Generalizations of the Chua Equations

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***Abstract*— We present two generalizations of the equations governing Chua's circuit. In the type-I generalization of Chua's equations we use a 2-D autonomous flow as a component in a 3-D autonomous flow in such a way that the resulting equations will have double-scroll attractors similar to those observed experimentally in Chua's circuit. The value of this generalization is that 1) it provides a *building block* approach to the construction of chaotic circuits from simpler 2-D components that are not chaotic by themselves. In so doing, it provides an insight into how chaotic systems can be built up from simple nonchaotic parts. 2) It illustrates a precise relationship between 3-D flows and 1-D maps. In the type-II generalized Chua equations we show that attractors similar to the Lorenz and Rössler attractors can be produced in a building block approach using only piecewise linear vector fields. As a result we have a method of producing the Lorenz and Rössler dynamics in a circuit without the use of multipliers. These results suggest that the generalized Chua equations are in some sense fundamental in that the dynamics of the three most important autonomous 3-D differential equations producing chaos are seen as variations of a single class of equations whose nonlinearities are generalizations of the Chua diode.**

I. Introduction

In this paper, we explore two generalizations of Chua's equations. These two generalizations are significant in that they show the following:

1) How chaotic flows in three dimensions can be constructed from simple nonchaotic parts
2) how 3-D flows can be analytically related to 1-D maps
3) how a very wide range of chaotic dynamical systems including systems similar to Rössler and Lorenz's can be constructed from piecewise linear flows.

This is significant for many reasons. Prior to these results, the number of chaotic dynamical systems available to guide research into the mechanisms of chaos were very few. Second, in order to construct new systems with given properties, there was no available methodology that started with simple components and, in a systematic way, used these components to build new chaotic systems. Third, there was no analytical connection between 3-D flows and 1-D maps that offered the prospect of linking the relatively extensive machinery of 1-D map theory to 3-D flows. The results of this paper provide an advance in all three of these research areas.

The dimensionless form of Chua's equations are given by

$$\begin{aligned} \dot{x} &= \alpha(y - x - f(x)) \\ \dot{y} &= x - y - z \\ \dot{z} &= -\beta y \end{aligned} \tag{1}$$

where

$$f(x) = \begin{cases} bx + a - b, & \text{for } x \geq 1.0 \\ ax, & \text{for } |x| \leq 1.0 \\ bx - a + b, & \text{for } x \leq 1.0 \end{cases} \tag{2}$$

is a three-segment piecewise linear function and α and β are dimensionless parameters.

We note from the history of Chua's circuit [3] that the function $f(x)$ represents a *nonlinear* resistor, and hence can be *any scalar* function of one variable. The choice of the piecewise linear function (2) is only for convenience in synthesizing the physical circuit. It is also obvious from [3] that $f(x)$ can be modified in many ways without changing the qualitative dynamics. In Section II we will choose the discontinuous "signum" function $\operatorname{sgn}(x)$ for $f(x)$.

Section II of this paper discusses the type-I generalized Chua equation.

In Section III we define the type-II generalized Chua equations.

II. Type-I Equations

In this section we derive the type-I generalized Chua equations, the single scroll, and relate them to 1-D maps. We show how to derive a double scroll from a large class of 2-D autonomous flows.

A. Simplification of Chua's Equation

The dimensionless Chua equations can be recast into the form:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} -\alpha & \alpha & 0.0 \\ 1.0 & -1.0 & 1.0 \\ 0.0 & -\beta & 0.0 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \alpha \begin{pmatrix} f(x) \\ 0.0 \\ 0.0 \end{pmatrix} \tag{3}$$

A more compact expression for $f(x)$ is given by $bx + 0.5(a - b)(|x+1.0| - |x-1.0|)$. Using this expression, (3) simplifies to:

$$\dot{\boldsymbol{x}} = \begin{cases} \boldsymbol{A}(\alpha, \beta, b)(\boldsymbol{x} - \boldsymbol{k}), & \text{for } x \geq 1 \\ \boldsymbol{A}(\alpha, \beta, a)\boldsymbol{x}, & \text{for } |x| \leq 1 \\ \boldsymbol{A}(\alpha, \beta, b)(\boldsymbol{x} + \boldsymbol{k}), & \text{for } x \leq -1 \end{cases} \tag{4}$$

where

$$\boldsymbol{A}(\alpha, \beta, c) = \begin{bmatrix} -\alpha(c+1) & \alpha & 0.0 \\ 1.0 & -1.0 & 1.0 \\ 0.0 & -\beta & 0.0 \end{bmatrix} \tag{5}$$

where

$$\boldsymbol{k} = \begin{bmatrix} k \\ 0 \\ -k \end{bmatrix}$$

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and $c = b$ or a depending on the value of x, and $k = (b-a)/(b+1)$. Thus the vector field determined by the matrix $\boldsymbol{A}$ varies depending on which of three regions the vector $\boldsymbol{x}$ is in.

We seek to simplify (4) to a set of equations having only two linear regions instead of three and such that the matrix $\boldsymbol{A}$ is the same for both regions. In doing this we want to preserve the qualitative dynamics found in the Chua circuit.

In particular we want to introduce a function in place of $f(x)$ for which the region where $c = a$ is such that $|x| \leq \epsilon$, where ϵ is a number we can control. If we can do this and then decrease ϵ to zero, while holding on to the global dynamics found in the Chua circuit, we will have a simpler equation with which to work. In doing this it can happen that the system can go unbounded for reasons that will be clear later. So if we decrease the middle region, we must also decrease the real part of the expanding eigenvalues (which are complex conjugate) in order to retain a region where there are bounded solutions of the ODE. We can carry out this strategy if we replace the piecewise-linear function

$$h(x) = 0.5(|x + 1.0| - |x - 1.0|) \tag{6}$$

that makes up part of $f(x)$ by the C^∞ function

$$g(x) = \frac{\exp(\gamma x) - 1.0}{\exp(\gamma x) + 1.0}. \tag{7}$$

(This function is known as a *sigmoid* function due to its "S"-shaped graph and it is prevalent in the theory of neural networks.)

Doing this we have the following equation:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} \alpha(b+1) & \alpha & 0.0 \\ 1.0 & -1.0 & 1.0 \\ 0.0 & -\beta & 0.0 \end{bmatrix} \begin{pmatrix} x - kg(x) \\ y \\ z + kg(x) \end{pmatrix}. \tag{8}$$

By replacing the piecewise-linear function, (2), with a sigmoid function, (7), and slightly increasing β we can obtain an equation that will produce chaotic dynamics very similar to those in the Chua equations, i.e., produce a double scroll.

After this replacement is made we study what happens as γ is increased. We begin by setting $\gamma = 10.0$, which compresses the middle region, and to offset this we increase β from 14.2857 to 15.0 in order to lower the rate of expansion. Having made our replacement successfully we increase γ to 100.0 (β will need no further adjustments). This equation has bounded solutions which are analytic, and for which the middle region, i.e., the region for which $|g(x)| < 1$, is very small, and, in fact, its size is inversely proportional to γ. It is a simple matter now to let $\gamma \to \infty$ to obtain the following simplification of Chua's original equation (3):

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} \alpha(b+1) & \alpha & 0.0 \\ 1.0 & -1.0 & 1.0 \\ 0.0 & -\beta & 0.0 \end{bmatrix} \begin{pmatrix} x - k\,\mathrm{sgn}(x) \\ y \\ z + k\,\mathrm{sgn}(x) \end{pmatrix}. \tag{9}$$

Equation (9) is a pointwise limit, rather than the uniform limit of (8), but that is not a limitation on its use since (9) satisfies all of our requirements, most important of which is that it has a double-scroll attractor just as in Chua's original equations. Even though when $\gamma = \infty$ the system has a discontinuity at $x = 0.0$, this discontinuity does not introduce chaos where it did not already exist: if we fix a specific time T in the future, there exists a γ defining a C^∞ equation having a solution arbitrarily close to the solutions of (9) for all time less than T. As a result of these numerical observations, we have every right to believe that (9) is a system which is representative of the chaotic dynamics of Chua equation.

All constants in (9) are the same as those in the original Chua equations, (3), with the exception that $\beta = 15.0$ rather than 14.2857. In particular, $a = -8/7, b = -5/7, \alpha = 9.0, -\alpha(b+1) = -18/7$, and $k = 1.5$. For comparison, we recast Chua's original equations (1) in the same form as (9) so that their differences be easily discerned:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} -\alpha(b+1) & \alpha & 0.0 \\ 1.0 & -1.0 & 1.0 \\ 0.0 & -\beta & 0.0 \end{bmatrix} \begin{pmatrix} x - kh(x) \\ y \\ z + kh(x) \end{pmatrix} \tag{10}$$

where $h(x)$ is given by (6).

The effect of suppressing the middle region makes the vector field discontinuous, but as we point out later, this does not change the complexity in the simplified version of the Chua equations from that in the original Chua equations.

The analytical advantage of working with (9) is that the matrix in (9) is constant while the function $\mathrm{sgn}(u)$ takes on only two values. In contrast, in (10), while the matrix $\boldsymbol{A}$ has been made constant, the function $h(u)$ is continuous and preserves the middle region.

For the matrix $\boldsymbol{A}$ in (9) we are able to introduce coordinate transformations that separate, or "decouple" from a circuit theoretic point of view, its invariant subspaces. These subspaces determine the stable and unstable manifolds for the linear vector field near the critical points $\boldsymbol{k}$ and $-\boldsymbol{k}$. These two points were also critical points for the Chua equations. The dynamics of (10) and (9) are the same at these points.

We know from linear algebra that we may write the matrix $\boldsymbol{A}$ in (9) as $\boldsymbol{JDJ}^{-1}$, where

$$\boldsymbol{J} = \begin{bmatrix} 1.0 & -1.287 & 1.0 \\ 0.143 & -1.513 & -0.148 \\ -2.37 & -2.14 & -0.569 \end{bmatrix}$$

and

$$\boldsymbol{D} = \begin{bmatrix} 0.0 & -9.876 & 0.0 \\ 1.0 & 0.334 & 0.0 \\ 0.0 & 0.0 & -3.9055 \end{bmatrix}.$$

Doing this and changing coordinates, (9) is transformed to

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} 0.0 & -9.876 & 0.0 \\ 1.0 & 0.334 & 0.0 \\ 0.0 & 0.0 & -3.9055 \end{bmatrix} \begin{pmatrix} x - a\,\mathrm{sgn}(u) \\ y - b\,\mathrm{sgn}(u) \\ z - c\,\mathrm{sgn}(u) \end{pmatrix} \tag{11}$$

where $a = 0.4455, b = -0.05445, c = 0.984$, and $u = x - 1.287y + z$. It should be noted that if we transform (10) by the same coordinates as we used to transform (9), the matrix $\boldsymbol{A}$ would be diagonalized, but the function $h(u)$ would not be improved, and therefore, the troublesome middle region would not be eliminated.

B. The Single Scroll

We now use the fact that the vector field defined by (10) and (11) is an odd symmetric vector field and is invariant under the flip map $\boldsymbol{x} \rightarrow -\boldsymbol{x}$. This allows us to view the double scroll in (11) as a single scroll. In particular, whenever $\operatorname{sgn}(u)$ changes sign we apply the flip map and use the linear ODE:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} 0.0 & -9.876 & 0.0 \\ 1.0 & 0.334 & 0.0 \\ 0.0 & 0.0 & -3.9055 \end{bmatrix} \begin{pmatrix} x-a \\ y-b \\ z-c \end{pmatrix} \tag{12}$$

to continue the orbit. Note that the value of a, b, and c are the same as in (11).

If as in the terminology of [2] we identify the last differential equation as an expanding linear twist, we see that we have *factored* (11) into the form FT where the map T is determined by selecting an initial condition $\boldsymbol{x}_0$, integrating the above linear ODE until the solution starting at this initial condition reaches the boundary determined by the function $\operatorname{sgn}(u)$ and using this *final* condition as the value of $T(\boldsymbol{x}_0)$. We continue the solution by applying the flip map F to this final condition and using this flipped value as the initial condition for the above ODE. The effect of doing this is exactly the same as the process used in the twist-and-flip maps of [2]. In this way we are able to plot the entire double scroll of (11) on one side of the plane determined by the function $u = x - 1.287y + z = 0.0$ around only one of the fixed points, just as happens in using the twist-and-flip map.

Doing this amounts, mathematically speaking, to folding the 3-D space in half along the plane $x - 1.287y + z = 0.0$ and identifying the points $\boldsymbol{x}$ and $-x$. That is, we consider the two points $\boldsymbol{x}$ and $-\boldsymbol{x}$ as the same point.

Since we have separated out the stable manifold direction in the transformed equations, we can examine the effect of increasing the contracting eigenvalue by considering the following equation instead of (11):

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} 0.0 & -9.876 & 0.0 \\ 1.0 & 0.334 & 0.0 \\ 0.0 & 0.0 & -\gamma \end{bmatrix} \begin{pmatrix} x - a\operatorname{sgn}(u) \\ y - b\operatorname{sgn}(u) \\ z - c\operatorname{sgn}(u) \end{pmatrix}. \tag{13}$$

In this equation the entry 3.9055 in (11) has been replaced by γ. (We are reusing the symbol γ in a manner that is unrelated to the γ in (7).)

The effect of increasing γ is to flatten the scroll onto a pair of parallel planes. If we combine this with the folding operation, we get the single scroll obtained from using (13) with $u > 0$ combined with the flip as was done with (12). Fig. 1 shows the double scroll obtained from (13) with the eigenvalue $\gamma = 100$.

The point of this analysis is to conclude that the source of chaos in (11) and (13) can be understood by analyzing a 2-D single scroll which we obtain by considering the limit of (13) as the contracting eigenvalue, $\gamma \rightarrow \infty$. As this happens we see that $z \rightarrow 0.984$ and we get a limiting 2-D single scroll on which all complex dynamics occur. The linear part of the 2-D single scroll is given by

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{bmatrix} 0.0 & -9.876 \\ 1.0 & 0.334 \end{bmatrix} \begin{pmatrix} x - 0.4455 \\ y + 0.054 \end{pmatrix}. \tag{14}$$

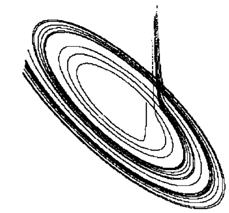

Fig. 1. Transformed SGN double scroll, $\beta = 15.0$, and eigenvalue$= -100.0$.

The nonlinear part is supplied by the condition that we apply the flip map when $\operatorname{sgn}(x - 1.287y + 0.984) < 0.0$.

Equation (14) is solved by

$$\begin{aligned} x(t) &= \exp(\alpha t/2)[(x_0 - a)\cos(\omega t) + C_1 \sin(\omega t)] + a \\ y(t) &= \exp(\alpha t/2)[(y_0 - b)\cos(\omega t) + C_2 \sin(\omega t)] + b \end{aligned} \tag{15}$$

where

$$\begin{aligned} C_1 &= -(0.5\alpha(x_0 - a) + \beta(y_0 - b))/\omega \\ C_2 &= -((x_0 - a) + 0.5(y_0 - b))/\omega \end{aligned}$$

and $\alpha = 0.334, \beta = 0.876, \omega = \sqrt{\beta - (0.5\alpha)^2}$, and a, b are as in (11).

C. 1-D Maps

The single scroll maps the line $y = (x - 0.984)/1.287$ to its image under the flip map. For initial conditions of the form $0.3 \le x \le 0.85$ and $y = (x - 0.984)/1.287$, a segment of this line is mapped into itself. There are two fixed points on this line segment: $(0.54, -0.347)$ and $(0.6875, -0.2293)$. We have now associated (9) with a 1-D map of a segment of the line $y = (x - 0.984)/1.287$ onto itself.

We review how this 1-D map works. We begin with an initial condition on the line $y = (x - 0.984)/1.287$ with the value of the x coordinate in the closed interval [0.3, 0.85]. We use (15) to produce a trajectory which expands outward until it meets the line $y = (x + 0.984)/1.287$. We then apply the flip map, which takes this point on the line $y = (x + 0.984)/1.287$ back to the line $y = (x - 0.984)/1.287$ where the x value will lie in the closed interval [0.3, 0.85]. This flipped point will then be used as the initial conditions for (15) to generate a new trajectory. Hence we see that this line segment is mapped onto itself. From this we conclude that the source of the complexity, or chaos, in (9), (11), and (13) can be traced to a 1-D map.

We may convert (13) into a smooth equation by replacing the "sgn" function by the sigmoid function, (7), thus obtaining a C^∞ vector field whose chaotic properties are closely tied to a given 1-D map as long as γ is large. The possibility that (13) could be reduced to a simple 1-D map was suggested by Prof. Morris Hirsch.

D. Misiurewicz's Single Scroll

The foregoing analysis and the work of Misiurewicz in [5] provides the motivation for introducing the definition of the type-I generalized Chua equations to designate an important class of 3-D autonomous ordinary differential equations very similar in their dynamics to the Chua equations. The key similarity is that this class of ODE's have attractors that look very similar to the double scroll. However, in contrast to the original Chua equations, the nonlinearity in these equations is composed of two linear components, instead of three.

We will define type-I generalized Chua equations in two steps. The first step is to include the following class of double scroll producing equations (motivated by [5]) within the definition of type-I generalized Chua equations:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} s & -1.0 & 0.0 \\ 1.0 & s & 0.0 \\ 0.0 & 0.0 & -\gamma \end{bmatrix} \begin{pmatrix} x - a\,\mathrm{sgn}(u) \\ y - b\,\mathrm{sgn}(u) \\ z - \mathrm{sgn}(u) \end{pmatrix} \tag{16}$$

where $u = z - x$, and a, b are any real constants, and s is a positive constant.

This form of the type-I generalized Chua equations based on the analysis of [5] reveals the role of the 2-D flow that defines the local unstable manifold located at the fixed point a, b, 1). This 2-D flow is given by the equation

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{bmatrix} s & -1.0 \\ 1.0 & s \end{bmatrix} \begin{pmatrix} x - a \\ y - b \end{pmatrix} \tag{17}$$

which defines a source which spirals outward from the critical point (a, b). If, as in [5], we use (17) as the 2-D single scroll in the previous section and in place of the line $y = (x - 0.984)/1.287$ we use the line $x = -1$ as the line at which we apply the flip map, then we obtain a mapping of the line $x = -1$ onto itself and the entire analysis of Misiurewicz is available for analyzing these equations.

The work of Misiurewicz suggests the need for a somewhat more formal definition of single scrolls. We note that [5] implicitly provides the rigorous mathematical definition for the single scroll, but this definition would require some translation for the nonspecialist and so we do not repeat it here.

Let the following differential equation define a vector field on $\boldsymbol{R}^2$:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = F\begin{pmatrix} x \\ y \end{pmatrix} \tag{18}$$

and assume that for any initial condition (x_0, y_0) in $\boldsymbol{R}^2$ this equation has a unique solution defined for all time $-\infty < t < \infty$. Also suppose that for each initial condition of the form $(-1.0, y_0)$ there is a time t at which the solution having this initial condition crosses the vertical line $x = 1$.

Working Definition: 2-D Single Scroll—The 2-D single scroll is defined by the curve formed by starting with an initial condition of the form $(-1, y_0)$ and following the solution of (18) above until it meets the line $x = 1$ and then applying the flip map, and continuing in this manner indefinitely.

When we refer to a 2-D single scroll we mean that we are talking about an ODE with the above characteristics with which we generate an orbit using this ODE and the flip map as described in the above definition.

The single scroll construction can be carried out with any 2-D flow which always crosses the line $x = 1$ given an initial condition on the line $x = -1$. Such flows are easy to construct and we illustrate this by using two familiar equations to construct two examples of new double scrolls.

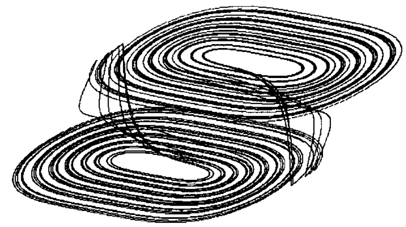

Fig. 2. Double scroll for the type-I generalized Chua equations using the Duffing oscillator as the 2-D vector field for the 2-D single scroll.

Example 1: For the first example we use the following variation on Duffing's equation:

$$\ddot{x} - s\dot{x} + x^3 = 0$$

where $s > 0$ rather than $s \leq 0$, which usually defines Duffing's equation. The result of choosing s positive is to make the critical point of the equation a source rather than a sink.

We rewrite Duffing's equation in a matrix form with the critical point translated to the point (a, b):

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{bmatrix} 0.0 & -1.0 \\ (x-a)^2 & s \end{bmatrix} \begin{pmatrix} x - a \\ y - b \end{pmatrix}. \tag{19}$$

The type-I generalized Chua equation from which we may obtain a double scroll based on this variation on Duffing's equation is given by

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} 0 & -1.0 & 0.0 \\ U & s & 0.0 \\ 0.0 & 0.0 & -\gamma \end{bmatrix} \begin{pmatrix} x - a\,\mathrm{sgn}(u) \\ y - b\,\mathrm{sgn}(u) \\ z - \mathrm{sgn}(u) \end{pmatrix} \tag{20}$$

where $U = (x - a\,\mathrm{sgn}(u))^2$, and where a, b, s, and u are the same as for (16).

Fig. 2 is the double scroll produced by (20). In this figure, $a = -0.5, b = 0.5, s = 0.17$, and $\gamma = 100.0$. The initial conditions are $(-1.0, 0.1, 1.0)$.

Example 2: What we have done with this variation on Duffing's equation we may do with the Van der Pol equation. The translated Van der Pol equation in matrix form is given by

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{bmatrix} 0.0 & -1.0 \\ 1.0 & s(1.2 - (x-a)^2) \end{bmatrix} \begin{pmatrix} x - a \\ y - b \end{pmatrix}. \tag{21}$$

The type-I generalized Chua equation from which we may obtain a double scroll based on the Van der Pol equation is given by

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} 0 & -1.0 & 0.0 \\ 1.0 & U & 0.0 \\ 0.0 & 0.0 & -\gamma \end{bmatrix} \begin{pmatrix} x - a\,\mathrm{sgn}(u) \\ y - b\,\mathrm{sgn}(u) \\ z - \mathrm{sgn}(u) \end{pmatrix} \tag{22}$$

where $U = s(1.2 - (x - a\,\mathrm{sgn}(u))^2)$, and where a, b, s, and u are the same as for (16).

Based on these examples and the work of Misiurewicz we formally define the type-I generalized Chua equations.

Definition (Type-I Generalized Chua Equations): An equation of the form

$$\begin{pmatrix}\dot{x}(t)\\ \dot{y}(t)\\ \dot{z}(t)\end{pmatrix} = \begin{bmatrix}a_{11} & a_{12} & 0.0\\ a_{21} & a_{22} & 0.0\\ 0.0 & 0.0 & -\gamma\end{bmatrix}\begin{pmatrix}x - a\,\mathrm{sgn}(u)\\ y - b\,\mathrm{sgn}(u)\\ z - \mathrm{sgn}(u)\end{pmatrix} \qquad \textbf{(I)}$$

where a, b, and s are any real numbers, $u = z - x$, and each a_{ij} is a function of (x, y) such that the vector field

$$\begin{pmatrix}\dot{x}(t)\\ \dot{y}(t)\end{pmatrix} = \begin{bmatrix}a_{11}(x, y) & a_{12}(x, y)\\ a_{21}(x, y) & a_{22}(x, y)\end{bmatrix}\begin{pmatrix}x - a\\ y - b\end{pmatrix}$$

defines a single scroll in $\boldsymbol{R}^2$, will be called a type-I generalized Chua equation.

III. Type-II Equations

In the previous section we have seen one method for extending the Chua equations based on using 2-D flows as building blocks of double scrolls. In this section we use the analysis of Section II to form a generalization in an entirely different direction.

The role of the sigmoid function and the sgn function in leading to a type-I generalized Chua equation which can be analyzed in terms of a 1-D map suggests that there is another direction of generalization for which, unlike the type-I generalized Chua equations, the only nonlinearities are those of sigmoid, piecewise linear, and sgn functions.

Definition (Type-II Generalized Chua Equations): Let the solutions of the following vector ODE be unique in a bounded region of $\boldsymbol{R}^n$:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}(\boldsymbol{x})(\boldsymbol{x} - F(\boldsymbol{x})) \qquad \textbf{(II)}$$

where $\boldsymbol{A}(\boldsymbol{x})$ is an $n \times n$ matrix function of $\boldsymbol{x}$, and F is a mapping of $\boldsymbol{R}^n$ to itself.

A type-II generalized Chua equation is an ODE of the above form in which the components of the matrix $\boldsymbol{A}$ and the vector function F are composed of finite linear combinations of sigmoid functions, piecewise linear functions, or sgn functions.

In addition to the usefulness of this equation for analysis, there are numerical advantages for modeling and simulation in that we are able to generate simple maps that are easy to evaluate on a computer and which have a wide range of dynamics. In fact, from the following examples we may conclude that we are able to construct a type-II generalized Chua equation having almost any dynamics we desire.

The key to the following constructions is found in how we view the various parts of the type-II generalized Chua equation, **(II)**. In duplicating the dynamics of a given vector field, the matrix $\boldsymbol{A}$ can be chosen to be the linearization of the given vector field at the nonzero fixed points. Hence if $\boldsymbol{x}_0$ is a nonzero fixed point, then the matrix is chosen to be the function of this point, $\boldsymbol{A}(\boldsymbol{x}_0)$, which is the linear part of the vector field at the fixed point $\boldsymbol{x}_0$. Similarly, F is chosen to be a function of the fixed points, $F(\boldsymbol{x}_0)$. For example, if there are just three fixed points, say $\boldsymbol{x}_0$, 0.0, and $-\boldsymbol{x}_0$, a single sigmoid or signum function will likely be sufficient to construct F.

When there are three fixed points, the fixed point at 0.0 need not play a direct role in the creation of chaos as shown by the analysis in Section II. What is important is the presence of at least two fixed points. Since, at this time, there is only the beginnings of a theory for type-II generalized Chua equations, we will use two examples to illustrate a general method for the construction of a dynamical system. Our two examples will be the Rössler dynamical system and the Lorenz dynamical system. What we will do in the next subsection is to construct type-II generalized Chua equations having dynamics very similar to these two systems.

A. The Rössler Dynamics

The Rössler dynamics can be obtained from a type-II generalized Chua equation as follows:

First, we write down the Rössler equations:

$$\begin{pmatrix}\dot{x}(t)\\ \dot{y}(t)\\ \dot{z}(t)\end{pmatrix} = \begin{bmatrix}0.0 & -1.0 & -1.0\\ 1.0 & 0.398 & 0.0\\ 0.0 & 0.0 & -4.0\end{bmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} + \begin{pmatrix}0.0\\ 0.0\\ 2.0 + xz\end{pmatrix}.$$

Next, we determine the fixed points. There are two:

$$x_0 = 2 \pm 1.7899 \quad y_0 = -x_0/0.398 \quad z_0 = -y_0.$$

For reasons of convenience that will be clear shortly we write x_0 as

$$x_0 = 2 \pm \lambda.$$

Conveniently, the coordinates of the two fixed points of the Rössler equation can be expressed by a single parameter, sgn(λ), the sign of λ.

Using the λ notation the linear part of the vector field at these fixed points is given by

$$\boldsymbol{A} = \begin{bmatrix}0.0 & -1.0 & -1.0\\ 1.0 & 0.398 & 0.0\\ (2 \pm \lambda)/0.398 & 0.0 & -2.0 \pm \lambda\end{bmatrix}$$

The function $F(\boldsymbol{x})$ serves to define the fixed points of the type-II generalized Chua equation. As noted earlier it is determined by the fixed points of the Rössler equations. Hence F is given by

$$F(\boldsymbol{x}) = \begin{bmatrix}(2.0 \pm \lambda)\\ -(2.0 \pm \lambda)/0.398\\ (2.0 \pm \lambda)/0.398\end{bmatrix}$$

depending on λ.

One way of writing this in a way that removes the symbol $\pm$ is to use a sigmoid or sgn function. We choose the sigmoid function defined in (7). Doing this we rewrite F as

$$F(\boldsymbol{x}) = \begin{bmatrix}(2.0 + \lambda g(u))\\ -(2.0 + \lambda g(u))/0.398\\ (2.0 + \lambda g(u))/0.398\end{bmatrix}$$

where $g(u)$ is as in (7) and u must now be chosen as a function of (x, y, z) so that the vector field will equal the linear part of the Rössler equation near each nonzero fixed point. In fact the equation we construct will be linear in two regions determined by the two fixed points of the Rössler equations. Unfortunately, the function u must be chosen as $u = x - y^2$, and we recognize that how we arrived at this choice is not straightforward. The analysis of [5] offers the best insights. But due to the elementary level of our present theory we cannot say more at this time. In the next example, the choice of u is more simple to determine. At a later time we anticipate that there will be a complete theory that determines u. The function u may be thought of as a transition function since it defines a surface in three spaces which separates the two fixed points, and for which the equation we are constructing is a different linear equation within each region. In the region where $x > y^2$, the vector field is determined by the fixed point where $x = 2 + \lambda$. When $x < y^2$, the vector field is determined by $x = 2 - \lambda$. As with (16), this surface provides the 2-D surface on which a Poincaré map may be defined. In each of these regions, the matrix $\boldsymbol{A}$ is also determined by these same conditions. One way to write $\boldsymbol{A}$ in a formula without the use of symbol $\pm$ is as follows:

$$\boldsymbol{A} = \begin{bmatrix} 0.0 & -1.0 & -1.0 \\ 1.0 & 0.398 & 0.0 \\ (2 + \lambda g(u))/0.398 & 0.0 & -2.0 + \lambda g(u) \end{bmatrix}$$

Combining F and $\boldsymbol{A}$ according to (**II**) gives the desired type-II generalized Chua equation.

A key factor in this construction is to note that the linear part of the vector field varies with the fixed point, and hence we forced the matrix $\boldsymbol{A}$ to vary accordingly by use of the function $g(u)$, which as before is defined in (7). In doing this we made the matrix $\boldsymbol{A}$ function like the twist matrix used in the twist-and-flip map definition [2].

B. The Lorenz Dynamics

We now produce the Lorenz-like dynamics from a type-II generalized Chua equation. The Lorenz equations are

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{bmatrix} -10.0 & 10.0 & 0.0 \\ 28.0 & -1.0 & 0.0 \\ 0.0 & 0.0 & -2.67 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0.0 \\ -xz \\ xy \end{pmatrix}.$$

The three fixed points are approximately

$$x_0 = y_0 = \pm 8.48, \quad z_0 = 27.0$$

and (0, 0, 0).

The linear part of the vector field for a fixed point is given by

$$\boldsymbol{A} = \begin{bmatrix} -10.0 & 10.0 & 0.0 \\ 1.0 & -1.0 & x_0 \\ -x_0 & -x_0 & -2.67 \end{bmatrix}$$

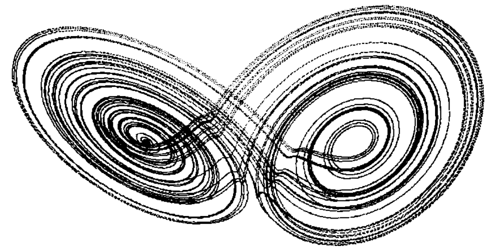

Fig. 3. Attractor for Lorenz-like dynamics in a type-II generalized Chua equation.

and we choose F as

$$F(\boldsymbol{x}) = \begin{bmatrix} x_0 g(x) \\ x_0 g(x) \\ 27.0 \end{bmatrix}.$$

We now generate the nonlinear matrix $\boldsymbol{A}$ from the linear part of the vector field:

$$\boldsymbol{A} = \begin{bmatrix} -10.0 & 10.0 & 0.0 \\ 1.0 & -1.0 & x_0 g(x) \\ -x_0 g(x) & -x_0 g(x) & -2.7 \end{bmatrix}$$

where, as in the Rössler map, $g(x)$ is given by (7), and $\gamma = 3.0$. Fig. 3 is the attractor for this map.

In this case, we were able to take $u = x$ since the transition from one linear region to the other takes place when $x = 0$, that is the surface of transition is the $y - z$ plane, and provides the natural surface for defining the Poincaré map. It is routine to replace the sigmoid with the sgn function and obtain a completely piecewise linear Lorenz equation. In this equation, the piecewise linear one, it should be possible to find the 1-D map that describes its dynamics numerically, and very possibly analytically. Its similarity to the 1-D map obtained from the axiomatic Lorenz equations will be of interest to determine.

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References

[1] R. Brown, "Generalizing the Chua Equations," *Int. J. Bifurcation and Chaos,* vol. 2, no. 4, 1992.

[2] R. Brown and L. Chua, "Generalizing the twist and flip paradigm," *Int. J. Bifurcation and Chaos,* vol. 1, no. 2, pp. 385–416, 1991.

[3] L. O. Chua, "The genesis of Chua's circuit," *Arch. Elektron. Ubertragungstech.,* vol. 46, no. 4, pp. 250–257, 1992.

[4] L. O. Chua, M. Komuro, and T. Matsumato, "The double scroll family," *IEEE Trans. Circuits Syst.,* vol. CAS-33, pp. 1073–1118, 1986.

[5] M. Misiurewicz, "Unimodal interval maps obtained from the modified Chua equations," *J. Circuits, Syst. Comput.,* (Special Issue on the Chua Circuit: A Paradigm for Chaos), 1993.

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