Multiparameter Martingale and Markov Process

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1. Introduction

One of the motivations in studying multiparameter martingales and Markov processes is to extend the results of filtering and detection problems in one dimension to the multidimensional (especially 2-dimensional) case. We begin with a brief review of these problems as follows:

Let $\{ \xi_t, \quad t_0 \leq t \leq t_1 \}$ be a stochastic process representing the *observation* and assume that it has the form

$$\xi_t = S_t + N_t \tag{1}$$

where $\{ S_t, N_t \}$ are stochastic processes with known distributions that represent *signal* and *noise* respectively. The (causal) filtering problem deals with the evaluation of

$$\hat{S}_t = E(S_t \,|\, \xi_s, \quad 0 \leq s \leq t) \tag{2}$$

while the detection problem concerns testing (1) as a hypothesis against the alternative that ξ_t consists of noise alone. In most cases, a likelihood ratio test would be used, and this involves the evaluation of the likelihood ratio

$$L_t = E_0\left(\frac{dP}{dP_0} \,\middle|\, \xi_s, \quad 0 \leq s \leq t \right) \tag{3}$$

where $\mathbf{P}$ is the probability measure associated with (1), and $\mathbf{P}_0$ is the probability measure under the hypothesis that ξ_t consists of noise only.

If we assume that the noise is a Gaussian white noise process, then the two problems: filtering and detection, are closely related, and one can show that the likelihood ratio can be expressed as

$$L_t = \exp\left\{ \int_0^t \hat{S}_\tau \, dX_\tau - \frac{1}{2} \int_0^t \hat{S}_\tau^2 \, d\tau \right\} \tag{4}$$

where $X_t = \int_0^t \xi_\tau \, d\tau$ and we have assumed that the white noise is normalized to have a unit spectral density. The first integral in (4) is an Ito integral. This formula, due to Duncan [DUN68,70] and Kailath [KAI69] is one of the highlights in applying martingale theory to problems in communication and control.

If a Markovian model is assumed for the signal S_t, then the problem of evaluating $\hat{S}_t$ can be formulated as a stochastic partial differential equation, viz., the Zakai equation [ZAK69]. If S_t is also Gaussian, then the problem can be reduced drastically, to a linear stochastic differential equation that is the Kalman filtering equation for continuous time.

Processing signals that depend on several parameters (space or space-time) is of considerable practical interest. Image processing, for example, involves signals and noise with a two dimensional parameter. In this particular case, the additive-noise model given by (1) is again appropriate if we take the observation to be the logarithm of the image intensity.

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Both analytically and computationally, the problem of processing multiparameter signals is more difficult than its one-dimensional counterpart. Thus, there is strong motivation to seek the kind of simplification made possible by martingale theory and Markovian models in one dimension. Over the last two decades, there has been a significant effort in developing a theory of multiparameter martingales and Markov processes. Although this effort has met with incomplete success, the results that have been found are most interesting and suggest that much more is yet to come. The objective of this paper is to review some of these results, and to indicate some directions where future efforts might be usefully deployed.

2. Martingales

Let $\mathbb{R}_t^2$ denote the positive quadrant of the plane with a partial ordering defined by

$$t > s \iff t_i \geq s \quad i = 1, 2 \tag{5}$$

Let $(\Omega, \mathbf{F}, \mathbf{P})$ denote a probability space with a filtration $\{\mathbf{F}_t,\ t \in \mathbb{R}_t^2\}$ that satisfies

$$t > s \iff \mathbf{F}_t \supset \mathbf{F}_s \tag{6}$$

We assume the $\mathbf{F}_4$-condition of Cairoli and Walsh [CAI 75], namely,

$$\mathbf{F}_{t_1,\infty} \ \text{ and } \ \mathbf{F}_{\infty,t_2} \ \text{ are independent given } \mathbf{F}_{t_1,t_2}$$

We define a Wiener process on the plane as a zero-mean Gaussian process $\{W_t,\ t \in \mathbb{R}_+^2\}$ with

$$E\,W_t W_s = \min(t_1, s_1) \cdot \min(t_2, s_2) \tag{7}$$

The properties of W_t are most easily seen by viewing W as

$$W_t = \int_{A_t} \xi(s)\, ds \tag{8}$$

where $\xi(s)$ is a Gaussian white noise and A_t is the rectangle in $\mathbb{R}_t^2$ bound by the origin and the point $t = (t_1, t_2)$.

Wong and Zakai [WON74] introduced stochastic integrals of two different types with respect to the Wiener process and showed that processes defined by these integrals:

$$M_t^{(1)} = \int_{A_t} \psi_s\, W(ds) \tag{9}$$

$$M_t^{(2)} = \int_{A_t \times A_t} \psi_{s,s'}\, W(ds)\, W(ds') \tag{10}$$

were martingales. These integrals were found to form a basis for representing martingales generated by a Wiener process. Specifically, if $\mathbf{F}_t$ is the σ-field generated by $\{W_s, s < t\}$ then every square-integrable $\mathbf{F}_t$-martingale has the representation

$$M_t = M_0 + M_t^{(1)} + M_t^{(2)} \tag{11}$$

where $M^{(1)}$ and $M_{(2)}$ are given by (9) and (10) respectively.

The papers [WON74] and [CAI75] provided a framework within which a considerable body of results on multi-parameter martingales has been developed. These include a formula of exponential type for the likelihood ratio [WON77] and a set of linear recursive filtering equations for signals that are Gaussian and Markovian with respect to rectangular boundaries [OGI81].

The definition of martingale and the ensuing results depend on the choice of partial ordering given by (15). In this sense multiparameter martingale theory is not "geometric." The partial ordering should be viewed as an artifact introduced to facilitate computation. In this respect, the situation is not significantly different from Ito's definition of stochastic integral in one dimension. It too is dependent on the choice of a "forward" direction and as such is not "geometric."

In one dimension the Ito differentiation formula provides much of the manipulative power of stochastic calculus, and its simplest version can be stated as follows. Let X_t be an *Ito process*, i.e., a process of the form

$$X_t = X_0 + \int_0^t \theta_s\, ds + \int_0^t \psi_s\, dW_s \tag{12}$$

where the last integral is a stochastic integral with respect to the one-parameter Wiener process W. Let $f: \mathbb{R} \to \mathbb{R}$ be a twice continuously-differentiable function. Then, $Y_t = f(X_t)$ is given by

$$Y_t = f(X_0) + \int_0^t f'(X_s)\, dX_s + \frac{1}{2}\int_0^t f''(X_s)\, d\langle X\rangle_s \tag{13}$$

where

$$dX_t = \theta_t\, dt + \psi_t\, dW_t \tag{14}$$

and

$$d\langle X\rangle_t = \psi_t^2\, dt \tag{15}$$

It is worth noting that both dX_t and $d\langle X\rangle_t$ can be computed from X directly, and are not dependent on its representation (12).

The differentiation formula (13) has several interpretations. First, it is a statement of the *closure* of Ito processes under C^2 transformations. As such, the most appropriate form of it is:

$$f(X_t) = f(X_0) + \int_0^t \left[f'(X_s)\,\theta_s + \frac{1}{2} f''(X_s)\,\psi_s^2 \right] ds + \int_0^t f'(X_s)\,\psi_s\, dW_s \tag{16}$$

Additionally, it is also a statement of how differentiation of functions of an Ito process must be modified from ordinary calculus. The most appropriate expression of this statement is:

$$df(X_t) = f'(X_t)\, dX_t + \frac{1}{2} f''(X_t)\, d\langle X\rangle_t \tag{17}$$

where the fact that the differential is intrinsic, i.e., independent of the representation (12), is of considerable importance.

In higher dimensional parameter spaces, these statements are no longer the same. The counterpart to the "closure" statement is the following: [WON76] Let *weak semi-martingale* be a process of the form

$$X_t = X_0 + \int_{A_t} \phi_s W(ds) + \int_{A_t^2} \psi_{s,s'} W(ds)\, W(ds') + \int_{A_t} \theta_s\, ds + \int_{A_t^2} \alpha_{s,s'} W(ds)\, ds' + \int_{A_t^2} \beta_{s,s'}\, ds\, W(ds') \tag{18}$$

Let $f: \mathbb{R} \to \mathbb{R}$ be four times continuously differentiable. Then $f(X_t)$ is again of the form (18). The explicit expression for $f(X_t)$ that is the counterpart to (16) is complicated, and the complexity increases with dimensionality as one attempts to generalize to processes with higher dimensional parameters. A major reason for this is that the stochastic integrals (9) and (10) are integrals over "volumes" in $\mathbb{R}^n$, and are not the inverse to differentials in dimensions higher than one.

Unlike (16), the differential form (17) of Ito's formula should admit a simple generalization that in its appearance is both coordinate and dimension independent. To do this, we need to develop an exterior calculus, but one in which the martingale property is reflected. This is one of the motivations in the development of stochastic differential forms to be discussed in section 4.

3. Markov Processes

Lévy [LEV56] defined a multiparameter Markov process as follows: Let ∂D be a simply connected $(n-1)$ surface dividing $\mathbb{R}^n$ into a bounded part D_- and an unbounded part D_+. A process $\{X_t,\ t \in \mathbb{R}^n\}$ is said to be Markov if $X_t, t \in D_+$ and $X_{t'}, t' \in D_-$ are conditionally independent given $\{X_s, s \in \partial D\}$. For $n > 1$, this is a rather restrictive condition, and it was shown in [WON68, 69] that no Gaussian, Markov, homogeneous and isotropic process could be Markov, unless the definition was relaxed to allow X to be a generalized process. In that case, a Gaussian generalized process with a covariance bilinear form given by

$$E\, X(\psi)\, \overline{X(\sigma)} = \int_{\mathbb{R}^n} \frac{\hat{\psi}(\nu)\, \overline{\hat{\sigma}}(\nu)}{1 + |\nu|^2}\, d\nu \tag{19}$$

is Markov. In (19), $\hat{\psi}$ and $\hat{\sigma}$ denote Fourier transforms. Indeed, except for scaling differences, this is the only example of isotropic-homogeneous Gauss-Markov process, and is widely known as the *free Euclidean field* [NEL73].

Generalized processes are usually defined as random functions parameterized by testing functions. As such, to define surface data: $\{X_s, S \in \partial D\}$ is difficult, though possible. A natural alternative is to introduce processes parameterized by k-dimensional sets, with $k < n$, in $\mathbb{R}^n$, and study Markovian properties for such processes. This is another motivation for introducing stochastic differential forms.

4. Stochastic Differential Forms [WON 87]

Intuitively, we want to define in a consistent way processes parameterized by k-dimensional sets in $\mathbb{R}^n$. We begin by considering *oriented k-rectangles* in $\mathbb{R}^n$ defined as follows: Let a_i denote an interval (left open, right closed) on the t_i axis of $\mathbb{R}^n$. Let $\sigma = a_{i_1} \wedge a_{i_2} \wedge \ldots \wedge a_{i_k}$ denote a rectangle with sides $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$. The orientation of σ is positive if $i = (i_1, i_2, \ldots, i_k)$ can be put into increasing order by an *even* permutation, and negative otherwise. We shall call i the direction of σ. A rectangular *k-chain* A is an algebraic sum

$$A = \sum_{\nu=1}^{m} \alpha_\nu \sigma_\nu \tag{20}$$

where $\alpha_\nu = \pm 1$ and σ_ν are oriented k-rectangles. We note that the boundary $\partial A = \sum \alpha_\nu\, \partial\sigma_\nu$ is a $(k-1)$-chain. A *random k-cochain* X is a random function defined on all *k*-chains such that

$$\begin{aligned} X(-A) &= -X(A) \\ X(A+B) &= X(A) + X(B) \end{aligned} \tag{21}$$

We note that a k-cochain is determined by its values on k-rectangles.

Chains can be used to approximate k-dimensional sets in $\mathbb{R}^n$ by introducing the *flat norm* $|A|^{\sim}$ as follows: Let $|\sigma|$ denote the k-dimensional volume of a k-rectangle σ. Let $|A|$ be defined by

$$\left|\sum \alpha_\nu \sigma_\nu\right| = \sum |\alpha_\nu|\, |\sigma_\nu| \tag{22}$$

and define

$$|A|^{\sim} = \inf \{\, |A - \partial B| + |B| \,\} \tag{23}$$

where the infimum is taken over all $(k+1)$-chains B. It can be shown that $|\ |^{\sim}$ is a norm with

$$|\partial A|^{\sim} \le |A|^{\sim} \le |A| \tag{24}$$

Using the flat norm, we can approximate continuous paths by 1-chains and smooth surfaces by 2-chains [WHI57].

We can now define *stochastic differential k-forms* as random co-chains X that are continuous in probability with respect to the flat norm, and thus can be extended to all limits of k-chains. If ∂D is an (n −1) surface and X is a k-form, with $k \leq n-1$, the surface data of X on ∂D are easily defined. An interesting problem is the study of Markovian k-forms.

We note that a natural example of stochastic differential forms is the white noise process on $\mathbb{R}^n$, which can be defined as a zero-mean n-form η in $\mathbb{R}^n$ such that for n-rectangles σ and σ'

$$E\, \eta(\sigma)\, \eta(\sigma') = \pm \text{ volume } (\sigma \cap \sigma')$$

where the sign is + if σ and σ' are similarly oriented and −otherwise. If η is Gaussian and A_t is the rectangle bounded by the origin and the point t in $\mathbb{R}^n_+$, then

$$W_t = \eta(A_t)$$

is a Wiener process, the two-dimensional version of which was used in section 2.

Defining stochastic differential forms as above leads to a simple definition for *exterior derivative*. For any k-form $X(k \leq n-1)$, we define dX as a (k + 1)-form such that

$$(dX)(\sigma) = X(\partial\sigma) \tag{25}$$

for all oriented rectangles σ. In short, we use Stokes theorem to define the exterior derivative. Because of (24), continuity of dX with respect to the flat norm is assured and dX is guaranteed to be a stochastic differential form. Thus, differential forms are closed under exterior differentiation.

Square integrable stochastic differential forms are always random currents in the sense of Ito [ITO 56], but not conversely. Thus, a stochastic differential form admits any operation that is defined on random currents. However, such an operation may produce only a current, not differential form. The Hodge star operator on a form, for example, will in general produce only a current. Take the case of a white noise n-form η. If $*\eta$ were a 0-form, it must be an ordinary function such that

$$\eta(\sigma) = \int_\sigma (*\eta)\, dt \tag{26}$$

No such function exists.

The notion of differential form can be combined with that of martingale in a fruitful way. For ease of exposition, we restrict our discussion to the case of $\mathbb{R}^2$. The general case is treated in [WON87]. Consider a stochastic differential 1-form X. It can be evaluated on 1-rectangles in two directions, and these take the form of line segments:

$$\sigma_1 = (t_1, t_2) \rightarrow (t_1 + a, t_2) \quad \text{and} \quad \sigma_2 = (t_1, t_2) \rightarrow (t_1, t_2 + a)$$

respectively. We say X is a *1-martingale* if for all σ_1

$$E\left[X(\sigma_1) \,\middle|\, F_{t_1\infty}\right] = 0 \tag{27a}$$

a *2-martingale* if for all σ_2

$$E\left[X(\sigma_2) \,\middle|\, F_{\infty, t_2}\right] = 0 \tag{27b}$$

and a *martingale* if it is both a 1-martingale and a 2-martingale.

A 2-form M in $\mathbb{R}^2$ can be evaluated on a general rectangle: $\sigma = [t_1, t_1 + a] \times [t_2, t_2 + b]$. We say M is an i-martingale (i = 1, 2) if for all σ

$$E\left[M(\sigma) \,\middle|\, F^i_t\right] = 0 \tag{28}$$

where $F^i_t = F_{t_1,\infty}$ or F_{∞, t_2} for i = 1 or 2 respectively. Again, we say M is a martingale if it is both a 1-martingale and a 2-martingale.

A Gaussian white noise η is clearly a martingale 2-form. Let W be a Wiener process as defined in (7). Then its exterior derivative dW is a martingale 1-form. The definition for martingales given in section 2 is equivalent to one of defining a martingale 0-form as one whose exterior derivative is a martingale 1-form. This is consistent with the one-dimensional case where the martingale property is most naturally associated with the *increments* of a process rather than with the process itself.

One of the most interesting ways in which martingale differential forms can be used is in a bilinear operation that is at once a generalization of stochastic integral and a generalization of the *exterior product* for ordinary differential forms. Let X and Y be martingale k and r forms respectively with respect to a fixed filtration $\{F_t,\ t \in \mathbb{R}^n\}$. Then we can define their exterior product $X \wedge Y$ as a martingale $(k + r)$-form. For example, suppose that m and M are 0-forms defined by two type-1 stochastic integrals:

$$m_t = \int_{A_t} \theta_s W(ds)$$

$$M_t = \int_{A_t} \psi_s W(ds)$$

Then, dm and dM are both martingale 1-forms, and $dm \wedge dM$ is a martingale 2-form that is related to a type-2 integral as follows:

$$\left(dm \wedge dM\right)(A_t) = \int_{A_t \times A_t} \left(\theta_s \psi_{s'} - \psi_s \theta_{s'}\right) W(ds)\, W(ds')$$

A natural and interesting question is whether the representation theorem for Wiener-martingales cited in section 2 can be re-expressed as representation theorems for martingale differential forms.

5. Markovian Random Currents

The notion of a random current was introduced in [ITO 56]. Let S^r denote the space of (ordinary) differential r-forms in $\mathbb{R}^n$ with coefficients in the Schwartz space of functions of rapid descent. Then, a *random k-current* X is defined as a continuous linear map of S^{n-k} into $L^2(\{\Omega, F, P\})$. Roughly speaking, a random k-current is a differential k-form with coefficients that are generalized n-parameter processes.

The space of random currents (of all order) is closed under both the exterior derivative

$$(dX)(\phi) = (-1)^k X(d\phi)$$

and the Hodge star operator

$$(*X)(\phi) = (-1)^{k(n-k)} X(*\phi)$$

If $\psi \in S^r$ and X is a random k-current, then the *exterior product* $X \wedge \psi$ can be defined as a $(k + r)$-current by

$$(X \wedge \psi)(\phi) = X(\psi \wedge \phi)$$

All stochastic k-forms, as defined in section 4, are random k-currents. Conversely, if $\psi \in S^{n-k}$ and X is a k-current, then $X \wedge \psi$ is a stochastic n-form. However, in general, for $r < n-k$, $X \wedge \psi$ is not a $(k + r)$-form.

We say a k-current is *localizable* if for all $\psi \in S^{n-k-1}$ and $\phi \in S^{k-1}$ both $X \wedge \psi$ and $*X \wedge \phi$ are $(n-1)$-forms. Given a $(n-1)$-surface ∂D, we take

$$\left\{(X \wedge \psi)(\partial D)\ ,\ (*X \wedge \phi)(\partial D)\ ;\ \psi \in S^{n-k-1},\ \phi \in S^{k-1}\right\}$$

to be the *surface data* of X on ∂D, and denote it by the abbreviated notation $X(\partial D)$. A localizable k-current X $(k \leq n-1)$ is said to be *Markov* if given an $(n-1)$ surface ∂D that separates $\mathbb{R}^n$ into a bounded part D^- and an unbounded part D^+, $X(\sigma), \sigma \subset D^+$ and $X(\sigma')$, $\sigma' \subset D^-$ are independent given $X(\partial D)$. This identifies an appropriate class of random functions for which the (simple) Markov property can be studied. The free Euclidean field, for example, is a Markov 0-current under our definition.

One of the simplest, yet interesting, classes of random currents is the class of isotropic and homogeneous currents. Let G denote the full group of isometries on $\mathbb{R}^n$ that preserve the Euclidean distance. A motion $g \in G$ induces a transformation τ_g on $\phi \in S'$ which in turn induces a transformation T_g on k-forms X:

$$(T_g X)(\phi) = X(\tau_g^{-1} \psi)$$

A random k-current is said to be *isotropic and homogeneous* if for all $g \in G$

$$E\, X(\phi)\, \overline{X}(\psi) = E\left[(T_g X)(\phi)\,(T_g \overline{X})(\phi)\right]$$

It was shown in [ITO 56] that every isotropic and homogeneous k-current ($1 \leq k \leq n-1$) is characterized by two Borel measures F_i and F_s on $[0, \infty)$. For $k = 0$ or n, only one such measure suffices. A natural question is: for a Gaussian isotropic and homogeneous k-current X, what must F_i and F_s be in order for X to be Markov?

Thus far, this question has only been partially answered: What we have been able to show is the following:

(1) For X to be Markov, F_i and F_s must be of the form:

$$F_i(d\lambda) = A \frac{\lambda^{n-1}\, d\lambda}{\alpha^2 + \lambda^2}$$

$$F_s(d\lambda) = B \frac{\lambda^{n-1}\, d\lambda}{\beta^2 + \lambda^2}$$

(2) If $\alpha^2 = \beta^2$, then A and B must be equal in which case X is indeed Markov.

A major open problem is to find examples of non-Gaussian isotropic and homogeneous Markov processes in $\mathbb{R}^n$. We think the introduction of stochastic differential forms may well aid this effort in several ways. First, it may allow "instantaneous" nonlinear operation to be defined on localizable currents. For example, "exterior product" would be such an operation. However, the martingale exterior product considered in section 4 is coordinate dependent and is unsuitable for the construction of Markovian currents.

Another approach is to use the exterior calculus available for currents to relate Markov processes to some basic elemental process such as "white noise." This may lead to a way of generating Markovian currents using stochastic differential equations of some kind.

6. Conclusions

In this paper we focus on the concepts of martingales and Markov processes as generalized to processes with a multidimensional parameter, and briefly review some of the known results on these two topics. We then introduce the recently developed notion of stochastic differential form and indicate how it can be related to martingales and Markov processes with a multidimensional parameter.

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