2. a. Series Filters

In general, use the voltage divider rule!

\[
\frac{V_o}{V_i} = \frac{Z_2}{Z_1 + Z_2}
\]

In contention for the most important useful ECE formula you will learn as an undergrad. (Some say Maxwell's, some say Euler's Formula. These people are squares)

Using \( z\)'s

<table>
<thead>
<tr>
<th>( V_o ) across R</th>
<th>( V_o = \frac{Z_R}{Z_R + Z_C} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V_o ) across C</td>
<td>( V_o = \frac{Z_C}{Z_R + Z_C} )</td>
</tr>
</tbody>
</table>

At \( s = 0 \), \( Z_c \) is \( \infty \) (open), no current, so \( V_o = 0 \).
At \( s = \infty \), \( Z_c \) is 0 (short), so \( V_o \) drops across R.

This is a **HPF**

Substituting \( s \)

| \( Z_R = R \) |
| \( Z_C = \frac{1}{CS} \) |

\[
\frac{V_o}{V_s} = \frac{R}{R + \frac{1}{CS}} \quad \text{cs} = \frac{RCS}{RCS + 1} \quad \frac{1}{RC}
\]

\[
\frac{V_o}{V_s} = \frac{S}{S + \frac{1}{RC}} \quad \text{pole at} \quad s = -\frac{1}{RC}
\]

\[
\left. \frac{V_o}{V_s} \right|_{s=0} = \frac{0}{0 + \frac{1}{RC}} = 0
\]

\[
\left. \frac{V_o}{V_s} \right|_{s=\infty} = \frac{1}{\infty} = 1 \quad \Rightarrow \quad \text{HPF}
\]

\[
\left. \frac{V_o}{V_s} \right|_{s=\frac{1}{RC}} = \text{pole at} \quad s = -\frac{1}{RC}
\]

\[
\left. \frac{V_o}{V_s} \right|_{s=0} = 1 \quad \Rightarrow \quad \text{LPF}
\]

\[
\left. \frac{V_o}{V_s} \right|_{s=\infty} = 0 \quad \Rightarrow \quad \text{LPF}
\]
What did this teach us about 1-pole circuits?

**Low pass:** \( \frac{P}{s+P} \) \( \rightarrow \) For DC gain of 1, time constant \( \frac{1}{p} \)

**High pass:** \( \frac{s}{s+P} \) \( \rightarrow \) \( p = \text{pole} \)

If we want a bandpass filter, we need another pole to allow for resonance effects... enter the RLC!
Using $Z_s$

**RLC**

\[ \frac{V_o}{V_s} = \frac{Z_c}{Z_s + Z_L + Z_c} \]

- **Vo across R**
  - At $s=0$, $Z_c = 0$, so $V_o = 0$
  - At $s=\infty$, $Z_c = \infty$, so $V_o = 0$
  - There is a nonzero response in between, so we have [BPF]

- **Vo across C**
  - At $s=0$, $Z_c = 0$, so $V_o/V_s = 1$
  - At $s=\infty$, $Z_c = \infty$, so $V_o/V_s = 0$
  - This is a [LPF]

- **Vo across L**
  - At $s=0$, $Z_L = 0$, so $V_o/V_s = 0$
  - At $s=\infty$, $Z_L = \infty$, so $V_o/V_s = 1$

**Substituting $s$**

\[ s = \frac{R}{Z_c} \]
\[ s = \frac{L}{R} \]
\[ s = \frac{1}{L} \]

\[ \frac{V_o}{V_s} = \frac{sZ_c}{sZ_s + sZ_L + sZ_c} \]

- **When $s=0$, $V_o/V_s = 0$**
- **When $s=\infty$, $V_o/V_s = 0$**

Notice the denominator is the same for all 3 cases:

\[ s^2 + \frac{R}{L} s + \frac{1}{LC} \quad \text{OR} \quad s^2 + (\omega_0/Q)s + \omega_0^2 \]

This is because the circuit topology and thus its dynamics (poles) is always the same. What changes is which element we tap for an output.

Since these are passive circuits, $V_o/V_s$ will never be greater than 1 (i.e., max DC gain = 1)
Parallel Filters

In general, use the current divider rule:

\[ I_0 = \frac{V_1}{Z_1} = I_1 \left( \frac{1}{Z_2 + Z_b} \right) \]

\[ I_0 = \frac{V_1}{Z_1} = I_1 \left( \frac{1}{Z_2 + Z_b} \right) \]

Let \( Y = \frac{1}{Z} \)

This is the dual of the voltage divider rule.

Using \( Y \)'

<table>
<thead>
<tr>
<th>Substituting ( S )</th>
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<tbody>
<tr>
<td><strong>RC</strong></td>
</tr>
<tr>
<td>Through R</td>
</tr>
<tr>
<td>[ I_o = I_s \to \frac{1}{Y_s} + Y_c ]</td>
</tr>
<tr>
<td>At ( s = \infty ), ( Y_c ) is open, ( I_o = I_s )</td>
</tr>
<tr>
<td>At ( s = 0 ), ( Y_c ) is short, ( I_o = 0 )</td>
</tr>
<tr>
<td>This is a LPF</td>
</tr>
<tr>
<td><strong>RLC</strong></td>
</tr>
<tr>
<td>Through R</td>
</tr>
<tr>
<td>[ I_o = I_s = \frac{1}{Y_s} + Y_c ]</td>
</tr>
<tr>
<td>At ( s = \infty ), ( Y_c ) is open, ( I_o = I_s )</td>
</tr>
<tr>
<td>At ( s = 0 ), ( Y_c ) is short, ( I_o = 0 )</td>
</tr>
<tr>
<td>This is a BPF</td>
</tr>
<tr>
<td>Through C</td>
</tr>
<tr>
<td>[ I_o = I_s = \frac{1}{Y_s} + Y_c ]</td>
</tr>
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<td>At ( s = \infty ), ( I_o = I_s ) (C is open)</td>
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</table>
4. Remember \( Y(s) = T(s) \cdot X(s) \)

Step 1: \( \mathcal{L}(x(t)) = X(s) \)
Step 2: \( Y(s) = X(s) \cdot T(s) \)
Step 3: \( y(t) = \mathcal{L}^{-1}(Y(s)) \)

\[
Y(s) = \left( \frac{V_a}{s} \right) \left( \frac{\frac{P_I}{s+P_I}}{s+P_I} \right)
\]

\[
\Rightarrow A = \frac{V_a}{s}
\]

\[
B = \frac{V_a P_I}{s+P_I}
\]

\[
-y(t) = \frac{V_a}{s} + \frac{-V_a}{s+P_I}
\]

\[
\mathcal{L}^{-1}(y(s)) = V_a u(t) - V_a e^{-P_I t} u(t)
\]

\[
\therefore y(t) = V_a (1-e^{-P_I t}) u(t)
\]

**steady state** \( \downarrow \)

Here, the **LPF** allows \( V_a u(t) \)

to pass with gain of 1.

The transient dies out eventually,
and the total solution approaches
the steady state.

Imagine input is \( V_a e^{st} \), where

\( s = 0 \). To get steady state gain,

find \( T, (0) = 1 \).

\[
T_1(s) = \frac{P_I}{s+P_I} \leftarrow \text{LPF}
\]

\[
T_2(s) = \frac{S}{s+P_I} \leftarrow \text{HPF}
\]

\[
Y(s) = \left( \frac{V_a}{s} \right) \left( \frac{\frac{S}{s+P_I}}{s+P_I} \right)
\]

\[
\Rightarrow A = \frac{V_a}{s}
\]

\[
B = \frac{V_a P_I}{s+P_I}
\]

\[
-y(t) = \frac{V_a}{s} + \frac{-V_a}{s+P_I}
\]

\[
\mathcal{L}^{-1}(y(s)) = V_a e^{-P_I t} u(t)
\]

\[
\therefore y(t) = V_a e^{-P_I t} u(t) \quad (\uparrow)
\]

Here, the **HPF** blocks \( V_a u(t) \),

so the steady state is 0.

The response that is left is

the transient that dies out due

to the applied step.

Imagine input is \( V_a e^{st} \), where

\( s = 0 \). To get steady state gain,

find \( T_2(0) = 0 \).
b) \( X(t) = V_a e^{-\alpha t} \rightarrow X(s) = \frac{V_a}{s+\alpha} \)

\[ T_1(s) \text{ (LPF)} \]

\[ Y(s) = \frac{V_a}{s+\alpha} \cdot \frac{P_i}{s+P_i} \]

\[ \Rightarrow \frac{A}{s+\alpha} + \frac{B}{s+P_i} = \frac{V_a P_i}{(s+\alpha)(s+P_i)} \]

\[ \Rightarrow A(s+\alpha) + B(s+P_i) = V_a P_i \]

\[ \begin{cases} (A+B)s = 0 & \text{of} \quad s \\ A P_i + B \alpha = V_a P_i \end{cases} \]

\[ \therefore A = -\frac{V_a P_i}{\alpha - P_i} \]
\[ B = \frac{V_a P_i}{\alpha - P_i} \]

\[ Y(s) = \frac{-V_a P_i}{(s+\alpha)(\alpha - P_i)} + \frac{V_a P_i}{(s+P_i)(\alpha - P_i)} \]

\[ \begin{bmatrix} Y(s) \end{bmatrix} = \frac{V_a P_i}{\alpha - P_i}[e^{-\alpha t} + e^{-P_it}] \]

\[ \therefore \begin{cases} y(t) = \frac{V_a P_i}{P_i - \alpha}[e^{-\alpha t} - e^{-P_it}]u(t) \\ \text{steady state transient} \end{cases} \]

Here, \( V_a e^{-\alpha t} \) is passed with gain \( T_1(-\alpha) = \frac{P_i}{P_i - \alpha} \). Since initial conditions are 0, the transient has opposite amplitude, such that \( y(t) = 0 \) when \( t = 0 \).

\[ T_2(s) \text{ (HPF)} \]

\[ Y(s) = \left( \frac{V_a}{s+\alpha} \right) \left( \frac{s}{s+P_i} \right) \]

\[ \Rightarrow \frac{A}{s+\alpha} + \frac{B}{s+P_i} = \frac{V_a P_i}{(s+\alpha)(s+P_i)} \]

\[ \Rightarrow A(s+\alpha) + B(s+P_i) = V_a P_i \]

\[ \begin{cases} (A+B)s = (V_a) s \\ A P_i + B \alpha = 0 \end{cases} \]

\[ \therefore A = -\frac{V_a \alpha}{P_i - \alpha} \]
\[ B = \frac{V_a P_i}{P_i - \alpha} \]

\[ Y(s) = \frac{V_a}{P_i - \alpha} \left[ \frac{-\alpha}{s+\alpha} + \frac{P_i}{s+P_i} \right] \]

\[ \begin{bmatrix} Y(s) \end{bmatrix} = \frac{V_a}{P_i - \alpha}[\alpha e^{-\alpha t} + P_i e^{-P_it}] \]

\[ \therefore \begin{cases} y(t) = \frac{V_a}{P_i - \alpha}[\alpha e^{-\alpha t} + P_i e^{-P_it}]u(t) \\ \text{steady state transient} \end{cases} \]

Here, \( V_a e^{-\alpha t} \) is passed with gain \( T_2(-\alpha) = -\frac{\alpha}{P_i - \alpha} \). Since initial conditions are 0, the transient has opposite amplitude, such that \( y(t) = 0 \) when \( t = 0 \).

(High pass filters will pass an instantaneous change in input, such as applying \( V_a e^{-\alpha t} \) at \( t = 0 \).)
c) \( X(t) = V_a \delta(t) \rightarrow X(s) = V_a \cdot 1 = V_a \)

\[ T_1(s) \text{ (LPF)} \]

\[
Y(s) = V_a \frac{P_1}{s + P_1} \\
\mathcal{L}^{-1}[Y(s)] = V_a p_1 e^{-p_1 t} u(t) \\
\therefore y(t) = V_a p_1 e^{-p_1 t} u(t) \quad \uparrow \text{transient} \quad \uparrow \text{steady state}
\]

The LPF blocks the impulse, then its transient dies down. This is the typical LPF unit impulse response.

\[ h(s) \]

\[ \text{V}_a \]

\[ + \]

\[ T_2(s) \text{ (HPF)} \]

\[
Y(s) = V_a \frac{s}{s + P_1} \quad \text{decomposition}... \\
= V_a \left(1 - \frac{P_1}{s + P_1}\right) \\
\mathcal{L}^{-1}[Y(s)] = V_a \delta(t) - V_a p_1 e^{-p_1 t} u(t) \\
\therefore y(t) = V_a \left[\delta(t) - p_1 e^{-p_1 t}\right] u(t) \quad \uparrow \text{steady state} \quad \uparrow \text{transient}
\]

The HPF passes the impulse, then its transient dies down. This is the typical HPF unit impulse response.
\[ x(t) = V_a \cos \omega t \]

In general, sinusoids have the form

\[ x(t) = V_a \cos (\omega t + \phi) \]

which can be written as

\[ V_a(\cos \omega t \cos \phi - \sin \omega t \sin \phi) \]

Take the Laplace transform \( \rightarrow \)

\[ X(s) = \frac{V_a}{s^2 + \omega^2} \]

In this case, \( \phi = 0 \), so

\[ X(s) = \frac{V_a}{s^2 + \omega^2} \]

Use the result in section 11-5, equation (11-21) that forced response is:

\[ \mathcal{L}[\cos(\omega t + \phi)] = \frac{s \cos \phi - \omega \sin \phi}{s^2 + \omega^2} \]

and the residues \( K + K' \) come from

\[ \mathcal{L}[\cos(\omega t)] = \frac{s}{s^2 + \omega^2} \]

\[ T_1(s) \quad (LPP) \]

\[ Y(s) = \frac{V_a}{s^2 + \omega^2} \cdot \frac{P_1}{s + P_1} \]

\[ \Rightarrow \quad \frac{K}{s + \omega^2} + \frac{K'}{s + \omega^2} \]

\[ \text{Since } \quad K = \frac{1}{2} V_a |T_1(j\omega)| e^{j\theta} = \frac{1}{2} V_a \left[ \frac{1}{j\omega P_1} \right] e^{j\theta} \]

\[ = \frac{1}{2} V_a \frac{1}{j\omega P_1} \cos(\omega t + \theta) \]

Steady state sinusoidal part is:

\[ y_\omega(t) = K'' \left[ \frac{s^2 P_1}{(s+j\omega)^2} \right] \]

\[ = V_a \frac{1}{s^2 P_1} \cos(\omega t + \theta) \]

To find residue \( A \), better just use the initial value theorem,

\[ y(0) = \lim_{s \to \infty} sY(s) = \lim_{s \to \infty} \frac{s^2 P_1}{(s+j\omega)^2} V_a = 0 \]

This means the transient, which is of the form \( Ae^{-Pt} \), cancels the steady state output at \( t = 0 \).

\[ y_p(0) + A = 0 \quad \Rightarrow \quad A = -y_p(0) \]

\[ y(t) = \frac{P_1}{J \omega P_1} \cos(\omega t + \theta) - \frac{P_1}{J \omega P_1} \cos(\omega t + \theta) e^{-Pt} \]

\[ T_2(s) \quad (HPP) \]

\[ Y(s) = \frac{V_a}{s^2 + \omega^2} \cdot \frac{s}{s + P_1} \]

\[ \Rightarrow \quad \frac{K}{s + \omega^2} + \frac{K'}{s + \omega^2} \]

\[ \text{Since } \quad K = \frac{1}{2} V_a |T_1(j\omega)| e^{j\theta} = \frac{1}{2} V_a \left[ \frac{1}{j\omega P_1} \right] e^{j\theta} \]

\[ = \frac{1}{2} V_a \frac{1}{j\omega P_1} \cos(\omega t + \theta) \]

This makes steady state output:

\[ y_\omega(t) = \frac{1}{\omega P_1} \left[ \frac{k}{s + \omega^2} + \frac{k'}{s + \omega^2} \right] \]

\[ = V_a \frac{1}{s^2 P_1} \cos(\omega t + \theta) \]

To find residue \( A \), better just use the initial value theorem.

\[ y(0) = \lim_{s \to \infty} sY(s) \]

\[ = \lim_{s \to \infty} \frac{s^2 P_1}{s + P_1} V_a = 0 \]

This means the transient at \( t \to 0 \) adds to \( y_p(0) \) to equal \( V_a \)

Therefore,

\[ A = V_a - y_p(0) \]

\[ y(t) = \frac{V_a}{s^2 P_1} \cos(\omega t + \theta) - \frac{V_a}{s^2 P_1} \cos(\omega t + \theta) e^{-Pt} \]

\[ + A e^{-Pt} \]