Problem 1: (13 points) Consider the situation pictured above.

a) (10 points) Derive the range \( d \) for this projectile in terms of \( \theta \), \( v \), \( g \), and \( h \).

b) (2 points) What angle \( \theta \) maximizes \( d \)? (If \( h = 0 \))

\[ \text{(a) } \begin{align*}
&1 \text{ Calculate Time of Flight } t \\
&y_0 = 0 \quad v_{oy} = v_0 \sin \theta \quad a_y = -g \\
&y = -h \quad y = y_0 + v_{ox}t + \frac{1}{2} a_y t^2 \\
&-h = v_0 \sin \theta t - \frac{1}{2} g t^2 \\
&t = \frac{v_0 \sin \theta \pm \sqrt{(v_0 \sin \theta)^2 - 4 \left(\frac{1}{2}g\right)(-h)}}{2 \left(\frac{1}{2}g\right)} = \frac{v_0 \sin \theta \pm \sqrt{(v_0 \sin \theta)^2 + 2gh}}{g}
\end{align*} \]

\[ \text{only the } + \text{ is valid (otherwise } t < 0) \]

\[ \text{(b) Find Distance traveled in } x \text{ over time } t. \]

\[ x_0 = 0 \quad v_{ox} = v_0 \cos \theta \quad a_x = 0 \]

\[ x = d \quad x = x_0 + v_{ox}^2 + \frac{1}{2} a_x t^2 \\
\]

\[ d = v_0 \cos \theta \left[ \frac{v_0 \sin \theta + \sqrt{(v_0 \sin \theta)^2 + 2gh}}{g} \right] \]

\[ \sin \theta \cos \theta = \frac{1}{2} \sin 2\theta \]

\[ d = \frac{1}{2} v_0^2 \sin 2\theta + \frac{v_0 \cos \theta \sqrt{(v_0 \sin \theta)^2 + 2gh}}{g} \]
b) \( d \big|_{h=0} = \frac{1}{2} \frac{V_0^2 \sin 2\theta}{g} + \frac{V_0 \cos \theta \sqrt{V_0^2 \sin^2 \theta + g}}{g} \)

\[
= \frac{1}{2} \frac{V_0^2 \sin 2\theta}{g} + \frac{1}{2} \frac{V_0^2 \sin 2\theta}{g}
\]

\[d = \frac{V_0^2 \sin 2\theta}{g}\]

To maximize, find \( \frac{d(d)}{dt} = 0 \)

\[\frac{d(d)}{dt} = \frac{2V_0^2 \cos 2\theta}{g} = 0\]

\[\cos 2\theta = 0\]

\[2\theta = \cos^{-1}(0) = \frac{\pi}{2}\]

\[\theta = \frac{\pi}{4} \text{ rad or } 45^\circ\]

Alternatively, note that \( \frac{V_0^2 \sin 2\theta}{g} \) is maximized when \( \sin (2\theta) = 1 \) (when \( \sin 2\theta \) is maximized). You know this occurs when \( \theta = \frac{\pi}{4} \) rad or \( 45^\circ \).
Problem 2: (13 points) You are an amusement park engineer. Your coworker claims to have improved the safety of the popular cylinder spinner by banking the sides by angle \( \theta \) as pictured.

a) (7 points) What is the angular velocity \( \omega_0 \) for the ride that would allow the cylinder to be made from a frictionless surface?

\[ \Sigma F_R = m \alpha_R = m \omega_0^2 r \]

\[ a_R = \frac{v^2}{r} = \omega_0^2 r \]

b) (6 points) What is the range of \( \omega \) that is safe if the ramp has coefficient of friction \( \mu_s \)?

a) Let the cylinder be made from a frictionless surface.

\[ \Sigma F_R = F_N \cos \theta = m \omega_0^2 r \]

We have to solve for \( F_N \)

\[ \Sigma F_y = 0 \Rightarrow F_N \cos \theta - mg = 0 \]

\[ mg = F_N \cos \theta \]

\[ F_N = \frac{mg}{\cos \theta} \]

\[ \Sigma F_R = \frac{mg}{\cos \theta} \sin \theta = m \omega_0^2 r \]

\[ \omega_0 = \sqrt{\frac{g}{r} \tan \theta} \]
b) Now we consider the situation with friction.

Force of friction will oppose the rider's motion from sliding up or down. Rider slides up if \( W \) is too fast. Rider slides down if \( W \) is too slow.

\( W \) is buffered by the friction force \( \mu_s F_N \) in each case, so the two situations are symmetric.

In each case, \( F_{fr} = F_N \mu_s \), let's examine one case closely.

Now, \( \Sigma F_y = F_N \cos \theta - F_{fr} \mu_s \sin \theta - mg = 0 \)

\[ F_N = \frac{mg}{\cos \theta - \mu_s \sin \theta} \]

Friction is contributing to \( F_{fr} \)!

\[ \Sigma F_x = F_N \sin \theta \pm F_{fr} \cos \theta = mW^2r \]

\[ = F_N \left( \sin \theta \pm \mu_s \cos \theta \right) = mW^2r \]

\( W \) will be the maximum and minimum allowable speeds.

\[ \frac{mg}{\cos \theta - \mu_s \sin \theta} \left( \sin \theta \pm \mu_s \cos \theta \right) = mW^2r \]

\[ r \left( \frac{W}{r} \right)^2 = \left[ \frac{\tan \theta \pm \mu_s}{1 - \mu_s \tan \theta} \right] \left( \frac{1}{\cos \theta} \right) \]

(to simplify...)

\[ \sqrt{\frac{\tan \theta \pm \mu_s}{1 - \mu_s \tan \theta}} < W < \sqrt{\frac{\tan \theta \pm \mu_s}{1 - \mu_s \tan \theta}} \]
**Problem 3:** (13 points) Three blocks are stacked on top of a frictionless table. The top and bottom blocks are attached by a string through a pulley with negligible mass. The coefficient of friction between blocks A and B is $\mu_A$ and between blocks B and C is $\mu_C$.

a) (11 points) When a force with magnitude $F_p$ is applied to the middle block, all three blocks accelerate. What is the acceleration of each block?

b) (2 points) What is the ratio $\frac{\mu_A}{\mu_C}$ that allows the middle block to be removed without blocks A or C accelerating?

\[
\begin{align*}
\Sigma F_y &= F_{NA} - m_A g = 0 \\
F_{NA} &= m_A g \\
\Sigma F_x &= \mu_A F_{NA} - T = m_A a_A \\
m_A a_A &= \mu_A m_A g - T \\
\Sigma F_y &= F_{NB} - F_{NA} - m_B g = 0 \\
F_{NB} &= (m_A + m_B) g \\
\Sigma F_x &= F_p - \frac{F_c}{m_B} - \frac{F_c}{m_C} = m_B a_B \\
& \text{friction b/w C+B due to normal force between them!}
\end{align*}
\]

\[
\begin{align*}
m_B a_B &= F_p - \mu_A m_A g - \mu_C (m_A + m_B) g \\
\frac{a_B}{m_B} &= \frac{F_p - \mu_A m_A g - \mu_C (m_A + m_B) g}{m_B}
\end{align*}
\]

\[
\begin{align*}
\Sigma F_y &= F_{NC} - F_{NB} - m_C g = 0 \\
F_{NC} &= (m_A + m_B + m_C) g \\
\Sigma F_x &= m_C F_{NB} - T = m_C a_C \\
& \text{friction is b/w C+B!} \quad \frac{m_C (m_A + m_B) g - T = m_C a_C}{-T = m_C a_C}
\end{align*}
\]

Since $m_A$ and $m_C$ connected by string, $|\vec{a}_A| = |\vec{a}_C| = a$

We now have a system of equations 1 and 2 with unknowns $T$ and $a$:

1. $T = \mu_A m_A g - m_A a$
2. $-T = -\mu_C (m_A + m_B) g - m_C a$

\[
0 = \mu_A m_A g - \mu_C (m_A + m_B) g + (m_C - m_A) a
\]

\[
a = \frac{a_A = a_C}{m_C - m_A}
\]

\[
a_A : a_C = \frac{-\mu_A m_A g + \mu_C (m_A + m_B) g}{m_C - m_A}
\]
b) Set $a_A = a_C = 0$

\[-u_A m_A g + \frac{u_C (m_A + m_B) g}{m_C - m_A} = 0\]

\[\Rightarrow u_C (m_A + m_B) g = u_A m_A g\]

\[\frac{u_A}{u_C} = \frac{m_A + m_B}{m_A}\]
Problem 4: (12 points) Newton's cannon is sitting on a perch \( r_0 \) above the surface of the earth. The cannon is fired horizontally with initial velocity \( v_0 \) so that its cannonball orbits earth at a constant height \( r_0 \), eventually hitting the cannon again.

a) (6 points) Express \( v_0 \) as a function of \( G, r_E, r_0, \) and \( m_E \).

b) (3 points) By what factor must \( v_0 \) be increased if \( m_E \) doubles and \( r_0 \) stays constant?

c) (3 points) The mass of the earth given its density \( \rho_E \) is \( m_E = \rho_E \frac{4}{3} \pi r_E^3 \). By what factor must \( v_0 \) be increased if the radius of the earth doubles and \( \rho_E \) and \( r_0 \) stay the same?

\[ a_R = \frac{v_0^2}{(r_0 + r_E)} \]

\[ F_G = G \frac{m_{\text{ball}} m_E}{(r_0 + r_E)^2} = m_{\text{ball}} a_R = m_{\text{ball}} \frac{v_0^2}{(r_0 + r_E)} \]

\[ V_0 = \sqrt{\frac{G}{(r_0 + r_E)}} \]

b) If \( m_E \) doubles, see by expression above that \( V_0 \) increases by a factor of \( \sqrt{2} \)

c) If \( m_E = \rho_E \frac{4}{3} \pi r_E^3 \) and \( r_{\text{new}} = 2r_E \)

\[ m_{E \text{ new}} = \rho_E \frac{4}{3} \pi (2r_E)^3 = 8 \rho_E \frac{4}{3} \pi r_E^3 = 8m_E \]

If \( m_E \) \( \uparrow \) by factor of 8, and \( r_0 + r_E \) \( \uparrow \) by 2, we get \( V_0 \) \( \uparrow \) by \( \sqrt{8/2} = \sqrt{4} = 2 \)
Good Luck at Berkeley!