Segmentation of Subspace Arrangements II – GPCA

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Representation of Subspace Arrangements

1. For a single subspace $V \subset \mathbb{R}^D$, if $\dim(V) = d$ and $r = D - d$:

$$(\mathbf{u}_1^T\mathbf{z} = 0) \wedge (\mathbf{u}_2^T\mathbf{z} = 0) \wedge \cdots \wedge (\mathbf{u}_r^T\mathbf{z} = 0) \Leftrightarrow \begin{cases} \mathbf{u}_1^T\mathbf{z} = 0 \\ \vdots \\ \mathbf{u}_r^T\mathbf{z} = 0 \end{cases}.$$

2. For a subspace arrangement $\mathcal{A} = V_1 \cup V_2 \cup \cdots \cup V_K$,

$$(V_1^{\perp T}\mathbf{z} = 0) \vee (V_2^{\perp T}\mathbf{z} = 0) \vee \cdots \vee (V_K^{\perp T}\mathbf{z} = 0).$$

This constraint can also be written as a system of polynomial constraints:

Example (Hyperplane Arrangements)

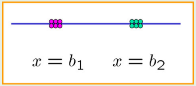
- A hyperplane is a subspace of $D - 1$ dimension $\Rightarrow r = 1$.
- For a hyperplane arrangement $\mathcal{A} = V_1 \cup V_2 \cup \cdots \cup V_K$:

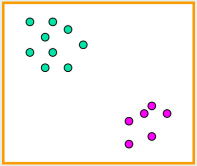
$$\begin{array}{rl} & (\mathbf{u}_{1,1}^T\mathbf{z} = 0) \vee (\mathbf{u}_{2,1}^T\mathbf{z} = 0) \vee \cdots \vee (\mathbf{u}_{K,1}^T\mathbf{z} = 0), \\ \Rightarrow & (\mathbf{u}_{1,1}^T\mathbf{z})(\mathbf{u}_{2,1}^T\mathbf{z}) \cdots (\mathbf{u}_{K,1}^T\mathbf{z}) = 0. \end{array}$$

- $p(\mathbf{z}) \doteq (\mathbf{u}_{1,1}^T\mathbf{z})(\mathbf{u}_{2,1}^T\mathbf{z}) \cdots (\mathbf{u}_{K,1}^T\mathbf{z})$ is degree-K homogeneous polynomial.
- For any polynomial vanishing on $\mathcal{A}$ (i.e., $\forall \mathbf{z} \in \mathcal{A}, p'(\mathbf{z}) = 0$), $p'(\mathbf{z}) = p(\mathbf{z})g(\mathbf{z})$ for some polynomial g.

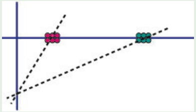
Example (Point Clusters)

- Point clusters can be treated as zero-dimensional affine subspaces.





- Recall the standard procedure: Homogenization.
  For the 1-D case, change the sample coordinates to: $\mathbf{x} = [x, 1]^T$.



- All noise-free samples $\mathbf{z} = [x, y]^T \in V_1 \cup V_2$ satisfy:

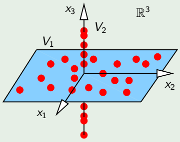
$$p(\mathbf{z}) \doteq (x - b_1 y)(x - b_2 y) = 0.$$

Kth Degree Vanishing Polynomials

Example (De Morgan's Law)

Let $\mathcal{A} = V_1 \cup V_2 \subset \mathbb{R}^3$, $\dim(V_1) = 2$, $\dim V_2 = 1$.

$$
\begin{aligned}
& (\mathbf{u}_{1,1}^T \mathbf{z} = 0) \vee \left\{ (\mathbf{u}_{2,1}^T \mathbf{z} = 0) \wedge (\mathbf{u}_{2,2}^T \mathbf{z} = 0) \right\} \\
\Rightarrow \quad & \left\{ (\mathbf{u}_{1,1}^T \mathbf{z} = 0) \vee (\mathbf{u}_{2,1}^T \mathbf{z} = 0) \right\} \wedge \left\{ (\mathbf{u}_{1,1}^T \mathbf{z} = 0) \vee (\mathbf{u}_{2,2}^T \mathbf{z} = 0) \right\} \\
\Rightarrow \quad & \begin{cases} p_1(\mathbf{z}) \doteq (\mathbf{u}_{1,1}^T \mathbf{z})(\mathbf{u}_{2,1}^T \mathbf{z}) = 0 \\ p_2(\mathbf{z}) \doteq (\mathbf{u}_{1,1}^T \mathbf{z})(\mathbf{u}_{2,2}^T \mathbf{z}) = 0 \end{cases}
\end{aligned}
$$



- De Morgan's law: The constraint for a subspace arrangement can be rewritten as

$$
({V_1^\perp}^T \mathbf{z} = 0) \vee ({V_2^\perp}^T \mathbf{z} = 0) \vee \cdots \vee ({V_K^\perp}^T \mathbf{z} = 0) \Leftrightarrow \begin{cases} p_1(\mathbf{z}) = 0 \\ \vdots \\ p_l(\mathbf{z}) = 0 \end{cases},
$$

  where $\deg(p_i) = K$, and $l \doteq r_1 r_2 \cdots r_K$.

- $p_1, \cdots, p_l$ are all Kth degree homogeneous polynomials. Define

$$
p' = c_1 p_1 + c_2 p_2 + \cdots + c_l p_l,
$$

  then $\forall \mathbf{z} \in \mathcal{A}$, $p'(\mathbf{z}) = 0$.

  Any scalar combination also vanishes on $\mathcal{A}$.

Vanishing Polynomials

Kth Component of Vanishing Polynomials

- Denote $J_K = \text{Span}(p_1, p_2, \cdots, p_l)$. J_K is a polynomial subspace.
- $h \doteq \dim(J_K)$, $h \leq l = r_1 r_2 \cdots r_K$.
- Completeness: Given Kth degree homogeneous f, if $\forall \mathbf{z} \in \mathcal{A}, f(\mathbf{z}) = 0$, then $f \in J_K$.

Vanishing Polynomials

- Define a vanishing polynomial of $\mathcal{A}$ as $f(\mathbf{z}) = 0$ for all $\mathbf{z} \in \mathcal{A}$.
- Is it possible that $\deg(f) > K$?
- Is it possible that $\deg(f) < K$?
- Particularly, given $p \in J_K$, $g(\mathbf{z})p(\mathbf{z}) = 0 \Rightarrow gp$ is a vanishing polynomial.
- All vanishing polynomials form a special polynomial set $I_{\mathcal{A}}$, called an *ideal* in algebra.

Properties of Vanishing Polynomials

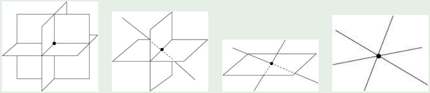
- $\mathcal{A}$ and $I_{\mathcal{A}}$ are completely determined by J_K.
- **[Derksen, 2005]** *If* $\mathcal{A} = V_1 \cup \ldots \cup V_K$ *is in general position, then,*

$$h \doteq \dim(J_K) = \sum_S (-1)^{|S|} \binom{K + D - 1 - c_S}{D - 1 - c_S},$$

where $c_S = \sum_{j \in S} c_j$ *and the sum is over all* $S \subseteq \{1, \ldots, n\}$ *(including the empty set) for which* $c_S < D$.

Example ($\dim(J_K)$ for three subspaces in $\mathbb{R}^3$)

From Derksen's equation, $\dim(J_3(\mathcal{A}))$ can only take four possible values:



r_1	r_2	r_3	h
1	1	1	1
1	1	2	2
1	2	2	4
2	2	2	7

Veronese Map

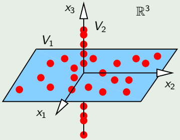
Since J_K completely determines $\mathcal{A}$, we consider estimating J_K from samples.

- J_K is a polynomial subspace, hence only need to get hold of a set of basis vectors.
- The basis vectors for J_K are homogeneous Kth degree polynomials that are *linearly independent*.

Example (Basis Vectors of J_K)

1. For hyperplane arrangements, $\dim(J_K) = 1$,
$$p(\mathbf{z}) = (\mathbf{u}_{1,1}^T \mathbf{z})(\mathbf{u}_{2,1}^T \mathbf{z}) \cdots (\mathbf{u}_{K,1}^T \mathbf{z}).$$
$\Rightarrow J_K = \text{Span}(p)$.

2. Let $\mathcal{A} = V_1 \cup V_2$,
then $\mathbf{u}_{1,1} = [0,0,1]^T$, $\mathbf{u}_{2,1} = [1,0,0]^T$, $\mathbf{u}_{2,2} = [0,1,0]^T$, and $\dim(J_2) = 2$.
$$p_1 = (\mathbf{u}_{1,1}^T \mathbf{z})(\mathbf{u}_{2,1}^T \mathbf{z}) = x_1 x_3,$$
$$p_2 = (\mathbf{u}_{1,1}^T \mathbf{z})(\mathbf{u}_{2,2}^T \mathbf{z}) = x_2 x_3.$$
$\Rightarrow J_K = \text{Span}(p_1, p_2)$.



- What is the space of all homogeneous polynomials of degree K containing J_K?
$$\mathbb{R}_K[x_1, x_2, \cdots, x_D] \doteq \text{Span}(x_1^K, x_1^{K-1}x_2, \cdots, x_D^K).$$
$$\Rightarrow \dim(\mathbb{R}_K[x_1, x_2, \cdots, x_D]) \doteq M_K^{[D]} = \binom{K+D-1}{D-1}$$

Definition (Veronese Map)

The Veronese map of order k is the map $\nu_k : \mathbb{R}^D \rightarrow \mathbb{R}^{M_k^{[D]}}$ given by

$$\nu_k([x_1, \cdots, x_D]^T) = [x_1^k, x_1^{k-1}x_2, x_1^{k-1}x_3, \cdots, x_D^k]^T,$$

where the list of $x_1^k, x_1^{k-1}x_2, x_1^{k-1}x_3, \cdots, x_D^k$ are all the monomials of degree k.

Recovering J_K via the Veronese map

1. Given the number of subspaces K known and N samples $V = \{\mathbf{z}_1, \cdots, \mathbf{z}_N\}$, construct the data matrix

$$L_K(V) = [\nu_K(\mathbf{z}_1), \nu_K(\mathbf{z}_2), \cdots, \nu_K(\mathbf{z}_N)] \in \mathbb{R}^{M_K^{[D]} \times N}.$$

2. Any Kth degree vanishing polynomial is expressed by the monomials.

$$p = \begin{bmatrix} c_1, \cdots, c_{M_K^{[D]}} \end{bmatrix} \begin{bmatrix} x_1^K \\ x_1^{K-1}x_2 \\ \vdots \\ x_D^K \end{bmatrix}.$$

3. Since $p(\mathbf{z}) = 0$ for all $\mathbf{z}_1, \cdots, \mathbf{z}_N$,

$$\begin{bmatrix} c_1, \cdots, c_{M_K^{[D]}} \end{bmatrix} L_K(V) = [p(\mathbf{z}_1), p(\mathbf{z}_2), \cdots, p(\mathbf{z}_N)] = 0_{1 \times N}.$$

4. Hence, the coefficients of a vanishing polynomial as a vector are recovered from $\text{Null}(L_K)$!
$\dim(\text{Null}(L_K)) = dim(J_K)$, and the basis vectors of $\text{Null}(L_K)$ correspond to the basis vectors of J_K.

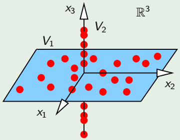
Derivatives of Polynomials

We have learned how to recover vanishing polynomials from the data. Next, how to recover the bases.

- Estimation of the bases for $V_1^\perp, \cdots, V_K^\perp$ depends on the *derivatives* of the vanishing polynomials.

Example

- The null space of $L_2(V)$ is
$$\begin{array}{l} \mathbf{c}_1 = [0,0,1,0,0,0] \\ \mathbf{c}_2 = [0,0,0,0,1,0] \end{array} \Rightarrow \begin{array}{l} p_1 = \mathbf{c}_1\nu_2(\mathbf{x}) = x_1x_3 \\ p_2 = \mathbf{c}_2\nu_2(\mathbf{x}) = x_2x_3 \end{array}.$$
$$P(\mathbf{x}) \doteq [p_1(\mathbf{x})\ p_2(\mathbf{x})] = [x_1x_3, x_2x_3]$$



- $\nabla_{\mathbf{x}} P = [\ \nabla_{\mathbf{x}} p_1\ \nabla_{\mathbf{x}} p_2\ ] = \begin{bmatrix} x_3 & 0 \\ 0 & x_3 \\ x_1 & x_2 \end{bmatrix}$.
- Suppose $\mathbf{z} = [1,1,0]^T \in V_1$, then $\nabla_{\mathbf{x}} P(\mathbf{z}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$. Suppose $\mathbf{z} = [0,0,1]^T \in V_2$, then $\nabla_{\mathbf{x}} P(\mathbf{z}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.
- Hence, $\nabla_{\mathbf{x}} P(\mathbf{z})$ returns a set of basis vectors for $V_1^\perp$ and $V_2^\perp$.

- How to compute derivatives of vanishing polynomials?
  Derivatives of monomials are created alongside with the Veronese map:

$$\nu_k(\mathbf{x}) = [\ x_1^k, x_1^{k-1}x_2, \cdots, x_D^k\ ] \leftrightarrow \nabla\nu_k(\mathbf{x}) = \begin{bmatrix} x_1^{k-1} & x_1^{k-2}x_2 & \cdots & 0 \\ 0 & x_1^{k-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & x_D^{k-1} \end{bmatrix}.$$

- Given a vanishing polynomial $p(\mathbf{x}) = c_1 x_1^K + \cdots + c_{M_K^{[D]}} x_D^K$,

$$\begin{aligned}\nabla p(\mathbf{x}) &= c_1 \nabla x_1^K + \cdots + c_{M_K^{[D]}} \nabla x_D^K \\ &= \begin{bmatrix} x_1^{k-1} & x_1^{k-2}x_2 & \cdots & 0 \\ 0 & x_1^{k-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & x_D^{k-1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{M_K^{[D]}} \end{bmatrix}.\end{aligned}$$

- Finally, evaluate $\nabla_{\mathbf{x}} P(\mathbf{z}) = \nabla_{\mathbf{x}}[p_1, \cdots, p_h]$ at one point per subspace, we then successfully recover $V_1^{\perp}, \cdots, V_K^{\perp}$!

Polynomial Differentiation Algorithm (PDA)