of the distortion index. In particular we have concentrated on a set of parameters which gives rise to a double-scroll like attractor and we have shown that its existence and characteristics can be easily predicted, via the spectral technique, without resorting to simulation. We remark that by following the same approach the dynamic behavior of the TDCC can be investigated in the whole of the parameter space. Moreover the above spectral technique allows to study the effects of the parasitics in those circuits described by difference equations, which have found applications in secure communications (see [1]).

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Experimental Poincaré Maps from the Twist-and-Flip Circuit

Guo-Qun Zhong, Leon O. Chua, and Ray Brown

Abstract—In this letter, we present a physical implementation of the twist-and-flip circuit containing a nonlinear gyrator. Many phase portraits and their associated Poincaré maps are observed experimentally from this circuit and presented in this paper.

I. INTRODUCTION

Fractals are one of many manifestations of complicated chaotic dynamics. The fractal phenomenon can occur not only in autonomous systems typical of Chua's circuit [1]–[3], but also in nonautonomous systems driven by time-varying signals and therefore described by a nonautonomous system of ordinary differential equations

$$\dot{\mathbf{x}} = f(\mathbf{x}, t) \quad (1)$$

where \mathbf{x} is a vector in an n -dimensional Euclidean space \mathbf{R}^n .

The twist-and-flip circuit offers one of the simplest paradigms for nonautonomous chaos. Indeed, the state equations associated with the twist-and-flip circuit are the only known nonautonomous system of ordinary differential equations whose Poincaré map can be derived in an *explicit analytic* form. Based on this property of the circuit an in-depth mathematical analysis of the twist-and-flip map has been carried out exhaustively and rigorously [4], [5]. The various fractals corresponding to several classes of nonlinear gyration conductance functions $g(v_1, v_2)$ from this map have been generated numerically [6].

In this letter we describe a physical implementation of the *twist-and-flip circuit* with a simple nonlinear gyration conductance function $g(v_1, v_2)$, driven by a square-wave voltage source. A variety of phase portraits and the corresponding Poincaré maps observed experimentally from this setup will be presented.

II. PHYSICAL IMPLEMENTATION OF THE TWIST-AND-FLIP CIRCUIT

The twist-and-flip circuit contains simply two linear capacitors C_1 and C_2 , a voltage source $s(t)$, and a *nonlinear* gyrator, as shown in Fig. 1(a). For voltage source $s(t)$ for driving the circuit is a square wave of amplitude a and angular frequency ω (or period $P = \frac{2\pi}{\omega}$), as shown in Fig. 1(b). The gyrator, which is the only nonlinear element in the circuit, is described by the equations

$$\left. \begin{aligned} i_1 &= g(v_1, v_2)v_2 \\ i_2 &= -g(v_1, v_2)v_1 \end{aligned} \right\} \quad (2)$$

where $g(v_1, v_2)$ is the associated gyration conductance [7]. In this letter we assume that

$$g(v_1, v_2) > 0, \quad \text{for } -\infty < v_1, v_2 < \infty.$$

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