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# A universal circuit for studying chaotic phenomena

By Leon O. Chua, Ladislav Pivka and Chai Wah Wu

*Electronics Research Laboratory and Department of Electrical Engineering and Computer Sciences, University of California, Berkeley, CA 94720, USA*

In this paper, an overview of the results on an autonomous chaotic electronic circuit, called Chua's oscillator, is given. Along with brief descriptions of numerical and analytical investigations on Chua's oscillator, we present some of its potential applications. The significance of the oscillator for the study of general dynamical systems is discussed.

## 1. Introduction

Many natural phenomena can be described in terms of models exhibiting regular oscillatory, or periodic, behaviour. In physics, a so-called state space of variables can be used to interpret trajectories describing the motion. It is natural to describe such oscillatory motions with as simple a model as possible while preserving all the essential properties of the physical systems being described. In electrical engineering Van der Pol's oscillator proved to be a classic, simple paradigm for the description of oscillatory phenomena involving only two state variables.

During the last 30 years, a new important type of behaviour—chaotic motion—has been identified. It turns out that at least three state variables are necessary to model such behaviour in autonomous systems, i.e. systems in which no external force is affecting their behaviour.

In 1963, the meteorologist E. N. Lorenz proposed the first three-dimensional autonomous system of equations (Lorenz 1963) exhibiting chaotic behaviour, which has since become a subject of intense research. It was not until 20 years later that a real physical object was discovered (Chua 1992), and later built (Zhong & Ayrom 1985), which is capable of reproducing chaotic phenomena known from the theory. *Chua's circuit* and its later generalization—*Chua's oscillator* (Madan 1993)—has since become the most widely studied paradigm for different types of dynamical, especially chaotic, behaviours. Figure 1 shows a diagram of the oscillator, whose state equations are given by

$$\left.\begin{aligned} \frac{\mathrm{d}v_1}{\mathrm{d}t} &= \frac{1}{C_1}[G(v_2 - v_1) - f(v_1)], \\ \frac{\mathrm{d}v_2}{\mathrm{d}t} &= \frac{1}{C_2}[G(v_1 - v_2) + i_3], \\ \frac{\mathrm{d}i_3}{\mathrm{d}t} &= -\frac{1}{L}(v_2 + R_0 i_3), \end{aligned}\right\} \tag{1.1}$$


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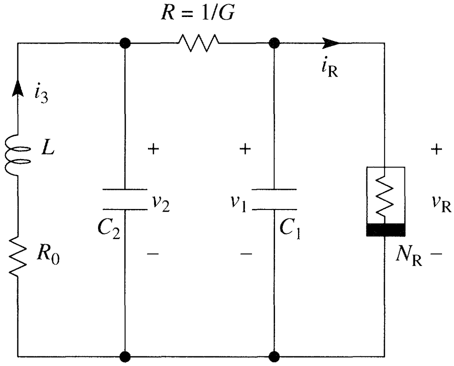


Figure 1. Circuit diagram of Chua's oscillator.

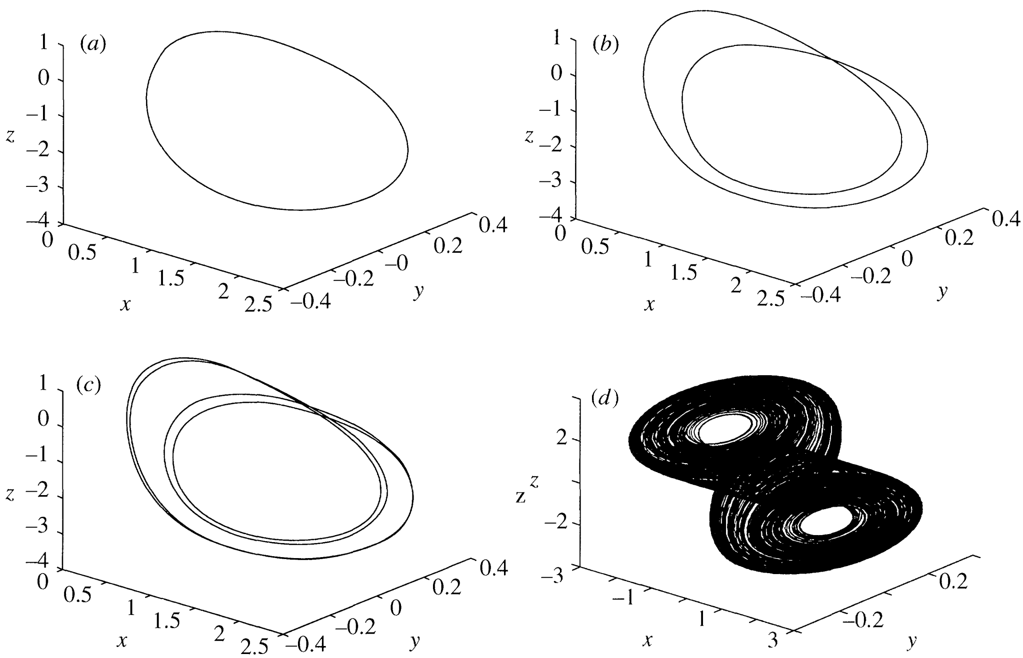


Figure 2. Period-doubling route to chaos. (*a*) Period-1 orbit. (*b*) Period-2 orbit. (*c*) Period-4 orbit. (*d*) Double-scroll attractor.

where $G = 1/R$, and $f(v_1) = G_b v_1 + \frac{1}{2}(G_a - G_b)\{|v_1 + E| - |v_1 - E|\}$ is the $v$–$i$ characteristic of the nonlinear resistor $N_R$ with a slope equal to $G_a$ in the inner region and $G_b$ in the outer region. By setting $R_0 = 0$ we obtain the original Chua's circuit.

By a change of variables, the state equations of Chua's oscillator (1.1) can be transformed into the following dimensionless form:

$$\left.\begin{aligned} \frac{dx}{d\tau} = k\alpha\,(y - x - f(x)), \quad \frac{dy}{d\tau} = k(x - y + z), \quad \frac{dz}{d\tau} = k(-\beta y - \gamma z), \\ f(x) = bx + \tfrac{1}{2}(a - b)\{|x + 1| - |x - 1|\}, \end{aligned}\right\} \tag{1.2}$$

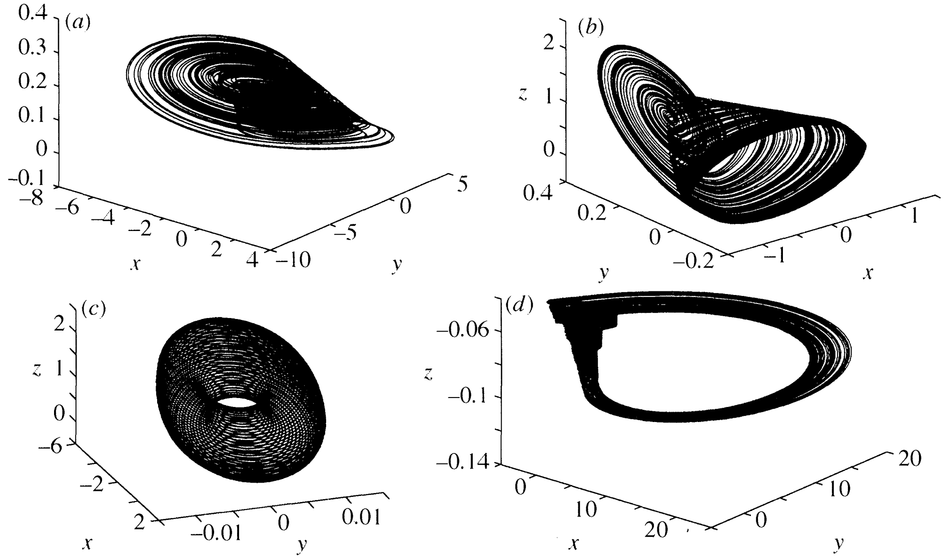


Figure 3. Chaotic attractors from Chua's oscillator (see table 1). (*a*) Attractor no. 4. (*b*) Attractor no. 18. (*c*) Torus attractor no. 22. (*d*) Attractor no. 28.

where

$$\left.\begin{array}{c} x = v_1/E, \quad y = v_2/E, \quad z = i_3(R/E), \\ \alpha = C_2/C_1, \quad \beta = R^2C_2/L, \quad \gamma = RR_0C_2/L, \\ a = RG_\mathrm{a}, \quad b = RG_\mathrm{b}, \quad \tau = t/|RC_2| \quad \text{and} \quad \begin{array}{ll} k = 1 & \text{if } RC_2 > 0, \\ k = -1 & \text{if } RC_2 < 0. \end{array} \end{array}\right\} \tag{1.3}$$

Note that there are more than one set of circuit parameters ($C_1$, $C_2$, etc.) that maps onto the same dimensionless equations (1.2). Furthermore by selecting the constant $RC_2$ we determine how 'fast' the real circuit is in comparison with the dimensionless system.

The history of the inception and evolution of Chua's circuit can be found in the review papers, *The genesis of Chua's circuit* (Chua 1992) and *Chua's circuit: ten years later* (Chua 1994), where an overview of the most important recent results is given along with extensive bibliographies.

In the present review we will follow the traditional path of development, from numerical experiments, through analytical results, to practical applications. We omit the experimental aspects since they are covered in detail in Kennedy (this volume) where also some definitions of basic concepts in the theory of dynamical systems, used throughout this paper, can be found.

## 2. Dynamical phenomena via numerical simulations

The chaotic nature of Chua's circuit was first observed experimentally by Zhong & Ayrom (1985). Because of its characteristic structure, the observed strange attractor was named the double-scroll Chua's attractor (see figure 2*d*). Several other chaotic attractors have since been observed from Chua's oscillator. Figure 3 shows a selection of four other chaotic attractors from the rich gallery summarized by table 1 in which scaled parameters, corresponding to equation (1.2), are used.

Table 1. *Parameter values of attractors in Chua's oscillator*

| attractor no. | $\alpha$ | $\beta$ | $\gamma$ | $a$ | $b$ | $k$ | Lyapunov exponents | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 9.3515908493 | 14.7903198054 | 0.0160739649 | −1.1384111956 | −0.7224511209 | 1.0 | 0.279 | 0 | −2.359 |
| 2 | −1.5590535687 | 0.0156453845 | 0.1574556102 | −0.2438532907 | −0.0425189943 | −1.0 | 0.03 | 0 | −0.282 |
| 3 | −1.458906 | −0.09307192 | −0.3214346 | 1.218416 | −0.5128436 | −1.0 | 0.013 | 0 | −0.677 |
| 4 | −1.2331692348 | 0.0072338195 | 0.0857850567 | −0.1767031151 | −0.0162669575 | −1.0 | 0.022 | 0 | −0.09 |
| 5 | 3.7091002664 | 24.0799705758 | −0.8592556780 | −2.7647222013 | 0.1805569489 | 1.0 | 0.368 | 0 | −1.297 |
| 6 | −6.69191 | −1.52061 | 0.0 | −1.142857 | −0.7142857 | 1.0 | 0.081 | 0 | −0.382 |
| 7 | 143.1037 | 207.34198 | −3.8767721 | −0.855372 | −1.09956 | −1.0 | 1.22 | 0 | −16.28 |
| 8 | −4.898979 | −3.624135 | −0.001180888 | −2.501256 | −0.9297201 | 1.0 | 0.144 | 0 | −1.071 |
| 9 | −1.3184010525 | 0.0125741900 | 0.1328593070 | −0.2241328978 | −0.0281101959 | −1.0 | 0.035 | 0 | −0.111 |
| 10 | −1.301814 | −0.0136073 | −0.02969968 | 0.1690817 | −0.4767822 | 1.0 | 0.012 | 0 | −0.042 |
| 11 | −1.3635256878 | −0.0874054928 | −0.3114345114 | 1.292150 | −0.49717 | −1.0 | 0.026 | 0 | −0.571 |
| 12 | 8.4562218418 | 12.0732335925 | 0.0051631393 | −0.7056296732 | −1.1467573476 | 1.0 | 0.252 | 0 | −1.451 |
| 13 | 6.5792294673 | 10.8976626192 | −0.0447440294 | −1.1819730746 | −0.6523354182 | 1.0 | 0.065 | 0 | −1.93 |
| 14 | 4.006 | 54.459671 | −0.93435708 | −0.855372 | −1.09956 | −1.0 | 0 | 0 | −0.066 |
| 15 | −4.08685 | −2.0 | 0.0 | −1.142857 | −0.7142857 | 1.0 | 0.071 | 0 | −0.755 |
| 16 | −75.0 | 31.25 | −3.125 | −0.98 | −2.4 | 1.0 | 1.01 | 0 | −69 |
| 17 | 15.6 | 28.58 | 0.0 | −1.142857 | −0.7142857 | 1.0 | 0.116 | 0 | −4.32 |
| 18 | −75.0 | 31.25 | −3.125 | −2.4 | −0.98 | 1.0 | 0.923 | 0 | −74 |
| 19 | 37.195804 | 73.049688 | −1.161224 | −0.855372 | −1.09956 | −1.0 | 0.83619 | 0 | −2.67 |
| 20 | 35.939189 | 75.700831 | −1.2033675 | −0.855372 | −1.09956 | −1.0 | 0 | −0.031 | −1.73 |
| 21 | 13.070921 | 53.612186 | −0.75087096 | −0.855372 | −1.09956 | −1.0 | 0.097 | 0 | −0.298 |
| 22 | −45012.877058 | −14125.787458626 | −0.2326833338 | −0.9995532369 | −1.0002837549 | 1.0 | 0 | 0 | −1.31 |
| 23 | 3.505 | 66.672752 | −0.94779892 | −0.855372 | −1.09956 | −1.0 | 0 | 0 | −0.066 |
| 24 | 12.141414 | 95.721132 | −0.8982235 | −0.855372 | −1.09956 | −1.0 | 0 | 0 | −0.45 |
| 25 | 1800.0 | 10000.00 | 0.0 | −1.026 | −0.982 | 1.0 | 0 | 0 | −0.145 |
| 26 | −1.7327033212 | 0.0421159445 | 0.2973436607 | −0.0974632164 | −0.2623276484 | −1.0 | 0.042 | 0 | −0.276 |
| 27 | −2.0073661199 | 0.0013265482 | 0.0164931244 | −0.5112930674 | 0.0012702165 | −1.0 | 0 | −0.00565 | −0.602 |
| 28 | −1.0837792952 | 0.0000969088 | 0.0073276247 | −0.0941189549 | 0.0001899298 | −1.0 | 0.0047 | 0 | −0.066 |

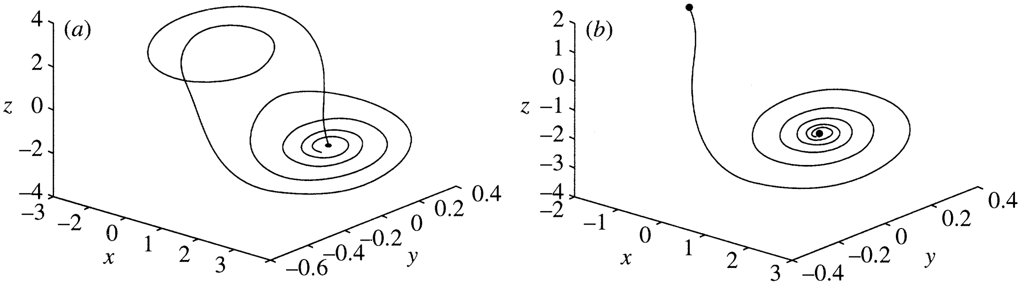


Figure 4. Examples of (*a*) homoclinic, and (*b*) heteroclinic trajectories from Chua's oscillator.

### (*a*) *Homoclinic and heteroclinic orbits in Chua's oscillator*

Closely related to the appearance of chaotic behaviour in dynamical systems in general are the so-called *homoclinic* and *heteroclinic* trajectories. A homoclinic trajectory is one whose limit point in both forward and backward times is the same saddle-type equilibrium point. On the other hand, two different equilibrium points are the limit points in forward and backward time, respectively, for a heteroclinic trajectory (see figure 4).

### (*b*) *Routes to chaos*

(i) When the parameter $R$ is changed, an equilibrium point loses its stability and a stable limit cycle emerges through an Andronov–Hopf bifurcation when one of the parameters (e.g. the resistance $R$) is changed. As the parameter is changed further, the stable limit cycle eventually loses stability, and a stable limit cycle of approximately twice the period emerges, which is usually referred to as a period-2 limit cycle. Similarly a period-4 limit cycle appears after the period-2 limit cycle loses its stability. This bifurcation occurs infinitely many times at ever-decreasing intervals of the parameter range which converges at a geometric rate, determined by the well-known *Feigenbaum constant*, to a limit (bifurcation point) at which point chaos is observed. This is called a *period-doubling route to chaos*, an example of which is shown in figure 2.

(ii) *Torus breakdown route to chaos* is one in which the system undergoes several Andronov–Hopf bifurcations. After two Andronov–Hopf bifurcations, we obtain a toroidal attractor. At the third Andronov–Hopf bifurcation, chaos is likely to appear. Both torus breakdown route to chaos and period-doubling route to chaos can be conveniently interpreted and explained in terms of the characteristic multipliers of the corresponding Poincaré map (Duchesne 1993).

(iii) *Intermittency route to chaos* is the phenomenon where the signal is virtually periodic except for some irregular (unpredictable) bursts. In other words, we have intermittently periodic behaviour and irregular periodic behaviour (Chua *et al.* 1993).

### (*c*) *Period-adding bifurcations*

In this phenomenon, windows of consecutive periods are separated by regions of chaos. In other words, as the parameter is varied, we obtain a stable period-$n$ orbit, $n = 1, 2, \ldots$, followed by a region of chaos, then a stable period-$(n+1)$ orbit, followed by chaos, and then a period-$(n+2)$ orbit and so on. Some numerical and experimental results with bifurcation diagrams are given in Pivka & Špány (1993) and Chua *et al.* (1993).

### *(d) Coexistence of attractors*

Coexistence of attractors is an interesting phenomenon in which the interaction of attractors can give rise to different dynamical phenomena described in the following two subsections. Recently, a coexistence of three distinct chaotic attractors has been reported (Lozi & Ushiki 1991), when two asymmetric attractors coexist with a symmetric one. Some other coexistence phenomena, including point attractors, periodic attractors, and chaotic attractors, can be found in Wu & Pivka (1993).

### *(e) Chaos–chaos intermittency and $1/f$ noise*

It has been known that interaction between chaotic attractors can give rise to intermittency: random switching process between attractors after long periods of 'laminar phases', when the trajectory stays near one of the attractors. A characteristic statistical property of the chaos-chaos type intermittency is the slope of its power spectrum in the low-frequency region. Such a property has also been observed (Anishchenko *et al.* 1994) in Chua's circuit for parameter values near the birth of the Chua double-scroll attractor. The power spectrum was numerically found to follow the law $S_x(w) \approx w^{-\delta}$, $\delta = 1.1 \pm 0.1$, i.e. the graph on the double logarithmic scale clings to the ideal $1/f$ line corresponding to $\delta = 1$. The $1/f$ spectrum has been observed previously in many processes of different origin, e.g. the fluctuations of the current in electron devices, the fluctuations of the Earth's rotation frequency, the fluctuation of the muscle rhythms in the human heart, etc., and has been found to obey the above universal law. The intermittency phenomenon can be used as a $1/f$ noise generator and can lead to a better understanding of the ubiquitous yet still poorly understood $1/f$ phenomenon.

### *(f) Stochastic resonance from Chua's circuit*

The phenomenon of stochastic resonance (SR) is observed in bistable nonlinear systems driven simultaneously by an external noise and a sinusoidal force. In this case, the signal-to-noise ratio (SNR) increases until it reaches a maximum at some optimum noise intensity $D$ which depends on the bistable system and on the frequency of the external sinusoidal force. In the absence of a periodic modulation signal, the noise alone results in a random transition between the two states. This random process can be characterized by the mean switching frequency $w_s$, depending on the noise intensity $D$ and the height of the potential barrier separating the two stable states. In the presence of an external modulation imposed by the sinusoidal signal $A\sin(wt)$, the potential barrier changes periodically with time. The modulation signal amplitude $A$ is assumed to be sufficiently small so that the input signal alone does not induce transitions in the absence of noise. A coherence between the modulation frequency $w$ and the mean switching frequency $w_s$ emerges when the system is simultaneously driven by a periodic signal and a noise source. As a result, a part of the noise energy is transformed into the energy of the periodic modulation signal so that the SNR increases. This phenomenon is qualitatively similar to the classical resonance phenomenon. However, unlike the classical circuit theory where one tunes the input frequency $w$ to achieve resonance in an RLC circuit, here $w$ is fixed at some convenient value and one tunes the noise intensity $D$ to achieve SR.

In Chua's circuit, the SR phenomenon can be observed (Anishchenko *et al.* 1993) in conjunction with the chaos–chaos type intermittency (Anishchenko *et al.* 1994) arising in a small vicinity of the bifurcation curve in the $\alpha$–$\beta$ parameter space when two spiral attractors merge to form the double-scroll attractor. In this case, the SNR of the amplified output signal is observed to be significantly greater than the SNR of the input signal, a novel phenomenon which cannot be achieved with a linear amplifier.

### (*g*) *Signal amplification via chaos*

Apart from the stochastic resonance phenomenon described above, another mechanism for achieving voltage gain (up to 50 dB has been demonstrated experimentally) from Chua's circuit has been discovered recently (Halle *et al.* 1992). The mechanism of this voltage gain is different from that of stochastic resonance because the effect is observed even when Chua's circuit is operating in a spiral Chua's attractor regime far from the bifurcation boundary where stochastic resonance takes place.

### (*h*) *Antimonotonicity phenomenon*

Antimonotonicity, concurrent creation and annihilation of periodic orbits, or inevitable reversals of period-doubling cascades, was shown to be a fundamental phenomenon for a large class of nonlinear systems (Kan & Yorke 1990). Experimental (Kočarev *et al.* 1993*a*) and numerical (Pivka & Špány 1993) evidence was given that this phenomenon is typical for a wide range of parameters in Chua's circuit and Chua's oscillator, respectively.

### (*i*) *Chua's circuit with smooth nonlinearity*

Most of the studies on Chua's circuit and Chua's oscillator assume piecewise-linear nonlinearity. Since the characteristics of nonlinear resistors in real circuits are always smooth, a question arises as to whether the phenomena in piecewise-linear and smooth models coincide. This question is approached in Khibnik *et al.* (1992) by demonstrating that most phenomena from piecewise-linear Chua's circuit (e.g. the double scroll) carry over to the smooth model with a cubic polynomial for the nonlinear function. Also most of the bifurcations (period-doubling, for instance) in the smooth model appear to be similar to those in the piecewise-linear model (see, for example, Zhong (1994), which also gives an implementation of a cubic polynomial $v$–$i$ characteristic using analogue multipliers).

### (*j*) *Universality and self-similarity: two-parameter bifurcation studies*

In the standard bifurcation scenarios described in the literature, usually only one control parameter is changed. In physics, engineering, and other fields, however, one often needs to control two or more parameters to obtain a broader view of the global geometry. Here we mention two cases of self-similar and universal structures, one for the autonomous and the other for the forced Chua's circuit.

#### (i) *Self-similar and universal structures in two-parameter study of transition to chaos*

Using the Poincaré map technique, the exact description of the system (1.1) can be reduced to a two-dimensional map which, in turn, can be approximated by a one-dimensional map (Chua & Tichonicky 1991) generally called Chua's one-dimensional map in the literature. Such an approximation is possible because of the strong dissipation of the system which 'flattens out' the dynamics. This map happens to be bimodal in certain parameter regions, which means that it has both a maximum and a minimum on an interval which is mapped onto itself. The condition is responsible

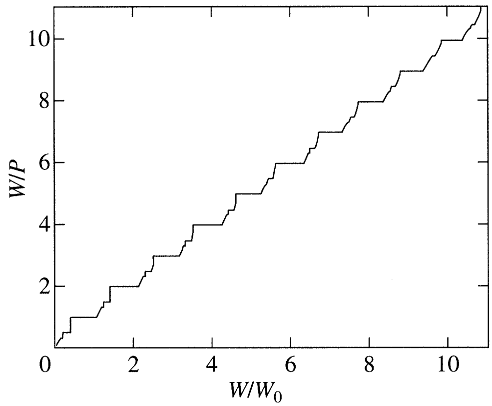


Figure 5. Devil's staircase from the sinusoidally driven Chua's circuit.

for the complicated structure of the boundary of chaos in a two-parameter bifurcation diagram.

In a typical one-parameter bifurcation sequence, if we tune only one parameter in Chua's circuit, we usually see a typical period-doubling cascade, which exhibits remarkable properties of quantitative universality (Feigenbaum 1979) and self-similarity, namely, an interval encompassing regions of different dynamical regimes reproduces itself under a change in scale by the universal factor $\delta = 4.6692\ldots$.

If we turn to a two-parameter study, we can no longer restrict ourselves to the Feigenbaum scenario which is a codimension-1 bifurcation phenomenon. In Kuznetsov *et al.* (1993) the construction of a binary tree of superstable orbits is performed for the one-dimensional Chua's map to show that beside the Feigenbaum critical lines, the boundary of chaos contains an infinite number of codimension-2 critical points, defined by a set of infinite binary codes. The topography of the parameter plane near the corresponding critical points reveals a property of two-parameter self-similarity: a two-dimensional structure of regions of different behaviour is reproduced under a scale change along appropriate axes in the parameter space. These self-similar two-dimensional patterns are universal (up to a linear parameter change) for all bimodal maps, and depend only on the code of the associated critical point. Moreover, two universal scaling numbers have been found for the two-parameter one-dimensional maps, which are generalizations of the Feigenbaum number.

### (ii) *Devil's staircase from the driven Chua's circuit*

One of the remarkable properties of nonlinear oscillators is their ability to lock onto certain subharmonic frequencies when driven by an external source of energy. Associated with the phase-locking property is usually the appearance of 'staircases' of phase-locked states when the parameters are varied over certain range. The picturesque name *devil's staircase* is used to describe the intricate, often fractal, structure of such staircases. Figure 5 shows the devil's staircase in Chua's circuit obtained by plotting the ratio of winding numbers and period numbers as a function of the normalized forcing angular frequency (see, for example, Pivka *et al.* 1994). The self-similar structure of the staircase tree and the devil's staircase become apparent when magnified pictures are drawn of the portions of the devil's staircase.

### (*k*) *Other dynamical phenomena from the driven Chua's oscillator*

Extensive computer simulations and physical experiments were performed in a two-parameter study (Anishchenko *et al.* 1995) to describe several types of transition to chaos in the non-autonomous Chua's circuit. Also in an experimental and numerical study (Itoh & Murakami 1994) of Chua's oscillator some new phenomena—frequency entrainment of chaos, period-preserving bifurcations—have been reported, along with many other phenomena previously observed from different oscillators.

## 3. Analytical results on Chua's oscillator

The first proof of the chaotic nature of the double-scroll attractor in Chua's circuit was given in Chua *et al.* (1986) by establishing the existence of a homoclinic loop of the saddle-focus at the origin, and by applying the Shil'nikov theorem. Another proof which also makes use of the Shil'nikov theorem, applied to the double-hook attractor, was given by Silva (1993). In addition to these proofs, many deep mathematical analyses of Chua's circuit and Chua's oscillator have been published. We now summarize some of these analytical results.

### (*a*) *Global bifurcation analysis*

An in-depth analysis of the global 2-parameter bifurcation structures of Chua's circuit was made by Komuro *et al.* (1991) in terms of homoclinic, heteroclinic, and periodic orbits. By using the normal form theory, developed earlier for three-dimensional, three-region, piecewise-linear systems, the following results were obtained:

(i) The parameter sets which give rise to homoclinic and heteroclinic orbits (homoclinic and heteroclinic bifurcation sets) were found to be all connected to each other via only one family of periodic orbits.

(ii) The structure of the windows of this family essentially determines the global structure of the periodic windows.

(iii) The bifurcation analyses were accomplished by deriving first the relevant bifurcation equations in exact analytic form, thus making it possible to construct high-resolution bifurcation diagrams without using numerical integration formulas.

### (*b*) *One-dimensional Chua's map*

The original Chua's one-dimensional map introduced in Chua *et al.* (1986) has been extensively investigated numerically (Chua & Tichonicky (1991) and analytically (Brown 1993; Sharkovsky *et al.* 1993; Misiurewicz 1993; Maistrenko *et al.* 1993). Using the generalized framework developed by Brown (1993), Misiurewicz (1993) has investigated maps of the real line into itself obtained from the modified Chua's equation. For a large range of parameters, Misiurewicz found the existence of *invariant* intervals as well as invariant sub-intervals on which the associated Chua's circuit is *unimodal* and resembles the well-known *logistic* map. Moreover, this map is found to have a negative Schwarzian derivative, implying the existence of at most one attracting periodic orbit. Moreover, Misiurewicz has proved that there is a set of parameters of positive measure for which chaos occurs.

### (*c*) *Universality in cycles of chaotic intervals*

The order of the bifurcation sequence in piecewise-linear maps is different from that of smooth maps. In the case of the piecewise-linear map associated with the Chua's circuit with time delay (Sharkovsky 1993), Maistrenko *et al.* (1993) have found that when a period-$n$ *point cycle* loses its stability, a 'rigid' period-doubling bifurcation occurs which leads to the emergence of *not* point cycles but *interval cycles* of double period having chaotic trajectories. This is followed by an inverse period-doubling bifurcation, i.e. interval cycles of period $2n$ are merged pairwise, giving birth to a period-$n$ interval cycle. Finally, in the next bifurcation all intervals of interval cycles will merge into the full *interval* cycle $I = [0, 1]$. In this case, there are no subintervals of $I$ which recur periodically under the map of $f$. Among many elegant mathematical properties concerning interval cycles, Maistrenko *et al.* (1993) has derived two *universal* constants analytically, and in explicit form. This result is most surprising since the well-known Feigenbaum universal constant was calculated only numerically.

### (*d*) *Global stability and instability of Chua's oscillator*

Recently, Leonov *et al.* (1993) has investigated Chua's oscillators as a feedback control system and derived a frequency-domain criterion for global stability and instability. This analytical study has led to a new version of the generalized Kalman's conjecture.

### (*e*) *The double-horseshoe theorem*

Using a new geometric model of Chua's circuit, Belykh & Chua (1993) have presented an analytical study of a new type of strange attractor generated by an odd-symmetric three-dimensional orbit at the origin. This type of attractor is intimately related to the double-scroll Chua's attractor. They have proved rigorously that the chaotic nature of this attractor is different from that of a Lorenz-type attractor, or a quasi-attractor. In particular, this attractor has the geometry of a *double horseshoe*. For certain nonempty intervals of parameters, this strange attractor has no stable orbits. Unlike other known attractors, the double horseshoe attractor contains not only a Cantor set structure of hyperbolic points typical of horseshoe maps, but unstable points (i.e. stable in reverse time) as well. This implies that the points from the stable manifolds of the hyperbolic points must necessarily attract the unstable points.

### (*f*) *Synchronization, trigger wave, and spatial chaos*

Several criteria for synchronizing two *mutually coupled* Chua's circuits operating under chaotic regimes are derived in Belykh *et al.* (1993) and Wu & Chua (1994). For a chain of Chua's oscillators, analytical results couched in terms of a moving coordinate system have been derived which guarantee the existence of *heteroclinic* orbits (Nekorkin *et al.* 1993). This analytical study is highly significant because it proves, among other things, the presence of a trigger wave along the chain. The proof of the existence of heteroclinic orbits represents a major breakthrough since it is generally extremely difficult, if not impossible, to derive such analytical results. In addition to trigger waves, this investigation also proves the existence of *spatial chaos* along a finite chain of Chua's oscillators.

### (*g*) *Fine structure of the double-scroll Chua's attractors*

Using the theory of *confinors* Lozi & Ushiki (1991, 1993) have developed an *analytical* approach, in sharp contrast to numerical integration methods, for examining the fine features of various Chua's attractors. The keystone of the original definition of confinors is that very often, changes in the shape of experimentally observed signals are more significant in characterizing the phase portrait, than any topological change between chaotic attractors. The theory of confinor takes into account the 'shape' of the signals, and is capable of modelling both transient and asymptotic regimes. Applying this unique approach to Chua's equation, Lozi & Ushiki (1991) have discovered the co-existence of three *distinct* double-scroll Chua's attractors in close proximity of each other for *the same value* of parameters. Without a precise knowledge of initial conditions, which the confinor theory can supply, it would be virtually impossible to pick these three attractors apart. This explains why in spite of the rather extensive numerical and experimental works of many researchers on Chua's circuit over the past 10 years, no one has ever observed the simultaneous existence of three chaotic attractors.

In addition to this discovery, Lozi & Ushiki (1993) have also provided the most precise characterizations of the structure of the double-scroll Chua's attractors via an exact two-dimensional Poincaré map. Moreover, they have discovered some very unusual bifurcation phenomena which are distinct from the usual period-doubling cascades. Since these results are all highly original and robust, they can be used as a guide for characterizing strange attractors of other chaotic systems, thereby demonstrating yet another application of Chua's circuit as a universal paradigm for chaos.

### (*h*) *The global unfolding theorem*

It is shown in Chua (1993) that the state equation of Chua's oscillator, or unfolded Chua's circuit, is topologically conjugate (i.e. equivalent) to a 21-parameter family of continuous, odd-symmetric, three-region, piecewise-linear equations in $R^3$. The corresponding circuit is uniquely determined by seven parameters and it is shown that no circuit with less than seven parameters has this property. The significance of the unfolded Chua's circuit is that the qualitative dynamics of every autonomous third-order system with one continuous, odd-symmetric, three-segment, piecewise-linear function can be mapped into this circuit, thereby making their separate analyses unnecessary. This unification reduces the investigation of the many heretofore unrelated publications on chaotic circuits and systems to the analysis of only one canonical circuit. Recently, Wu & Chua (1995*b*) have extended the global unfolding theorem in three ways. First, the vector field can be of arbitrary dimension. Second, the vector field need not be odd-symmetric. Third, the vector field need not be piecewise-linear. In particular, it was shown that the $n$-dimensional Chua's oscillator is topologically conjugate to almost all vector fields in Luré form.

### (*i*) *Chaos from a time-delayed Chua's circuit*

A generalization of Chua's circuit to infinite dimensions can be obtained by replacing the parallel LC 'resonator' by a lossless transmission line, terminated by a short circuit. The resulting 'time-delayed Chua's circuit' whose time evolution is described by a pair of linear partial differential equations with a nonlinear boundary condition. If we neglect the linear capacitance across the Chua diode which is described by a non-symmetric piecewise-linear $v_R$–$i_R$ characteristic, the resulting idealized 'time-delayed Chua's circuit is described *exactly* by a *scalar nonlinear difference equation* with continuous time, which makes it possible to characterize its associated nonlinear dynamics and spatial chaotic phenomena.

From a mathematical view point, circuits described by *ordinary* differential equations can generate only *temporal* chaos, while the time-delayed Chua's circuit can generate *spatial-temporal* chaos. Except for stepwise-periodic oscillations, the *typical solutions* of the idealized time-delayed Chua's circuit consist of either *weak turbulence*, or *strong turbulence*, which are examples of 'ideal' (or 'dry') turbulence. In both cases, we can observe infinite processes of spatial-temporal coherent formations (Sharkovsky 1993). Also, after deriving the corresponding one-dimensional map, it is possible to determine without any approximation the analytical equation of the stability boundaries of cycles of every period $n$. Since the stability region is nonempty for each $n$, this provides a rigorous proof that the time-delayed Chua's circuit exhibits the period-adding phenomenon where every two consecutive cycles are separated by a chaotic region (Sharkovsky *et al.* 1993).

### (*j*) *Dynamics of Chua's circuit in a Banach space*

Blázquez & Tuma (1993) derived generalized theorems of the Shil'nikov type for evolution equations in Banach spaces of infinite dimension. These theorems can be applied to the Chua equations in infinite dimension for a description of the behaviour of subsystem of solutions in a neighbourhood of double homoclinic orbits to the same saddle-focus point.

### (*k*) *Future problems*

Shil'nikov (1993) summarizes some of the rigorous results on Chua's circuit and discusses the differences between so-called 'stochastic' attractors of the hyperbolic and Lorenz type, which are amenable to study by statistical methods, and 'quasi-stochastic' attractors found in Chua's oscillator whose behaviour is much more complicated. It is pointed out that although a complete description of the dynamics and bifurcations in the Chua equations is impossible, quasi-attractors could be studied by adding a small noise to 'spread' the stable periodic orbits as well as structurally unstable periodic orbits.

An extension of the present results to infinite dimensions is nontrivial and provides opportunities for discovering many essentially new effects, e.g. coexistence of 'large' attractors, periodic, quasi-periodic, and 'small' strange attractors of a very different nature. There are also many unsolved mathematical problems associated not only with Chua's oscillator but also its higher-dimensional generalizations to CNN arrays of such oscillators (see, for example, Pérez-Muñuzuri *et al.*, this volume).

## 4. Applications

Not a long time ago, chaotic phenomena were recognized as a curiosity phenomenon, interesting and worth studying only from an academic point of view, without a prospect of immediate practical applications. Most practically oriented studies were directed toward avoiding chaos as a phenomenon which is undesirable in real-life applications. Following some breakthrough discoveries, especially of chaotic synchronization (Pecora & Carroll 1990) and stochastic resonance (see, for example, Anishchenko *et al.* 1993) phenomena, this attitude changed dramatically. In this section we outline some potential applications in which chaotic behaviour plays an essential role.

### (*a*) *Secure communication via chaos*

The key concept in the application to secure communication is that of synchronization. Given two (or more) nonlinear systems

$$\dot{\boldsymbol{x}}_k = \boldsymbol{f}_k(\boldsymbol{x}_k), \quad \boldsymbol{x}_k \in R^n, \quad k = 1, 2, \ldots, N$$

we are interested in conditions leading to the synchronization of the solutions, i.e. the convergence $(\boldsymbol{x}_l - \boldsymbol{x}_j) \to 0$, as $t \to \infty$, for $l \neq j$. For the special case of linear coupling of two systems

$$\dot{\boldsymbol{x}} = \boldsymbol{f}_1(\boldsymbol{x}), \quad \dot{\boldsymbol{y}} = \boldsymbol{f}_2(\boldsymbol{y}) + \Delta(\boldsymbol{x} - \boldsymbol{y}), \quad \Delta = \mathrm{diag}(\delta_1, \ldots, \delta_n)^{\mathrm{T}}, \quad \boldsymbol{x}, \boldsymbol{y} \in R^n$$

some general conditions are known (Kočarev *et al.* 1993*b*) to ensure synchronization. To ensure synchronization in another way, Pecora & Carroll (1991) suggested a drive-response concept as follows.

Consider an $n$-dimensional autonomous system with the state equation,

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}),$$

and divide the system into two parts, the driving subsystem D, and the response subsystem R:

$$\dot{\boldsymbol{x}}_{\mathrm{D}} = \boldsymbol{g}(\boldsymbol{x}_{\mathrm{D}}, \boldsymbol{x}_{\mathrm{R}}), \quad \dot{\boldsymbol{x}}_{\mathrm{R}} = \boldsymbol{h}(\boldsymbol{x}_{\mathrm{D}}, \boldsymbol{x}_{\mathrm{R}}).$$

By adding an identical copy of the response subsystem, we obtain

$$\dot{\boldsymbol{x}}_{\mathrm{D}} = \boldsymbol{g}(\boldsymbol{x}_{\mathrm{D}}, \boldsymbol{x}_{\mathrm{R}}), \quad \dot{\boldsymbol{x}}_{\mathrm{R}} = \boldsymbol{h}(\boldsymbol{x}_{\mathrm{D}}, \boldsymbol{x}_{\mathrm{R}}), \quad \dot{\boldsymbol{x}}'_{\mathrm{R}} = \boldsymbol{h}(\boldsymbol{x}_{\mathrm{D}}, \boldsymbol{x}'_{\mathrm{R}}).$$

Then for $\boldsymbol{x}_{\mathrm{R}}$ and $\boldsymbol{x}'_{\mathrm{R}}$ to synchronize it is necessary that the conditional Lyapunov exponents (depending on $\boldsymbol{x}_{\mathrm{D}}$) be all negative (Pecora & Carroll 1990). Several approaches have been suggested so far to use the synchronization effect for data transmission by using Chua's circuit as the chaos generator.

(i) *Secure communication via chaotic masking*

The idea of chaotic masking has been proposed in Oppenheim *et al.* (1992); Kočarev *et al.* (1992), where the information-carrying signal is added to the masking chaotic signal. Such an approach was used in an experimental setup for secure communication, where Chua's circuits were used as synchronizing blocks. Figure 6 shows a schematic diagram, in which the transmitter part is a Chua's circuit and the receiver part is two partial Chua's circuits. The first subcircuit is a decoding key and synchronizes only when exactly matched with the transmitter circuit, thus reproducing the $y_1$ signal. The second subcircuit is used for obtaining the variable $x_2$ needed for recovering the information signal through subtraction as shown in the diagram of figure 6.

(ii) *Chaos shift keying*

In the chaotic synchronization approach used above, the parameters in the transmitter and in the receiver must be matched exactly. Slight parameter mismatch will cause synchronization error to appear. This property is exploited in the *chaos shift keying* scheme (Parlitz *et al.* 1992) where a binary signal is encoded in terms of two different attractors existing for two different system parameter values. The transmitter and receiver is similar to the chaotic masking scheme, but rather than adding the information signal onto the chaotic signal, a waveform corresponding to one attractor is transmitted when a '1' occurs in the information stream and a waveform corresponding to the other attractor is transmitted when a '0' occurs in the information stream. Although the two attractors appear similar, the receiver will synchronize only to the attractor with the same parameter set. Whether the receiver synchronizes or not determines whether a '1' or a '0' is transmitted. For a more reliable system, a second receiver with the second parameter set can be used. The feasibility of this

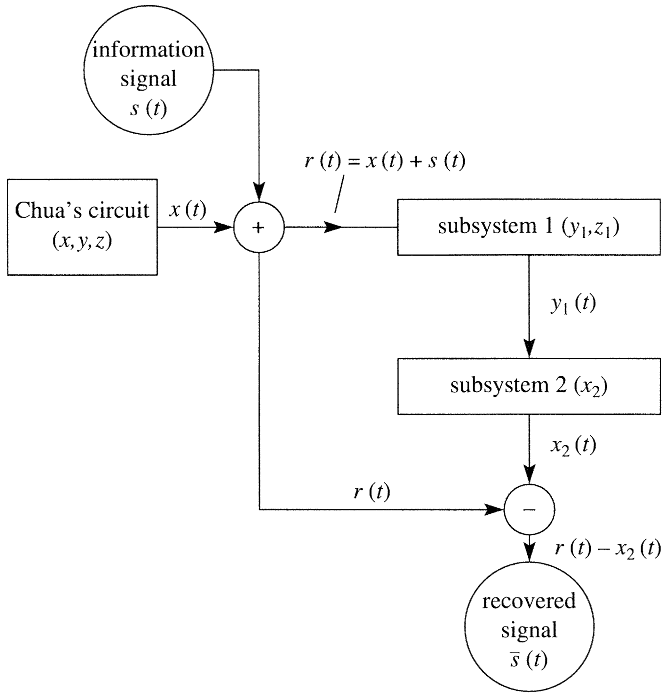


Figure 6. Diagram for secure communication system using chaotic masking.

approach for secure communication has been demonstrated experimentally in Parlitz *et al.* (1992).

(iii) *Spread-spectrum secure communication via chaotic masking*

Halle *et al.* (1993) provide another approach for information transmission by making use of the broad spectrum of the chaotic carrier signal. The transmitter and the receiver each contain an identical Chua's circuit. In the transmitter a current signal $I(t)$ is injected into the circuit to modify the voltage across capacitor $C_1$. An invertible coding function $c$ is chosen so that $I(t) = c(v_s(t))$ where $v_s(t)$ is the input signal to be transmitted. The detected output signal $I_d$ is then decoded from $v_r(t) = c^{-1}(I_d(t))$. In the spread spectrum scheme, the coding function is multiplication with the chaotic signal $x(t)$ which spreads the spectrum of the input signal. For proper operation, it is necessary that $v_r(t) \approx v_s(t)$ and the coding function should be chosen in such a way that the transmitted signal remains chaotic. The voltage across the capacitor $C_1$ is transmitted to the receiver circuit and is used as a forcing voltage on the second Chua's circuit capacitor $\tilde{C}_1$. If all circuit components of the transmitter and receiver are matched exactly, we have (Halle *et al.* 1993) $I_d(t) \to I(t)$ and $v_r(t) \to v_s(t)$ for $t \to \infty$, which means that the current flowing into the second Chua's circuit will eventually equal the current injected into the first Chua's circuit. One of the features of this scheme is that when the transmitter and receiver are completely matched and there is no distortion in the channel, then the information signal can be recovered at the receiving end without any noise, whereas in figure 6, there will be some residual noise in the receiver ($\bar{s}(t) \not\to s(t)$). For a dual scheme which also has this noise-free property, see Wu & Chua (1993). The full account of the laboratory implementation of the transmission system can be found in Halle *et al.* (1993).

### (*b*) *Using Chua's circuit for recognition tasks*

#### (i) *Trajectory recognition via array of Chua's circuits*

Recently, Altman (1993) uses the centre manifold and normal form theory to relate the local behaviour of Chua's circuit to some input trajectory to be recognized. This mathematical problem arises in the recognition of hand gestures in the design of artificial intelligence, where the hand position as a function of time is used to drive Chua's circuit to an attracting surface. Since Chua's circuit is known to undergo a series of bifurcations from fixed points, to limit cycles, to a cascade of period-doubling oscillations leading to chaotic oscillations in the vicinity of the centre manifold surface, the rapid entrainment of the chaotic system to an external signal having a trajectory near the centre manifold surface provides the basic mechanism for trajectory recognition. The recognition of many trajectories can be achieved by using a two-dimensional array of Chua's circuits. In this case, the variation of responses to the common input trajectory creates a *spatial pattern* which can be used to recognize the input trajectory. The above approach to trajectory recognition is both novel and fascinating.

#### (ii) *Handwritten character recognition using Chua's oscillator*

A neural network architecture and learning algorithm for associative memory storage of analogue patterns, continuous sequences, and chaotic attractors via a network of Chua's oscillators has recently been designed by Baird & Hirsch (1993). Their design is used in the application to the problem of real-time handwritten digit recognition. They have demonstrated that several of the attractors from Chua's oscillator have out-performed the previously studied Lorenz attractor system in terms of both accuracy and speed of convergence.

### (*c*) *Music from Chua's circuit*

While investigating Chua's system, several distinct features were observed (Mayer-Kress *et al.* 1993) that can be described in musical terms: the tendency to produce pitch due to its isochronic behaviour, the simultaneous presence of noise along with embedded periodic orbits, and sounds resembling the formant structures of acoustic instruments. The interplay of these factors can create steady-state waveforms that resemble steady-state portions of acoustic instrument sounds. Chua's circuit has parameter regions where noisy frequency- and amplitude-modulated sounds are generated, each of which is related to a certain transition to chaos, e.g. period-doubling, intermittency, torus breakdown, etc. Interestingly, almost harmonic pitch changes can be produced through a period-adding sequence of bassoon-like sounds. Transient dynamics were found to be important in the context of percussion-like sounds.

The great variety of attractors that can be generated from Chua's circuit makes this circuit an excellent candidate for a universal signal-generating standard. These properties have been succesfully used for sound synthesis and design (Mayer-Kress *et al.* 1993; Rodet 1993) by tuning different circuit parameters (Zhong *et al.* 1994). Rodet (1993) describes real-time simulations of Chua's circuit on a digital workstation allowing for easy experimentation with the properties and behaviours of the circuit and of the sounds. Rich and novel musical sounds have been obtained. Also, the audification of the local properties of the parameter space allowed an easy determination of very complex structures which would not be simple to determine by other ways.

In another set-up (Rodet 1993), the time-delayed Chua's circuit was found to be able to model the basic behaviour of an interesting class of musical instruments, namely those of clarinet, consisting of a massless reed coupled to a linear system. Other instruments are expected to be obtained from Chua's circuit, e.g. brass, voice, flute, and strings. A possibility of creating completely new electronic instruments is considered in Mayer-Kress *et al.* (1993) where a method (convex sums of vector fields) of generating new attractors that inherit the properties of parent attractors is proposed.

### (*d*) *Unified framework for synchronization and control*

Recently, there has been much interest in the dynamics of coupled chaotic circuits and systems. Of particular interest is synchronization and control phenomena of chaotic systems. As seen in § 3 *a* applications based on chaotic sychronization such as secure communication systems have emerged in recent years. In Wu & Chua (1994) a unified framework was given to unify many of these ideas on chaotic synchronization and control by relating them to asymptotic stability of related systems. We give the main ideas here with regards to Chua's oscillator. Detailed statements and proofs can be found in Wu & Chua (1994).

The main theorem can be stated as follows:

**Theorem 1.** *Consider the system*

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{x}, \boldsymbol{y}, t), \tag{4.1}$$

$$\dot{\boldsymbol{y}} = \boldsymbol{f}(\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{y}, t), \tag{4.2}$$

*where* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, *and* $\boldsymbol{f}$ *is defined on* $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$. *Suppose that for every* $\eta_1(t)$ *and* $\eta_2(t)$ *continuous functions into* $\mathbb{R}^n$, *the system*

$$\dot{\boldsymbol{z}} = \boldsymbol{g}(\boldsymbol{z}, t) = \boldsymbol{f}(\boldsymbol{z}, \eta_1(t), \eta_2(t), t) \tag{4.3}$$

*is uniform-asymptotically stable. Then* $\|\boldsymbol{y}(t) - \boldsymbol{x}(t)\| \to 0$ *as* $t \to \infty$.

The main idea to achieve synchronization of two chaotic systems is to 'extract' the parts of the system such that the rest of the system is uniform-asymptotically stable, and use the extracted parts as coupling. More precisely, we start off with the system $\dot{\boldsymbol{x}} = \boldsymbol{g}_1(\boldsymbol{x}, t)$ which is chaotic. Next we decompose the arguments in the vector field $\boldsymbol{g}_1$ into four components such that it can be written as $\boldsymbol{g}_1(\boldsymbol{x}, t) = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{x}, \boldsymbol{x}, t)$. The decomposition is chosen in such a way that the system $\dot{\boldsymbol{x}} = \boldsymbol{g}_2(\boldsymbol{x}, t) = \boldsymbol{f}(\boldsymbol{x}, \eta_1(t), \eta_2(t), t)$ is asymptotically stable for all $\eta_1(t)$ and $\eta_2(t)$. The second and third arguments to $\boldsymbol{f}$ is the 'extracted' part. When we couple two identical systems as in equations (4.1)–(4.2), the two systems will synchronize so that they are decoupled at the synchronized state.

For the class of systems which has vector fields with uniformly bounded Jacobians, strong enough linear coupling between two identical systems will cause them to be globally synchronized. In Chua's oscillator for the standard parameters values, the only active element is the nonlinear resistor. Therefore taking that part out of the system, the resulting system will consist of linear passive elements and is uniform-asymptotically stable. Applying this to two coupled Chua's oscillators we obtain the following result where the coupling is separable:

**Proposition 1.** *Consider the system:*

$$\begin{aligned}
\dot{x} &= \alpha\left(y - x - f(x) + g_1(\tilde{x}) - g_1(x)\right), \\
\dot{y} &= x - y + z, \\
\dot{z} &= -\beta y - \gamma z, \\
\dot{\tilde{x}} &= \alpha\left(\tilde{y} - \tilde{x} - f(\tilde{x}) + g_2(x) - g_2(\tilde{x})\right), \\
\dot{\tilde{y}} &= \tilde{x} - \tilde{y} + \tilde{z}, \\
\dot{\tilde{z}} &= -\beta\tilde{y} - \gamma\tilde{z}.
\end{aligned}$$

*Suppose* $\alpha, \beta, \gamma > 0$. *If the function* $f + g_1 + g_2$ *is strictly increasing, then*

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} \rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

*as* $t \rightarrow \infty$.

In linear diffusive coupling, $g_1$ and $g_2$ are chosen to be multiplication by a positive constant, i.e. $g_i(x) = c_i x$. If $f(x)$ is the piecewise-linear function $f(x) = bx + \frac{1}{2}(a - b)[|x + 1| - |x - 1|]$, then $c_1 + c_2 > \max(-a, -b)$ will cause the coupled system to synchronize.

Certain schemes of controlling trajectories of chaotic systems to lie on unstable periodic cycles can also be cast into this framework, where we take $g_1 = 0$, and take the first system to be moving along the unstable periodic cycle. Because there is no coupling from the second system to the first system, the first system can be replaced by a memory device which recalls a particular (unstable) trajectory of the system (Pyragas 1993).

In Wu & Chua (1994) it was also shown that synchronization between two Chua's oscillators in master-slave configuration is robust. Strictly speaking, when the conditions of Proposition 1 are satisfied with $g_1 = 0$, arbitrarily small changes in the component values $C_1$, $C_2$, $R$, $L$, $R_0$ (which result in arbitrarily small changes in $\alpha$, $\beta$, $\gamma$) in one of the systems will result in arbitrarily small synchronization error. This result suggests that practical physical implementations where parameter mismatches are unavoidable will still synchronize approximately. Moreover, in Wu & Chua (1995*a*) it was shown that in an array of identical Chua's oscillators under standard parameter values, if the array is connected together via linear resistors (of resistance $R_c$) across the nonlinear resistors, the array will synchronize for small enough $R_c$.

## 5. Conclusion

Chua's oscillator has proved to be an excellent paradigm for the generation of a multitude of different dynamical phenomena. Because of its generality, many more phenomena can be observed in addition to those in Chua's circuit, as it corresponds to a global unfolding of Chua's circuit, encompassing almost all three-dimensional, piecewise-linear, continuous, three-region, odd-symmetric systems by being topologically conjugate, for suitable parameter values, to any such vector field (except for a set of measure zero). Beside Chua's oscillator, individual vector fields which are continuous, odd-symmetric, three-region, and piecewise-linear, can be generated by the members of Chua's circuit family (Wu 1987). Many members of Chua's circuit family have been synthesized and built. Except for the canonical circuit reported in Chua & Lin (1990), the dynamics of the circuits, including the original Chua's circuit are not sufficiently general in the sense that certain phenomena observed from one such member of Chua's circuit family cannot be observed from another member, regardless of the choice of circuit parameters. From the circuit-theoretical point of view, it is desirable to synthesize the simplest circuit topology which is capable of reproducing the qualitative phenomena exhibited by every member of Chua's circuit family. Chua's oscillator was shown to meet these requirements, so that its significance can be summarized as follows:

*Chua's oscillator is structurally the simplest and dynamically the most complex member of Chua's circuit family.*

The significance of Chua's oscillator transcends beyond nonlinear circuit theory. It can be used successfully to mimic the behaviour of other three-dimensional dynamical systems, both piecewise-linear and smooth. Chua's oscillator has unified the nonlinear dynamics of the entire 21-parameter family of piecewise-linear vector fields into a single system defined by equation (1.1), hence it is not necessary for beginners in nonlinear dynamics to study all those papers with diverse notations and jargons. Furthermore, Wu & Chua (1995*b*) show that this unification can be extended to the class of Luré systems.

Even more significantly, arrays of Chua's oscillators appear to be a suitable candidate for important applications ranging from image processing to the simulations of biological processes. The building of the monolithic IC chip of Chua's circuit (Rodriguez-Vazquez & Delgado-Restituto 1993; Cruz & Chua 1993) is an important step toward building large arrays via VLSI technology, and will make it possible to reproduce, in real time, almost all reaction-diffusion situations, described in the literature with a relatively simple low-cost system.

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