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On Robust Chaos Suppression in a Class of Nondriven Oscillators: Application to the Chua's Circuit

Ricardo Femat, José Alvarez-Ramírez, Bernardino Castillo-Toledo, and Jesús González

***Abstract*—This paper deals with a feedback control strategy for chaos suppression. The proposed strategy is an input–output control scheme which comprises an uncertainty estimator and an asymptotic linearizing feedback. The developed control scheme allows chaos suppression in spite of modeling errors and parametric variations.**

***Index Terms*— Chaos, Chua's circuit, synchronization, uncertain systems.**

I. Introduction

Essential elements in chaos control are the following: 1) suppression of erratic dynamics in a given system and 2) stabilization of a chaotic system about a given reference trajectory. Many results have been reported in the literature. For instance, Lyapunov-based methods [1], adaptive strategies [2], [3], discrete-time control [4], chaos suppression via reconstruction of invariant manifolds [5], and robust asymptotic linearization [6].

In this work, we propose an input–output controller based on geometrical control theory [7]. The main idea is to lump the uncertainties in a nonlinear function which can be interpreted as a new state in an externally dynamically equivalent system. Thus, the new state is estimated by means of a state observer. The state observer provides the estimated value of the lumping nonlinear function (and consequently of the uncertainties) to the linearizing feedback control.

II. Background

Consider the following class of nonlinear systems whose trajectories are contained in a chaotic attractor $\dot{x} = f(x) + g(x)u$ [where $x(t) \in \mathbb{R}^n$ is a states vector, $u \in \mathbb{R}$ is a scalar input, and $f(x)$ and $g(x)$ are smooth vector fields]. Assume that $y = h(x) \in \mathbb{R}$ is the system output [$h(x)$ is a smooth function]. If ρ is a smallest integer such that the following conditions hold at $x = x_0$, the above system is said to have a relative degree ρ at x_0. The conditions are: 1) $\mathcal{L}_g\mathcal{L}_f^i h(x) = 0$, $i = 1, 2, \cdots, \rho - 2$ and 2) $\mathcal{L}_g\mathcal{L}_f^{\rho-1} h(x) \neq 0$ [where $\mathcal{L}_f^i h(x) = \mathcal{L}(\mathcal{L}_f^i h(x))$, $\mathcal{L}_f^i h(x)$ is the Lie derivative of $h(x)$ with respect $f(x)$] [7]. If the smoothness assumption and the above conditions are satisfied, we can define ρ new coordinates $z_{i+1} = \mathcal{L}_f^i h(x)$, $i = 0, 1, \cdots, \rho - 1$. The above system can be written in the following canonical form [7]:

$$
\begin{aligned}
\dot{z}_1 &= z_{i+1}, \qquad i = 1, 2, \cdots, \rho - 1\\
\dot{z}_\rho &= \alpha(z, \nu) + \gamma(z, \nu)u\\
\dot{\nu} &= \zeta(z, \nu)\\
y &= z_1
\end{aligned}
\tag{1}
$$

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R. Femat is with CIEP-FCQ, Universidad Autónoma de San Luis Potosí, 78231, San Luis Potosí, SLP, México (e-mail: rfemat@apollo.tc.uaslp.mx).

J. Alvarez-Ramırez and J. González are with the Universidad Autónoma Metropolitana, 09000, México D.F.

B. Castillo-Toledo is with CINVESTAV Guadalajara, Guadalajara, Jal., México.

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where $\alpha(z, \nu) = \mathcal{L}_f^{\rho} h(x)$ and $\gamma(z, \nu) = \mathcal{L}_g \mathcal{L}_f^{\rho-1} h(x)$. Thus, the following feedback $u = -[\mathcal{L}_f^{\rho} h(x) + V(x)]/\mathcal{L}_g \mathcal{L}_f^{\rho-1} h(x)$ is a linearizing control law.

Without loss of generality, we can suppose that the reference signal is $y_r = 0$. Then, in coordinates (z, ν) the linearizing feedback control becomes $u = [\alpha(z.\nu) + K^T z]/\gamma(z, \nu)$ where K's are such that the polynomial $s^{\rho} + K_{\rho} s^{\rho-1} + \cdots + K_2 s + K_1$ is Hurwitz. Nevertheless, if the vector fields $f(x)$ and $g(x)$ are uncertain, the coordinates transformation $z = T(x)$, bringing the original system into the canonical form (1), is uncertain. In principle, since the coordinates transformation is a diffeomorphism, one can suppose that: 1) the uncertain transformation exists and 2) it is invertible. However, since $T(x)$ is uncertain, the nonlinear functions $\alpha(z, \nu)$ and $\gamma(z, \nu)$ are also uncertain, hence, they cannot be directly used in the linearizing feedback. Moreover, the linearizing control law has been designed, assuming that the states $\nu \in \mathbb{R}^{n-\rho}$ are available for feedback. This is not a reasonable assumption. We use the linearizing feedback only as an intermediate control law toward the final controller.

III. Feedback Stabilization Under Uncertain Vector Fields

Let us assume the following.

- A1 Only the system output $y = z_1$ is available for feedback.
- A2 $\gamma(z, \nu)$ is bounded away from zero.
- A3 System (1) is minimum phase.
- A4 (A4) The nonlinear functions $\alpha(z, \nu)$ and $\gamma(z, \nu)$ are uncertain. However, an estimate $\hat{\gamma}(z)$ of $\gamma(z, \nu)$, satisfying $\text{sign}(\hat{\gamma}(z)) = \text{sign}(\gamma(z, \nu))$, is available for feedback.

Now, let us define $\delta(z, \nu) = \gamma(z, \nu) - \hat{\gamma}(z)$, $\Theta(z, \nu, u) = \alpha(z, \nu) + \delta(z, \nu)u$ and $\eta = \Theta(z, \nu, u)$. In addition, let us consider the following dynamical system:

$$\begin{aligned} \dot{z}_i &= z_{i+1}, \qquad 1 \le i \le \rho - 1 \\ \dot{z}_{\rho} &= \eta + \hat{\gamma}(z)u \\ \dot{\eta} &= \Xi(z, \eta\nu, u) \\ \dot{\nu} &= \zeta(z, \nu) \\ y &= z_1 \end{aligned} \tag{2}$$

where $\Xi(z, \eta, \nu, u) = \Sigma_{k=1}^{\rho-1} z_{k+1} \partial_k \Theta(z, \nu, u) + [\eta + \hat{\gamma}(z)u] \partial_{\rho} \Theta(z, \nu, u) + \delta(z, \nu)\dot{u} + \partial_v \Theta(z, \nu, u) \zeta(z, \nu)$, $\partial_k \Theta(z, \nu, u) = \partial \Theta(z, \nu, u)/\partial x_k$.

Proposition 1: The manifold $\Psi(z, \eta, \nu, u) = \eta - \Theta(z, \nu, u) = 0$ is invariant under the trajectories of system (2).

Proof: It suffices to prove that $d\Psi(z, \eta, v, u)/dt = 0$ along the trajectories of system (2) which, using the definition $\eta = \Theta(z, \nu, u)$, is straightforward. □

Proposition 2: System (2) is dynamically externally equivalent to system (1). This is, for all differentiable input $u \in \mathbb{R}$. System (2) has the same solution as the system (1) module $\pi\ (z, \eta, \nu) \to (z, \nu)$.

Proof: From the equality $\Psi(z, \eta, \nu, u) = 0$ and the condition $d\Psi(z, \eta, \nu, u)/dt = 0$, one can take the first integral [8] of system (3) to get $\eta = \Theta(z, \nu, u)$. When the first integral is back substituted in system (2), we obtain the solution of system (1). This implies that the solution $z(t) \in \mathbb{R}^{\rho}$ of system (1) is the solution of the upper subsystem (2), hence, $\pi \bullet (z, \eta, \nu) = (z, \nu)$. □

Remark 1: The augmented state, η, provides the dynamics of the uncertain function $\Theta(z, \nu, u)$ and, consequently, of the uncertain terms $\alpha(z, \nu)$ and $\gamma(z, \nu)$. From the minimum-phase assumption, the following result is not difficult to prove.

Proposition 3: Under the feedback $u = (-\eta + K^T z)/\hat{\gamma}(z)$, where K's are the coefficients of a Hurwitz polynomial, the states of system (2) converge asymptotically to zero.

An important advantage of system (2) is the following. The dynamics of the states (z, η) can be reconstructed from the output [8], [9]. We propose the following observer:

$$\begin{aligned} \dot{\hat{z}}_i &= + L^i \kappa_i (z_1 - \hat{z}_1), \qquad 1 \le i \le \rho\, 1 \\ \dot{\hat{z}}_{\rho} &= \hat{\eta} + \hat{\gamma}(\hat{z})u + L^{\rho} \kappa_{\rho}(z_1 - \hat{z}_1) \\ \dot{\hat{\eta}} &= + L^{\rho+1} \kappa_{\rho+1}(z_1 - \hat{z}_1) \end{aligned} \tag{3}$$

where $(\hat{z}, \hat{\eta})$ are estimated values of (z, η), respectively. Note that the uncertain term $\Xi(z, \eta, \nu, u)$ has been neglected in the construction of the observer (3).

Theorem 1: Let $e \in \mathbb{R}^{\rho+1}$ be an estimation error vector whose components are defined as follows: $e_i = L^{\rho-i}(z_i - \hat{z}_i)$, $i = 0, 1, \cdots, \rho - 1$ and $e_{\rho+1} = \eta - \hat{\eta}$. For a sufficiently large value of the high-gain parameter L, the dynamics of the estimation error e converge asymptotically to zero.

Proof: Combining systems (3) and (2), the dynamics of the estimation error can be written as follows: $\dot{e} = LA(\kappa)e + \Gamma(z, \eta, \nu, u)$ where $\Gamma(z, \eta, \nu, u) = [0, \Xi(z, \eta, \nu, u)]^T$ and the companion matrix is given by

$$A = \begin{bmatrix} -\kappa_1 & 1 & 0 & \cdots & 0 \\ -\kappa_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ -\kappa_{\rho} & 0 & 0 & \cdots & 1 \\ -r\kappa_{\rho+1} & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{4}$$

where $r = \gamma(z, \nu)/\hat{\gamma}(\hat{z})$. The matrix (4) is Hurwitz if $r > 0$ for all $t \ge 0$. According to Assumption A4), this condition is satisfied. In addition, since the trajectories $x(t)$ are contained in a chaotic attractor, hence, $\Gamma(z, \eta, \nu, u)$ is bounded. Consequently, for any $L > L^* > 0$, $e(t) \to 0$ as $t \to \infty$, which implies that $(\hat{z}, \hat{\eta}) \to (z, \eta)$. □

Corollary 1: Now, consider the following linearizing-like control law: $u = [-\hat{\eta} + K^T \hat{z}]/\hat{\gamma}(\hat{z})$. Under the above feedback system (3) is asymptotically stable for $L > L^* > 0$.

Remark 2: High-gain observers can induce undesirable dynamics effects such as the peaking phenomenon [11]. To diminish these effects, the control law can be modified by means of

$$u = \text{Sat}\{[-\hat{\eta} + K^T \hat{z}]/\hat{\gamma}(\hat{z})\}$$

where Sat: $\mathbb{R} \to \mathcal{B}$ is a saturation function and $\mathcal{B} \subset \mathbb{R}$ is a bounded set [10].

IV. Robust Chaos Supression in the Chua's Oscillator

The Chua's oscillator is widely used to study the suppression and synchronization of chaos [12]. The circuit equations can be written in dimensionless form as follows [3]:

$$\begin{aligned} \dot{x}_1 &= \gamma_1[x_2 - x_1 - f(x_1)] + u \\ \dot{x}_2 &= x_1 - x_2 + x_3 \\ \dot{x}_3 &= -\gamma_2 x_2 \\ y &= x_1 \end{aligned} \tag{5}$$

where $f(x) = bx_1 + 1/2(a - b)[|x_1 + 1| - |x_1 - 1|]$. Defining the invertible change of coordinates $z_1 = x_1$, $\nu_1 = x_2$, and $\nu_2 = x_3$, the dynamical system (5) can be transformed into the canonical form (1) and its equivalent form (3) [with $\eta = \Theta(z_1, \nu)$ as the augmented state]. Note that z_1 is the voltage cross capacitor C_1, which is bounded. Thus, the zero dynamics can be written as $\dot{\nu} = C\nu + Dz_1$ where $D = [1, 0]^T$ and

$$C = \begin{bmatrix} -1 & 1 \\ -\gamma_2 & 0 \end{bmatrix} \tag{6}$$

which is Hurwitz if $\gamma_2 > 0$. Hence, system (5) is minimum phase.

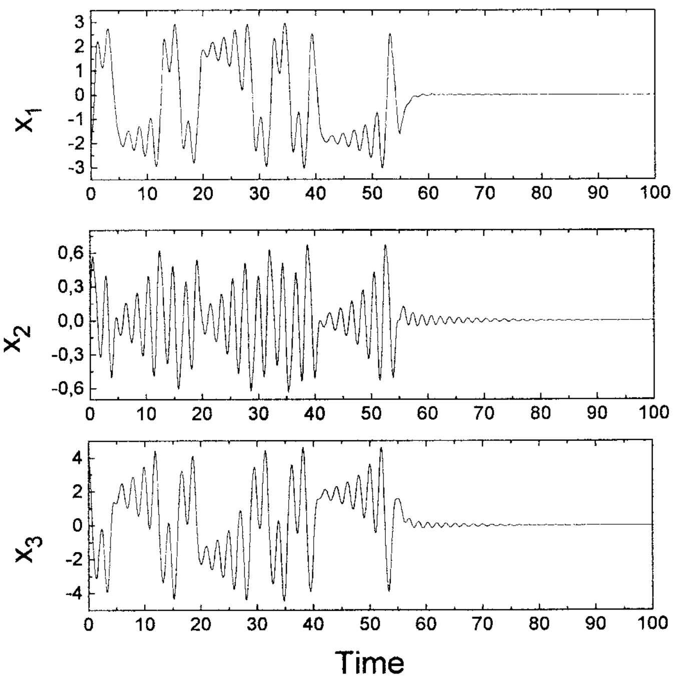

Fig. 1. Chaos suppression for the Chua's oscillator: $L = 30$.

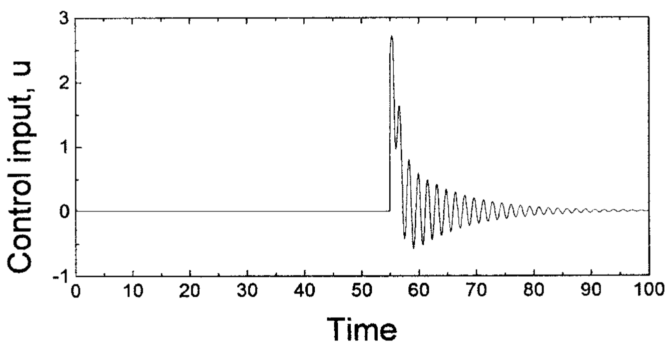

Fig. 2. Performance of the control input without saturation.

Then the assumptions A1)–A4) are satisfied. Thus, the asymptotic controller becomes

$$\dot{\hat{z}}_1 = \hat{\eta} + u + L\kappa_1(z_1 - \hat{z}_1) \tag{7.1}$$

$$\hat{\eta} = + L^2\kappa_2(z_1 - \hat{z}_1)$$

$$u = -\hat{\eta} + K_1\hat{z}_1. \tag{7.2}$$

Fig. 1 shows the stabilization of the Chua oscillator at the origin. The initial conditions for system (5) were $(x_1(0), x_2(0), x_3(0)) = (-2.0, 0.02, 4.0)$ and for the observer (8.1) $(\hat{z}_1(0), \hat{\eta}(0)) = (1, 15)$. The model parameters values were chosen as in [11]. The control gain $K_1 = 1.0$, the estimation constants $(\kappa_1, \kappa_2) = (2.0, 1.0)$, and the high-gain estimation parameters value $L = 30$ were chosen. The controller (8) was activated at $t = 55.0$ s. The performance of the control input is presented in Fig. 2. The effect of the input overshoot in the output can be diminished by means of a saturation function of the feedback controller (see Fig. 3).

V. Conclusions

A control scheme for chaos suppression has been presented. The main idea is to lump the uncertainties in a nonlinear function which

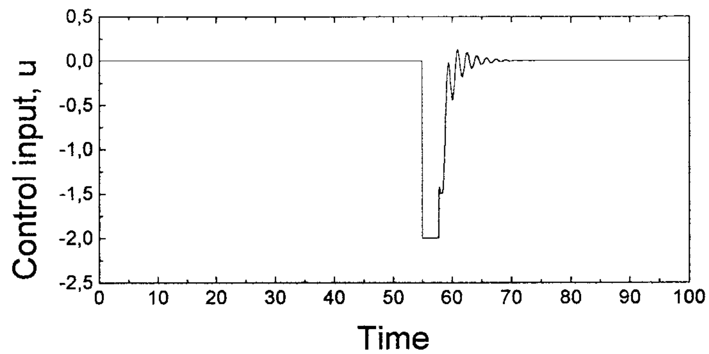

Fig. 3. Performance of the saturated version of the control input, $L = 80$.

can be interpreted as an augmented state in a dynamically equivalent nonlinear system. A state estimator provides an estimated value of the augmented state and, consequently, of the uncertainties. Thus, the controller comprises two parts: 1) a state observer and 2) a linearizing-like control law.

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