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Markov Processes on the Plane

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In this paper we introduce a Markov property for processes parameterized by paths in the plane, and illustrate the property by examples related to the Brownian sheet and the free Euclidean field. Transformation of processes and transformation of probability measures that preserve the Markov property are studied.

KEY WORDS: Markov fields, random fields, two parameter processes.

1. INTRODUCTION

A natural definition of the Markov property for multiparameter random processes is the following. Let $\{X_z, z\in\mathbb{R}^n\}$ be a multiparameter random process taking values in some nice space. For any set D in $\mathbb{R}^n$ let $\mathscr{F}_D$ denote the sigma-field generated by $\{X_z, z\in D\}$, i.e. $\mathscr{F}_D=\sigma\{X_z, z\in D\}$. The process X_z is said to be Markov (or Markov of degree 1, cf. [13, 15]) if for any bounded set U in $\mathbb{R}^n$ with smooth boundary and containing the origin $z=0$, $\mathscr{F}_U$ and $\mathscr{F}_{U^c}$ are conditionally independent given $\mathscr{F}_{\partial U}$ where U^c and ∂U denote the complement and boundary of U respectively. The Brownian sheet on $\mathbb{R}_+^2$, $\{W_z, z \in \mathbb{R}_+^2\}$, is a zero mean Gaussian process with $EW_zW_{z'} = \min(s, s')\cdot\min(t, t')$ where $z=(s, t)$, $z'=(s', t')$ and possessing continuous sample functions. Is the Brownian sheet Markov? At first sight it seems that the answer should be positive since the Brownian sheet is "the integral of white Gaussian noise". More specifically, consider a connected bounded set whose boundary is a finite number of vertical and horizontal line segments, and containing the origin: then obviously W_z has the Markov property with respect to this boundary. To quote Walsh [20] "...intuitively, this should be a Markov process if any process is." However, as shown in [20] (cf. also p. 161 of [19]), the Brownian sheet is not Markov in the sense of the above definition. A proof of this fact is as follows. Let U be the triangle with corners $(0,0)$, $(0,1)$, $(1,0)$, then $\mathscr{F}_{\partial U}=\sigma\{W_{\theta,1-\theta};\ 0\leqq\theta\leqq 1\}$. Note that

$$E(W_{1,1}|\mathscr{F}_U)=W(U)=\int_U dW_\zeta,$$

and assuming, temporarily, that W is Markov with respect to the set U then it follows that

$$E(W_{1,1}|\mathscr{F}_{\partial U})=W(U).$$

Since, in any case,

$$E(W_{1,1}|\mathscr{F}_{\partial U})=E(W(U)|\mathscr{F}_{\partial U}),$$

the assumption that W is Markov with respect to U implies that

$$E(W(U)-E(W(U)|\mathscr{F}_{\partial U}))^2=0. \tag{1}$$

Now, since $\{W_{\theta,1-\theta}, 0\leqq\theta\leqq 1, W(U)\}$ are zero mean and jointly Gaussian, $E(W(U)|\mathscr{F}_{\partial U})$ is linear in $\{W_{\theta,1-\theta}, 0\leqq\theta<1\}$ and is characterized by the orthogonality condition

$$E\{(W(U)-E(W(U)|\mathscr{F}_{\partial U}))W_{\theta,1-\theta}\}=0 \tag{2}$$

for all θ in $(0, 1)$. A direct calculation shows that setting

$$E(W(U)|\mathscr{F}_{\partial U})=2\int_0^1 W_{\theta,1-\theta}\,d\theta$$

satisfies (2) but not (1) therefore the Brownian sheet is not Markov in the sense of the above definition.

In order that the class of multiparameter Markov processes not be too small it is customary to modify the definition of the Markov property and instead of conditioning on the sigma fields generated by the values of the process on to the boundary, to condition on richer sigma fields (cf. [14], appendix A of [1], [19]). A (very) rich sigma field is obtained as follows, define the germ field $\Sigma_{\partial U}$ associated with the boundary ∂D of a set D by

$$\Sigma_{\partial D} = \cap \sigma\{X_t, t \in O_{\partial D}\}$$

where the intersection is over all the open subsets $O_{\partial D}$ that contain ∂D. Now replace $\mathscr{F}_{\partial D}$ by $\Sigma_{\partial D}$ as the splitting field in the definition of the Markov property, namely: the random process $\{X_t, t \in \mathbb{R}^n\}$ has the germ field Markov property if for every bounded set with smooth boundary and containing the origin, $\mathscr{F}_D$ and $\mathscr{F}_{D^c}$ are conditionally independent given $\Sigma_{\partial D}$ (cf. [14] for equivalent definitions). Obviously, the Brownian sheet has the germ field Markov property. Note that the germ-field Markov concept is easily extended to generalized processes [10].

In order to point out the difference between the Markov and germ-field Markov properties, consider a continuous one parameter Markov process $X_t, t \geqq 0$. Let $Y_t = \int_0^t X_s \, ds$ then Y_t is not Markov but it is germ-field Markov and the germ-field is $\Sigma_t = \sigma\{Y_t, dY_t/dt\}$. Another example is the following, let $\mathscr{E}$ denote the class of functions $\{X_t, -\infty < t < \infty\}$ that are the restriction to the real line of functions that are entire functions on the complex plane; then, for any probability law on $\mathscr{E}$, the process $\{X_t, -\infty < t < \infty\}$ is germ-field Markov.

The theory of one parameter Markov processes deals almost exclusively with processes that are Markov in the ordinary sense and has very little to say on processes that are Markov in a generalized sense (such as germ-field Markov or processes that are the projection of a Markov process). On the other hand, the theory of multiparameter processes is based mainly on the Markov property in a generalized sense and deals mainly with Gaussian processes ([1, 19]). For other definitions of the Markov property in the plane cf. [2, 8, 12, 16].

The purpose of this paper is to consider another definition of the Markov property for multiparameter processes. The idea is as follows, instead of considering processes that are a collection of random variables parameterized by points in $\mathbb{R}^2$ or $\mathbb{R}_+^2$ we consider processes parameterized by smooth curves in $\mathbb{R}^2$. The splitting sigma-field for the Markov property is now the sigma-field generated by curves lying in the boundary. Similarly a Markov process on $\mathbb{R}^3$ ($\mathbb{R}^n$) can be defined by considering a collection of random variables parametrized by smooth curves and surfaces (or r-cells where $0 \leq r \leqq n-1$) and the splitting sigma-field for the Markov property is that generated by points, curves, surfaces lying in the boundary. Stochastic processes parametrized by paths (or cells) that have a certain additivity property can be considered as stochastic differential 1-forms (r-forms) and are discussed in [27]. It is believed that the Markov property introduced here is a natural extension of the one parameter Markov property and is of particular interest in the analysis of non-Gaussian multiparameter processes.

Only the case of $\mathbb{R}^2$ will be considered in this paper.

In the next section we introduce definitions of the Markov property in the plane and show that the Brownian sheet is Markov under these definitions. We also note (as was first noted in [22]) that the free Euclidean field, which is a generalized process, can be considered a regular process parametrized by paths, and as such enjoys the Markov property under our definition. In Section 3 we consider transformations of measures via multiplicative functionals under which the Markov property is preserved. As an application to the results of this section it is shown that the solution to the stochastic differential equation $X(dz) = g(X_z)\, dz + W(dz)$ is Markov in the sense defined in the paper. Section 4 deals with transformations of the state space and the parameter z preserving the Markov property. As an application it is shown that the solution to $\partial_s X_{s,t} = -\alpha X_{s,t}\, ds + \partial_s W_{s,t}$ (the "infinite dimensional Ornstein–Uhlenbeck process", cf. [18]) is Markov. Let $z_1 = (s_1, t_1)$, $z_2 = (s_2, t_2)$ be points in the plane, introduce the partial order $z_1 \leqq z_2$ if $s_1 \leqq s_2$ and $t_1 \leqq t_2$. Let D be a connected set in $\mathbb{R}_+^2$ containing the origin, assume that D has the property if $z_2 \in D$ and $z_1 \leqq z_2$ then also $z_1 \in D$, the boundary of D^c is called a separating line. Section 5 deals with the Markov property with respect to separating lines, and it is shown that the solution to $\partial_s X_{s,t} = g(X_{s,t})\, ds + \partial_s W_{s,t}$ is Markov with respect to separating lines.

The Markov property with respect to separating lines can be considered as the stochastic version of Huygen's principle. This section is concluded with a remark on the extension of the notion of the Markov property with respect to separating lines to the notion of the Markov property with respect to random separating lines, i.e. the strong Markov property.

Remark In addition to the partial ordering $z_1 \leqq z_2$ defined earlier, and $z_1 < z_2$ if $s_1 < s_2$ and $t_1 < t_2$, we will use $z_1 \wedge z_2$ to denote $s_1 \leqq s_2$ and $t_1 \geqq t_2$.

2. PATH PARAMETERIZED MARKOV PROCESSES

Let γ denote a continuous finite nondecreasing path in $\mathbb{R}_+^2$, i.e. γ is defined by the function $\gamma(\theta)$ from $[0, 1]$ to $\mathbb{R}_+^2, \gamma(\theta)$ is assumed to be bounded continuous and $\gamma = \{z: z = \gamma(\theta), 0 \leqq \theta \leqq 1; \gamma(\theta_1) \leqq \gamma(\theta_2)$ whenever $\theta_1 < \theta_2\}$. Set $\gamma(\theta) = (s(\theta), t(\theta))$; $\gamma_0 = \gamma(0)$, $\gamma_1 = \gamma(1)$ are the endpoints of γ and let $A(\gamma)$ denote the vertical shadow of γ, i.e.

$$A(\gamma) = \{(\sigma, \tau): \sigma = s(\theta), \tau \leqq t(\theta), 0 \leqq \theta \leqq 1\}.$$

Similarly, $B(\gamma)$ is defined to be the horizontal shadow of γ:

$$B(\gamma) = \{(\sigma, \tau): \sigma \leqq s(\theta), \tau = t(\theta), 0 \leqq \theta \leqq 1\}.$$

Let

$$W(A(\gamma)) = \int_{A(\gamma)} W(d\xi); \quad W(B(\gamma)) = \int_{B(\gamma)} W(d\xi). \tag{3}$$

Also, let γ denote a finite continuous nonincreasing path defined by $\gamma = \{z: z = \gamma(\theta), 0 \leqq \theta \leqq 1, \gamma(\theta_1) \wedge \gamma(\theta_2)$ whenever $\theta_1 < \theta_2\}$. The endpoints are again $\gamma_0 = \gamma(0)$ and $\gamma_1 = \gamma(1)$, the shadows $A(\gamma)$, $B(\gamma)$ are defined as before and so are $W(A(\gamma)), W(B(\gamma))$; in this case $W(A(\gamma))$ and $W(B(\gamma))$ are not independent but this will not be important to us. Let $Y_\gamma = (W(A(\gamma)), W(B(\gamma))$, we want to consider the Markov properties of such path parametrized processes, for this purpose we first generalize as follows, let $(\Omega, \mathscr{F}, P)$ be a probability space and let Y_γ be a collection of random variables parametrized by paths in $\mathbb{R}_+^2$

that are continuous and either increasing or decreasing. Also, let X_z, $z \in \mathbb{R}_+^2$ be a collection of random variables parametrized by points z in $\mathbb{R}_+^2$. Let U denote a set in $\mathbb{R}_+^2$ and let $\Gamma(U)$ denote the collection of all continuous paths γ that are either increasing or decreasing and $\gamma \in U, \bar{U}$ will denote the closure of U. Let $\mathscr{G}_U$ denote the σ-field generated by Y_γ where γ runs over $\Gamma(\bar{U})$:

$$\mathscr{G}_U = \sigma\{Y_\gamma, \gamma \in \Gamma(\bar{U})\} \tag{4}$$

and

$$\begin{aligned} \mathscr{H}_U &= \mathscr{G}_U \vee \sigma\{X_z, z \in \bar{U}\} \\ &= \sigma\{Y_\gamma, X_{\gamma_0}, X_{\gamma_1}, \gamma \in \Gamma(\bar{U})\}. \end{aligned} \tag{5}$$

A boundary of ∂D of a set D will be said to be piecewise monotone if ∂D is the union of a finite number of paths γ_i, $i = 1, \ldots, m < \infty$ and $\gamma_i \in \Gamma(\mathbb{R}_+^2)$.

Definitions (a) A path parameterized process $\{Y_\gamma, \gamma \in \Gamma(\mathbb{R}_+^2)\}$ will be said to be γ Markov if for every connected open set D with piecewise monotone boundaries $\mathscr{G}_{\partial D}$ splits $\mathscr{G}_D$ and $\mathscr{G}_{D^c}$, i.e. $\mathscr{G}_D$ and $\mathscr{G}_{D^c}$ are conditionally independent given $\mathscr{G}_{\partial D}$.

b) The process $\{Y_\gamma, X_{\gamma_0}, X_{\gamma_1}, \ \gamma \in \Gamma(\mathbb{R}_+^2)\}$ will be said to be $\gamma +$ Markov if for every connected set with piecewise boundaries, $\mathscr{H}_{\partial D}$ splits $\mathscr{H}_D$ and $\mathscr{H}_{D^c}$. Obviously, a γ Markov process is also $\gamma +$ Markov with $X_z \equiv 0$. We prefer however, to define both concepts since the Brownian sheet induces both a γ Markov process and a γ + Markov process and moreover certain generalized processes can be reparametrized to become γ Markov processes.

Setting $Y_\gamma = (W(A(\gamma)), W(B(\gamma)))$ we will show now that $\{Y_\gamma, \gamma \in \Gamma(\mathbb{R}_+^2)\}$ is γ Markov (Proposition 1) and $\{Y_\gamma, W_{\gamma_0}, W_{\gamma_1}, \gamma \in \Gamma(\mathbb{R}_+^2)\}$ is $\gamma +$ Markov (Proposition 2). Proposition 2 is actually a rewording of Theorems 3.11 and 3.12 of Walsh [20] in the context of path parametrized processes, and Proposition 1 is a modification of it.

Remarks (a) It will be clear from the proof of Proposition 1 that $\{W(A(\gamma)), \gamma \in \Gamma(\mathbb{R}_+^2)\}$ alone, or $\{W(B(\gamma)) \in \Gamma(\mathbb{R}_+^2)\}$ alone, is also Markov.

b) If γ is increasing then $W_{\gamma_1}=W_{\gamma_0}+W(A(\gamma))+W(B(\gamma))$ and if γ is decreasing then $W_{\gamma_1}+W(B(\gamma))=W_{\gamma_0}+W(A(\gamma))$. Consequently the phrase "$Y_\gamma, W_{\gamma_0}, W_{\gamma_1}$" is equivalent to "$Y_\gamma, W_{\gamma_0}$" etc.

PROPOSITION 1 *Let $D\subset\mathbb{R}_+^2$ be a bounded connected set with piecewise monotone boundaries and $Y_\gamma=(W(A(\gamma)), W(B(\gamma)))$ then $\mathscr{G}_{\partial D}$ is the minimal splitting field for $\mathscr{G}_D$ and $\mathscr{G}_{D^c}$.*

Proof We will be considering subsigma fields generated by zero mean Gaussian random variables and therefore orthogonality and independence are equivalent. Let O denote the collection of all bounded open subsets of $\mathbb{R}_+^2$. Set

$$\tilde{\mathscr{G}}_D=\sigma\{W(U\cap D), U\in O\}$$

note that $\tilde{\mathscr{G}}_D$ and $\tilde{\mathscr{G}}_{D^c}$ are independent and

$$\begin{aligned}\mathscr{G}_D&=\tilde{\mathscr{G}}_D\vee\mathscr{G}_{\partial D}\\ \mathscr{G}_{D^c}&=\tilde{\mathscr{G}}_{D^c}\vee\mathscr{G}_{\partial D}.\end{aligned}\tag{6}$$

Let

$$\mathscr{G}_{\partial D}^{\text{in}}=\sigma\{W(D\cap A(\gamma)), W(D\cap B(\gamma)), \gamma\in\Gamma(\partial D)\}$$

$$\mathscr{G}_{\partial D}^{\text{out}}=\sigma\{W(D^c\cap A(\gamma)), W(D^c\cap B(\gamma)), \gamma\in\Gamma(\partial D)\}.$$

Then $\mathscr{G}_{\partial D}^{\text{in}}$ and $\mathscr{G}_{\partial D}^{\text{out}}$ are independent and

$$\mathscr{G}_{\partial D}\subseteq\mathscr{G}_{\partial D}^{\text{in}}\vee\mathscr{G}_{\partial D}^{\text{out}}.$$

In the converse direction note that for $\gamma\in\Gamma(\partial D)$, the set $(A(\gamma)\cap\partial D)$, can be decomposed into a finite union of paths γ_i, $i=1,\ldots,k$ such that $\gamma_i\in\Gamma(\partial D)$ and $(A(\gamma)\cap\partial D)\sim\bigcup_i\gamma_i$ and $W(D\cap A(\gamma))$ can be represented as a linear combination of $W(A(\gamma_i))$ with non-random coefficients. Therefore,

$$\mathscr{G}_{\partial D}^{\text{in}}\subseteq\mathscr{G}_{\partial D}$$

similarly $\mathscr{G}^{\text{out}}_{\partial D} \subseteqq \mathscr{G}_{\partial D}$ hence,

$$\mathscr{G}_{\partial D} = \mathscr{G}^{\text{in}}_{\partial D} \vee \mathscr{G}^{\text{out}}_{\partial D}.$$

Note that $\mathscr{G}^{\text{in}}_{\partial D} \subseteqq \tilde{\mathscr{G}}_D$ and $\mathscr{G}^{\text{out}}_{\partial D} \subseteqq \tilde{\mathscr{G}}_{\partial D}$, therefore (6) can be rewritten as

$$\begin{aligned} \mathscr{G}_D &= \tilde{\mathscr{G}}_D \vee \mathscr{G}^{\text{out}}_{\partial D} \\ \mathscr{G}_{D^c} &= \tilde{\mathscr{G}}_{D^c} \vee \mathscr{G}^{\text{in}}_{\partial D} \end{aligned} \tag{7}$$

and $\mathscr{G}^{\text{in}}_{\partial D}$ is independent of $\tilde{\mathscr{G}}_{D^c}$ and $\mathscr{G}^{\text{out}}_{\partial D}$ is independent of $\tilde{\mathscr{G}}_D$. Consider a path $\gamma \in \Gamma(D^c)$, then $W(A(\gamma)) = W(A(\gamma) \cap D) + W(A(\gamma) \cap D^c)$. Now, $W(A(\gamma) \cap D)$ can be represented by a finite sum $\sum_1^\kappa \alpha_i W(A(\gamma_i))$ with α_i non-random and $\gamma_i \subseteqq \Gamma(\partial D)$ therefore $W(A(\gamma) \cap D)$ is $\mathscr{G}_{\partial D}$ adapted. Turning to $W(A(\gamma) \cap D^c)$, it is $\tilde{\mathscr{G}}_{D^c}$ adapted, independent of $\tilde{\mathscr{G}}_D$ and can be decomposed into the sum of two Gaussian random variables one being $\mathscr{G}^{\text{out}}_{\partial D}$ adapted and the other orthogonal to $\mathscr{G}^{\text{out}}_{\partial D}$ and to $\tilde{\mathscr{G}}_D$ hence by (7) orthogonal to $\mathscr{G}_D$. Therefore $E(W(A(\gamma)) \mid \mathscr{G}_D)$ is $\mathscr{G}_{\partial D}$ adapted and equal to $E(W(A(\gamma)) \mid \mathscr{G}_{\partial D})$, this proves that $\mathscr{G}_{\partial D}$ splits $\mathscr{G}_D$ and $\mathscr{G}_{D^c}$. In order to show that $\mathscr{G}_{\partial D}$ is the minimal splitting field note that by (6), $\mathscr{G}_{\partial D} \subseteqq \mathscr{G}_D \cap \mathscr{G}_{D^c}$ and $\mathscr{G}_{\partial D} \subseteqq \mathscr{G}_D$. Since every splitting field for $\mathscr{G}_D$ and $\mathscr{G}_{D^c}$ includes $\mathscr{G}_D \cap \mathscr{G}_{D^c}$ (Corollary 2.2 of [15]) it follows that $\mathscr{G}_{\partial D}$ is minimal.

Proposition 2 ([20] Theorems 3.11, 3.12) *Let $D \subset \mathbb{R}^2_+$ be a bounded connected set with piecewise monotone boundaries, $Y_\gamma = (W(A(\gamma)), W(B(\gamma)))$ and $X_z = W_z$ then $\mathscr{H}_{\partial D}$ is the minimal splitting field for $\mathscr{H}_D$ and $\mathscr{H}_{D^c}$.*

Proof Note first that

$$\begin{aligned} \mathscr{H}_D &= \tilde{\mathscr{G}}_D \vee \mathscr{H}_{\partial D} \\ \mathscr{H}_{D^c} &= \tilde{\mathscr{G}}_{D^c} \vee \mathscr{H}_{\partial D}. \end{aligned} \tag{8}$$

Let R_z denote the rectangle $\{\xi : 0 \leqq \xi \leqq z\}$ and let

$$\mathscr{H}^{\text{in}}_{\partial D} = \sigma\{W(D \cap A(\gamma)), W(D \cap B(\gamma)), W(D \cap R_z); \gamma \in \Gamma(\partial D), z \in \partial D\}$$

$$\mathscr{H}^{\text{out}}_{\partial D} = \sigma\{W(D^c \cap A(\gamma)), W(D^c \cap B(\gamma)), W(D^c \cap R_z); \gamma \in \Gamma(\partial D), z \in \partial D\}$$

then $\mathscr{H}^{\text{in}}_{\partial D}$ and $\mathscr{H}^{\text{out}}_{\partial D}$ are independent and $\mathscr{H}_{\partial D}=\mathscr{H}^{\text{in}}_{\partial D} \vee \mathscr{H}^{\text{out}}_{\partial D}$. Furthermore,

$$\begin{aligned} \mathscr{H}_D &= \tilde{\mathscr{G}}_D \vee \mathscr{H}^{\text{out}}_{\partial D} \\ \mathscr{H}_{D^c} &= \tilde{\mathscr{G}}_{D^c} \vee \mathscr{H}^{\text{in}}_{\partial D} \end{aligned} \tag{9}$$

and the rest of the proof follows along the same lines as the proof of Proposition 1.

We now turn to another Gaussian example. Let $X_z, z\in\mathbb{R}^2$ a two-parameter free Euclidean field defined as a generalized Gaussian process with zero mean and a covariance function given by: $K_0(r)$ where K is the modified Bessel function. Equivalently, its spectral density function is given by:

$$S(v)=\frac{\alpha}{1+\|v\|^2}$$

where α is a positive constant [22, 17]. For a path γ in $\mathbb{R}^2$,

$$X(\gamma)=\int_\gamma X_z\,dl_z$$

is a Gaussian random variable with variance

$$E|X(\gamma)|^2=\int_{\mathbb{R}^2}\frac{\alpha}{1+\|v\|^2}\left|\int_\gamma \exp i(v,z)\,dl_z\right|^2 dv.$$

For a γ of finite non-zero length, the variance is finite. Thus, as a path-parametrized process, X is an ordinary process, not a generalized process. As was shown in [22], suitably interpreted, the free Euclidean field has a Markov property in the sense of Lévy [13]. In terms of the definition introduced in this paper, the path parametrized process X_γ is γ Markov.

3. TRANSFORMATIONS OF PATH PARAMETRIZED MARKOV PROCESSES I

The transformation of Markov processes via an absolutely continuous transformation that leaves the Markov property invariant

([6], Chapter 10, Section 4, [7]) will be considered in this section. The transformation of measures is induced by the exponential of additive functionals and known results on stochastic integration in the plane yield a large class of such functionals. This will be applied to show that the solution to the stochastic differential equation $X(dz)=g(X_z)\,dz+W(dz)$ is $\gamma+$ Markov.

Let $(Y_\gamma, X_{\gamma_0}, X_{\gamma_1}, \gamma\in\Gamma(\mathbb{R}_+^2))$ be a $\gamma+$ Markov process and let Q denote the collection of all connected open subsets D of $\mathbb{R}_+^2$ with piecewise monotone boundaries and their complements. Let $\{\alpha(D), D\in Q\}$ be a set parametrized collection of random variables defined for every $D\in Q$ and such that for every $D_1, D_2, D\in Q$ with $D_1\cap D_2=\phi$:

1) $\alpha(D_1\cup D_2)=\alpha(D_1)+\alpha(D_2)$

2) $E\{\exp\alpha(D)\}<\infty, E\{\exp\alpha(\mathbb{R}_+^2)\}<\infty$

3) $\alpha(D)$ is $\mathbb{H}_D$ adapted.

Let

$$L=\frac{\exp\alpha(\mathbb{R}_+^2)}{E\{\exp\alpha(\mathbb{R}_+^2)\}}.$$

PROPOSITION 3 *If $\{Y_\gamma, X_{\gamma_0}, X_{\gamma_1}, \gamma\in\Gamma(\mathbb{R}_+^2)\}$ is $\gamma+$ Markov under the original measure P then it is also $\gamma+$ Markov under $\tilde{P}=LP$.*

Proof The proof follows along the same lines as in the one parameter case or the germ-field Markov property in the multi-parameter case ([6, 7]) and is as follows. Let E_0, E_1 denote expectations with respect to the measures P and $\tilde{P}$ respectively. Let D be a bounded open set in $\mathbb{R}_+^2$ with piecewise monotone boundaries and let Z be a bounded random variable adapted to $\mathscr{H}_D$. Then $\alpha(\mathbb{R}_+^2)=\alpha(D)+\alpha(D^c)$ where $\alpha(D)$ is $\mathscr{H}_D$ adapted and $\alpha(D^c)$ is $\mathscr{H}_{D^c}$ adapted. Therefore

$$E_1(Z|\mathscr{H}_{D^c})=\frac{E_0(LZ|\mathscr{H}_{D^c})}{E_0(L|\mathscr{H}_{D^c})}$$

$$=\frac{E_0(Z\exp(\alpha(D)+\alpha(D^c))\,|\,\mathscr{H}_{D^c})}{E_0(\exp(\alpha(D)+\alpha(D^c))\,|\,\mathscr{H}_{D^c})}$$

$$=\frac{E_0(Z\exp\alpha(D)\,|\,\mathscr{H}_{D^c})}{E_0(\exp\alpha(D)\,|\,\mathscr{H}_{D^c})}$$

$$=\frac{E_0(Z\exp\alpha(D)\,|\,\mathscr{H}_{\partial D})}{E_0(\exp\alpha(D)\,|\,\mathscr{H}_{\partial D})}$$

$$=\frac{E_0(Z\exp\alpha(\mathbb{R}^2_+)\,|\,\mathscr{H}_{\partial D})}{E_0(\exp\alpha(\mathbb{R}^2_+)\,|\,\mathscr{H}_{\partial D})}$$

$$=E_1(Z\,|\,\mathscr{H}_{\partial D})$$

and therefore $\{Y_\gamma, X_{\gamma_0}, X_{\gamma_1}, \gamma\in\Gamma(\mathbb{R}^2_+)\}$ is $\gamma+$ Markov under $\tilde{P}$.

Similarly, consider the γ Markov process $\{Y_\gamma, \gamma\in\Gamma(\mathbb{R}^2_+)\}$. Let $\alpha(D)$ and L be as defined above but with (3) replaced by: (3′) $\alpha(D)$ is $\mathscr{G}_D$ adapted. Then it follows by the same arguments (or by specializing Proposition 3) that

Proposition 4 *If $\{Y_\gamma, \gamma\in\Gamma(\mathbb{R}^2_+)\}$ is γ Markov under the original measure P then it is also γ Markov under $\tilde{P}=LP$.*

As an example to the application of Proposition 3, let $\mathscr{F}_z$ denote the σ-field generated by $(W_\xi, \xi\leqq z)$. Let $(\theta_z, z\in\mathbb{R}^2_+)$ be a measurable random process adapted to $\mathscr{F}_z$ and

$$E\int_{\mathbb{R}^2_+}\theta_z^2\,dz<\infty.$$

Let

$$\int_{R_z}\theta_\xi W(d\xi),\ \iint_{R_z\times R_z}\theta_{\xi\wedge\xi'}W(d\xi)\cdot W(d\xi')=\int_{R_z}\theta_\xi\,\partial_2W_\xi\,\partial_1W_\xi$$

$$\int_{R_z}\theta_\xi\,\partial_2W_\xi\,ds,\ \int_{R_z}\theta_\xi\,dt\,\partial_1W_\xi\qquad(\xi=(s,t))$$

denote the stochastic integrals of the first, second and mixed types (cf. [23, 24]; [3]). Let $\psi_i(\cdot,\cdot)$, $i=1,\dots,4$ denote measurable functions on $\mathbb{R}_+^2 \times \mathbb{R}$ such that

$$E \int_{\mathbb{R}_+^2} (\psi_i(\xi, W_\xi))^2 \, d\xi < \infty.$$

Set

$$\alpha(D) = \int_D \psi_0(\xi, W_\xi) \, d\xi + \int_D \psi_1(\xi, W_\xi) W(d\xi)$$
$$+ \int_D \psi_2(\xi, W_\xi) \, \partial_2 W_\xi \, \partial_1 W_\xi + \int_D \psi_3(\xi, W_\xi) \, \partial_2 W_\xi \, ds$$
$$+ \int_D \psi_4(\xi, W_\xi) \, dt \, \partial_1 W_\xi. \tag{10}$$

Assuming now that $\psi_i(\cdot, \cdot)$, $i=0, \ldots, 4$ were chosen so as to satisfy $E \exp \alpha(D) < \infty$, $\alpha(D)$ as defined by (10) yields a large class transforming the Brownian sheet into a (generally non-Gaussian) $\gamma+$ Markov process. If the $\psi_i(\xi, W_\xi)$ in (10) are replaced by non-random $\phi_i(\xi)$ then (10) yields transformations of the Brownian sheet into γ Markov processes.

Returning to the case of $\gamma+$ Markov processes, consider the stochastic differential equation on $\mathbb{R}_+^2$:

$$X(dz) = g(z, X_z) \, dz + W(dz), \qquad X_{(0,0)} = 0, \tag{11}$$

This is a special case of the equation $dX_z = g(z, X_z) \, dz + \sigma(z, X_z) \, dW_z$ which has been considered by several authors (e.g. [2, 9]).

COROLLARY 1 *Let $g(z, a)$, $z \in \mathbb{R}^2$, $a \in \mathbb{R}^1$ be a bounded Borel function on $\mathbb{R}^2 \times \mathbb{R}^1$ and $g(z, \cdot) = 0$ for z outside a finite rectangle R_{z_0} where $z_0 \in \mathbb{R}_+^2$. Further assume that $g(\cdot, \cdot)$ satisfies a uniform Lipshitz condition in z, i.e.*

$$|g(z_1, a) - g(z_2, a)| \leqq K((s_1 - s_2)^2 + (t_1 - t_2)^2)^{1/2} \qquad z_i = (s_i, t_i)$$

then the process

$$\{X(A(\gamma)), X(B(\gamma)), X_{\gamma 0}, X_{\gamma 1}, \gamma \in \Gamma(\mathbb{R}_+^2)\}$$

is $\gamma +$ Markov

Proof The existence of a solution to (11) follows by standard arguments (e.g. [9]). Let W be the Brownian sheet under $\tilde{P}$, set

$$L_1=\exp\left(-\int_{\mathbb{R}^2_+} g(\xi, X_\xi)\,dW_\xi-\tfrac{1}{2}\int_{\mathbb{R}^2_+} g^2(\xi, X_\xi)\,d\xi\right) \tag{12}$$

then $EL_1=1$ and $\{X_\xi, \xi\in\mathbb{R}^2_+\}$ is Wiener under the measure $P=L_1\tilde{P}$ (cf. [25], note that $g(\cdot,\cdot)$ was assumed to be bounded). Consider now

$$\alpha(D)=\int_D g(\xi, X_\xi)\,dX_\xi-\tfrac{1}{2}\int_D g^2(\xi, X_\xi)\,d\xi \tag{13}$$

then under P we have $E\exp\alpha(\mathbb{R}^2_+)=1$, setting $L=\exp\alpha(\mathbb{R}^2_+)$ and $P_1=LP$ then $L=L_1^{-1}$, $P_1=\tilde{P}$ and $X_z-\int_{R_z} g(\xi, X_\xi)\,d\xi$ is Wiener under $\tilde{P}$. Therefore, for L defined via (13) the assumptions of Proposition 3 are satisfied which completes the proof.

4. TRANSFORMATION OF PATH PARAMETRIZED MARKOV PROCESSES II

Two types of transformations are considered in this section. In the first we consider a mapping $z=f(\xi)$ of $\mathbb{R}^2_+$ onto $\mathbb{R}^2_+$; this mapping induces a reparametrization of points and paths and the transformation of the Markov property under this reparametrization is considered. The second transformation deals with the case in which the Markov property of the process $\{\int_{A(\gamma)} dF(W_\xi), \int_{B(\gamma)} dF(W_\xi), F(W_{\gamma_0}\}$ is considered. The results will be applied to show that the Ornstein–Uhlenbeck process $\partial_s X_{s,t}=-\alpha X_{s,t}\,ds+\partial_s W_{s,t}$ is $\gamma+$ Markov.

A mapping $z=f(\xi)$ of a subset of $\mathbb{R}^2_+$ onto $\mathbb{R}^2_+$ will be said to be order preserving if $\xi_1\leqq\xi_2$ implies $z_1\leqq z_2$, $\xi_1\wedge\xi_2$ implies that $z_1\wedge z_2$ and $\xi_1\neq\xi_2$ implies that $z_1\neq z_2$. Let γ be the path $\gamma=\{\gamma(\theta), 0\leqq\theta\leqq 1\}$ then $f^{-1}(\gamma)$ will denote the path $\{f^{-1}(\gamma(\theta), 0\leqq\theta\leqq 1\}$. Note that any order preserving map is of the form $z=(f_1(\xi_1), f_2(\xi_2))$ and therefore it transforms horizontal (vertical) paths into horizontal (vertical) paths. Let $\{Y_\gamma, X_{\gamma_0}, X_{\gamma_1}\}$ be a $\gamma+$ Markov process and $z=f(\xi)$ order preserving. Let

$$\hat{X}_z=X_{f^{-1}(z)}, \qquad \hat{Y}_\gamma=Y_{f^{-1}(\gamma)}$$

then, obviously, $\{\hat{Y}_\gamma, \hat{X}_{\gamma_0}, \hat{X}_{\gamma_1}, \gamma\in\Gamma(\mathbb{R}^2_+)\}$ is also $\gamma+$ Markov.

Let $\{X_z, z \in \mathbb{R}_+^2\}$ be a real valued random process, assume that $\mathscr{F}_z$ is a collection of subsigma fields satisfying the assumptions of [3] and X_z is $\mathscr{F}_z$ adapted. Further assume that the stochastic integral in quadratic mean $\int \phi_\xi X(d\xi)$ with respect to the deterministic integrand ϕ is well defined, therefore the integral of $X(d\xi)$ over the horizontal and vertical shadows of γ is well defined. Set $X_\gamma = (\int_{A(\gamma)} X(d\xi), \int_{B(\gamma)} X(d\xi))$, X_γ will be said to be the path parametrized process induced by X_z. Since X_γ is defined on paths, it may be considered analogous to the integral of a differential one-form along paths; furthermore, as X_γ is induced by X_z which may be considered as a zero form, X_γ may be considered as the integral along paths of the exterior derivative of X_z, cf. [27]. We will use $\tilde{X}$ to denote the path parametrized process induced by the point parametrized process $\{X_z, z \in \mathbb{R}_+^2\}$.

LEMMA 1 *If X induces $\tilde{X}$, $z = f(\xi)$ is order preserving, and $\hat{X}$ induces $X\tilde{\hat{X}}$ then $\tilde{\hat{X}} = \hat{\tilde{X}}$ (and, as pointed out earlier if $\tilde{X}$ is $\gamma+$ Markov, so is $\hat{\tilde{X}}$).*

The proof is straightforward since $f(\xi)$ transforms rectangles with sides parallel to the axes into rectangles with sides parallel to the axes.

Turning now to the second transformation, let $F(s, t, x)$, $0 \leqq s < \infty$, $0 \leqq t < \infty$, $-\infty < x < \infty$ be a real valued function of its three variables, we want to show that if $F(\cdot, \cdot, \cdot)$ is sufficiently smooth and under some additional conditions, the process $X_z = F(s, t, W_z)$, $z = (s, t)$, F non-random, induces a $\gamma+$ Markov process. For this purpose we prepare the following lemmas. Note, however, the remark after the proof of Proposition 5.

LEMMA 2(a) *Let $F(\cdot,\cdot,\cdot)$ be defined on $\mathbb{R}_+ \times \mathbb{R}_+ \times$ and assume that F and its partial derivatives up to the fourth order are continuous and polynomially bounded. Let $\partial F/\partial\sigma$, $\partial F/\partial\tau$ denote the partials with respect to the first and second variable and F', F'', F''', $F^{(iv)}$ denote the first four derivatives, with respect to the third variable. Let $X_z = F(s, t, W_z)$, $z = (s, t)$ then, for every finite increasing path γ, ($\xi = (\sigma, \tau)$):*

$$X(A(\gamma)) = \int_\gamma \frac{\partial F}{\partial \sigma}(\sigma(\theta), \tau(\theta), W_{\xi(\theta)})\, d\sigma(\theta)$$

$$+\int_{\gamma} F'(\sigma(\theta), \tau(\theta), W_{\xi(\theta)})\, \partial_\theta W(A(\gamma) \cap R_{\xi(\theta)})$$

$$+\tfrac{1}{2}\int_{\gamma} F''(\sigma(\theta), \tau(\theta), W_{\xi(\theta)}) \cdot d\sigma(\theta) \tag{14}$$

and

$$X(B(\gamma)) = \int_{\gamma} \frac{\partial F}{\partial \tau}\, d\tau(\theta) + \int_{\gamma} F' \cdot \partial_\theta W(B(\gamma) \cap R_{\xi(\theta)}) + \tfrac{1}{2}\int_{\gamma} F'' \cdot d\tau(\theta). \tag{15}$$

Remark The stochastic line integrals can also be interpreted as stochastic surface integrals with respect to "weakly adapted" integrands (Theorem 2.3 and Section 4 of [3]).

Proof The Itô formula for two parameter processes [25] yields in this case:

$$F(s, t, W_z) - F(0, 0, 0) = \int_{R_z} \frac{\partial^2 F(\sigma, \tau, W_\xi)}{\partial\sigma\, \partial\tau}\, d\xi + \int_{R_z} F'(\sigma, \tau, W_z) W(d\xi)$$

$$+ \int_{R_z} F'' J(d_\xi) + \tfrac{1}{2}\int_{R_z} F''\, d\xi + \int_{R_z} \frac{\partial}{\partial\sigma} F'\, \partial_\tau W_\xi\, d\sigma$$

$$+ \int_{R_z} \frac{\partial}{\partial\tau} F'\, d\tau\, \partial_\sigma W_\xi + \tfrac{1}{2}\int_{R_z} \frac{\partial}{\partial\sigma} F''\, d\tau\, d\sigma$$

$$+ \tfrac{1}{2}\int_{R_z} \frac{\partial}{\partial\tau} F''\, d\tau\, d\sigma + \tfrac{1}{2}\int_{R_z} F'''\, d\tau \partial_\sigma W_\xi$$

$$+ \tfrac{1}{2}\int_{R_z} F'''\, \partial_\tau W_\xi\, d\sigma + \tfrac{1}{4}\int_{R_z} F^{(\mathrm{iv})}\, d\xi \tag{16}$$

where J_z denotes

$$J_z = \int_{R_z}\int_{R_z} W(d\xi) W(d\xi')$$

and the arguments of F', F'' etc. are (σ, τ, W_ξ), and $\xi = (\sigma, \tau)$.

Therefore

$$X(A(\gamma)) = \int_{A(\gamma)} dF = \int_{A(\gamma)} \frac{\partial^2 F(\sigma, \tau, W_\xi)}{\partial\sigma\,\partial\tau}\, d\xi + \int_{A(\gamma)} F'(\sigma, \tau, W_\xi) W(d\xi)$$

$$+ \int_{A(\gamma)} F'' J(d\xi) + \tfrac{1}{2} \int_{A(\gamma)} F''\, d\xi + \int_{A(\gamma)} \frac{\partial}{\partial\sigma} F'\, \partial_\tau W_\xi\, d\sigma$$

$$+ \int_{A(\gamma)} \frac{\partial}{\partial\tau} F'\, d\tau\, \partial_\sigma W_\xi + \tfrac{1}{2} \int_{A(\gamma)} \frac{\partial}{\partial\sigma} F''\, d\tau\, d\sigma$$

$$+ \tfrac{1}{2} \int_{A(\gamma)} \frac{\partial}{\partial\tau} F''\, \partial_\tau W_\xi\, d\sigma + \tfrac{1}{2} \int_{A(\gamma)} F'''\, d\tau\, \partial_\sigma W_\xi$$

$$+ \tfrac{1}{2} \int_{A(\gamma)} F'''\, d\tau\, W_\xi\, d\sigma + \tfrac{1}{4} \int_{A(\gamma)} F^{(\mathrm{iv})}\, d\xi. \tag{17}$$

Turning now to (14) and applying the one-parameter Itô formula to $\partial F/\partial\sigma$ yields for the first term in the right-hand side of (14)

$$\int_\gamma \frac{\partial F}{\partial\sigma}(\sigma(\theta), \tau(\theta), W_{\xi(\theta)})\, d\sigma(\theta) = \int_{A(\gamma)} \frac{\partial^2 F}{\partial\sigma\,\partial\tau}(\sigma, \tau, W_\xi)\, d\xi$$

$$+ \int_{A(\gamma)} \frac{\partial F'}{\partial\sigma}(\sigma, \tau, W_\xi)\, \partial_\tau W_\xi\, d\sigma$$

$$+ \tfrac{1}{2} \int_{A(\gamma)} \frac{\partial F''}{\partial\sigma}(\sigma, \tau, W_\xi)\, d\xi. \tag{18}$$

For the second term in the right-hand side of (14) we apply Green's formula of [4]:

$$\int_\gamma F'(\sigma(\theta), \tau(\theta), W_{\xi(\theta)})\, \partial_\theta W(A(\gamma) \cap R_{\xi(\theta)}) = \int_{A(\gamma)} F'(\sigma, \tau, W_\xi) W(d\xi)$$

$$+ \int_{A(\gamma)} F''\, dJ_\xi + \tfrac{1}{2} \int_{A(\gamma)} F'''\, d\tau\, \partial_\sigma W$$

$$+ \int_{A(\gamma)} F' \cdot d\tau\, \partial_\sigma W_\xi \tag{19}$$

and for the last term in (14) we have

$$\tfrac{1}{2}\int_{\gamma} F''(\sigma(\theta),\tau(\theta),W_{\xi(\theta)})\,d\sigma(\theta)=\tfrac{1}{2}\int_{A(\gamma)} F''(\sigma,\tau,W_{\xi})\,d\xi$$

$$+\tfrac{1}{2}\int_{A(\gamma)}\frac{\partial F''}{\partial\tau}d\xi+\tfrac{1}{2}\int_{A(\gamma)} F'''\,\partial_{\tau}W_{\xi}\,d\sigma$$

$$+\tfrac{1}{4}\int_{A(\gamma)} F^{(\mathrm{iv})}\,d\xi. \tag{20}$$

Substituting (18–20) into the right-hand side of (14) and comparing with (17) proves (14) and (15) follows by a similar argument.

Lemma 2(b) *Under the assumptions of Lemma 2(a), for every finite decreasing path* γ, $X(A(\gamma))$ *and* $X(B(\gamma))$ *are given by:*

$$X(A(\gamma))=\int_{\gamma}\frac{\partial F}{\partial\sigma}(\sigma(\theta),\tau(\theta),W_{\xi(\theta)})\,d\sigma(\theta)$$

$$+\int_{\gamma} F'(\sigma(\theta),\tau(\theta),W_{\xi(\theta)})\,\partial_{\theta}W(A(\zeta)\cap R_{\sigma(\theta),\infty})$$

$$+\tfrac{1}{2}\int_{\gamma} F''(\sigma(\theta),\tau(\theta),W_{\xi(\theta)})\,d\sigma(\theta) \tag{21}$$

and, with $\lambda=1-\theta$

$$X(B(\gamma))=\int_{\gamma}\frac{\partial F}{\partial\tau}(\sigma(\theta),\tau(\theta),W_{\xi(\theta)})\,d\tau(\theta)$$

$$+\int_{\gamma} F'(\sigma(\lambda),\tau(\lambda),W_{\xi(\lambda)})\,\partial_{\lambda}W(B(\gamma)\cap R_{\infty,\tau(\lambda)})$$

$$+\tfrac{1}{2}\int_{\gamma} F''(\sigma(\lambda),\tau(\lambda),W_{\xi(\lambda)})\,d\tau(\lambda). \tag{22}$$

The proof is the same as that of Lemma 2(a) and therefore omitted.

Proposition 5 *Let* $F(\cdot,\cdot,\cdot)$ *satisfy the assumptions of Lemma 2(a), further assume that* $F'(\cdot,\cdot,\cdot)$ *does not vanish on* $\mathbb{R}_{+}\times\mathbb{R}_{+}\times\mathbb{R}$ *and for*

$\hat{s}=\min(s_1, s_2)$, $\hat{t}=\min(t_1, t_2)$, it follows by a direct calculation that

$$E(X_{s_1,t_1}\cdot X_{s_2,t_2})=\hat{t}\,e^{-\alpha(s_1+s_2)}\int_0^{\hat{s}} e^{2\alpha\theta}\,d\theta$$

$$=\frac{\hat{t}}{2\alpha}(e^{-\alpha|s_1-s_2|}-e^{-\alpha(s_1+s_2)}).$$

Consider now the process [20]:

$$V_{s,t}=\frac{1}{\sqrt{(2\alpha)}}e^{-\alpha s}W_{e^{2\alpha s}-1,t}.$$

Then $EV_{s_1,t_1}V_{s_2,t_2}=EX_{s_1,t_1}\cdot X_{s_2,t_2}$ and since the two processes are zero mean and Gaussian they are identical in law. Setting $(\sigma, t)=(e^{2\alpha s}-1, t)$ we have

$$V_{s,t}=\frac{1}{\sqrt{(2\pi)}}\cdot\frac{1}{1+\sigma}W_{\sigma,t}.$$

Since $W_{\sigma,t}$ induces a $\gamma+$ Markov process, by Proposition 5, $(2\pi)^{-1/2}(1+\sigma)^{-1}W_{\sigma,t}$ is also $\gamma+$ Markov. Since $(s,t)=((2\alpha)^{-1}\log(1+\sigma), t)$ is order preserving it follows that $V_{s,t}$ and hence $X_{s,t}$ are $\gamma+$ Markov. We do not know whether the solution to $\partial_s X=g(x_s)\,ds+\partial_s W$ is $\gamma+$ Markov, a weaker Markov property of this process will be proved in the next section (Proposition 6).

5. THE MARKOV PROPERTY WITH RESPECT TO SEPARATING LINES

A path $L=\{z(\theta), 0<\theta<1\}$ in $\mathbb{R}_+^2$ will be said to be a separating line if it is (a) non-increasing; (b) as $\theta\to 0$ either $s(\theta)\to 0$ or $t(\theta)\to\infty$; and (c) as $\theta\to 1$ either $t(\tau)\to 0$ or $s(\theta)\to\infty$. Let $L=\{z(\theta), 0<\theta<1\}$ be a separating line and let

$$z^+(\theta)=\{z:z\geqq z(\theta)\},\qquad 0<\theta<1$$

$$z^-(\theta)=\{z:z\leqq z(\theta)\},=R_z(\theta),\qquad 0<\theta<1.$$

Set

$$L^+ = \bigcup_{0<\theta<1} z^+(\theta)$$
$$L^- = \bigcup_{0<\theta<1} z^-(\theta). \tag{23}$$

A separating line is, therefore, a non-increasing path separating $\mathbb{R}^2_+$ into "past" L^-, "present" L and "future" L^+.

Throughout this section, γ will denote a *decreasing path* and $\tilde{\Gamma}(D)$ will denote the collection of *decreasing paths* contained in D. A path parametrized process $\{Y_\gamma, X_{\gamma_0}, X_{\gamma_1}, \gamma \in \tilde{\Gamma}(\mathbb{R}^2_+)\}$ is said to be Markov with respect to separating lines if for every separating line L, $\sigma\{(Y_\gamma, X_{\gamma_0}, X_{\gamma_1}, \gamma \in \tilde{\Gamma}(L)\}$ splits $\sigma\{(Y_\gamma, X_{\gamma_0}, X_{\gamma_1}, \gamma \in \tilde{\Gamma}(L^+)\}$ and $\sigma\{(Y_\gamma, X_{\gamma_0}, X_{\gamma_1}, \gamma \in \tilde{\Gamma}(L^-)\}$. Obviously every process that is $\gamma+$ Markov is also Markov with respect to separating lines.

Two separating lines L_1 and L_2 will be said to satisfy the relation $L_1 < L_2$ if $L_1 \cap L_2 = \varnothing$ and $L_1 \subset L_2^-$. Let $L_\eta, 0 \leqq \eta \leqq 1$ be a one-parameter increasing collection of separating lines, i.e., $L_{\eta_1} < L_{\eta_2}$ whenever $\eta_1 < \eta_2$. Let

$$\sigma_{L_\eta} = \sigma\{(Y_\gamma, X_{\gamma_0}, X_{\gamma_1}, \gamma \subset L_\eta\}. \tag{24}$$

Then σ_{L_η}, $0 \leqq \eta \leqq 1$, $0 \leqq \eta \leqq 1$ is Markov in the sense that σ_{L_η} splits $\bigvee_{\rho<\eta} \sigma_{L_\rho}$ and $\bigvee_{\rho>\eta} \sigma_{L_\rho}$. Conversely, if for every increasing one-parameter collection of separating lines L_η, $0 \leqq \eta \leqq 1$, σ_{L_η}, $0 \leqq \eta \leqq 1$ is Markov then $(Y_\gamma, X_{\gamma_0}, X_{\gamma_1})$ is Markov with respect to separating lines.

PROPOSITION 6 *Consider the process X_z, $z \in \mathbb{R}^2_+$ defined by*

$$\partial_s X_{s,t} = g(X_{s,t})\, ds + \partial_s W_{s,t} \tag{25}$$

where $g(\cdot)$ satisfies a global Lipshitz condition and $X_{0,t}$ is a smooth non-random function of t. For $z = z(\theta)$, $0 \leqq \theta \leqq 1$ set:

$$Y_\gamma = X(A(\gamma)) = \int_\gamma g(X_{\sigma(\theta),\tau(\theta)})\, d\sigma(\theta) + W(A(\gamma)). \tag{26}$$

Then $(Y_\gamma, X_{\gamma_0}, X_{\gamma_1})$ is Markov with respect to separating lines.

Proof Note first that for every separating line L, $\sigma\{W(A(\gamma)), \gamma \in L\}$ splits $\sigma\{W(A(\gamma)), \gamma \in \tilde{\Gamma}(L^-)\}$ and $\sigma\{W(A(\gamma)), \gamma \in \tilde{\Gamma}(L^+)\}$ and

$$\sigma\{W(A(\gamma)), \gamma \in \tilde{\Gamma}(L^-)\} = \sigma\{X(A(\gamma)), X_{\gamma_0}, X_{\gamma_1}, \gamma \in \tilde{\Gamma}(L^-)\}.$$

Therefore $\sigma\{W(A(\gamma)), \gamma \subset L\}$ splits $\sigma\{W(A(\gamma)), \gamma \in \tilde{\Gamma}(L^+)\}$ and $\sigma\{Y_\gamma, X_{\gamma_0}, X_{\gamma_1}, \gamma \in \tilde{\Gamma}(L^-)\}$. A lemma of F. Knight states:

LEMMA [11] *Let $\mathscr{A}, \mathscr{B}, \mathscr{S}$ be subsigma fields of a sigma field and assume that $\mathscr{S}$ splits $\mathscr{A}$ and $\mathscr{B}$ then (a) if $\mathscr{S}_1$ satisfies $\mathscr{S} \subseteqq \mathscr{S}_1 \subseteqq \mathscr{S} \vee \mathscr{B}$ then $\mathscr{S}_1$ splits $\mathscr{A}$ and $\mathscr{B}$; (b) if $\mathscr{B}_1$ satisfies $\mathscr{B}_1 \subseteqq \mathscr{B} \vee \mathscr{S}$ then $\mathscr{S}$ splits $\mathscr{A}$ and $\mathscr{B}_1$.*

Applying part (a) of Knight's lemma and (26), it follows that $\sigma\{Y_\gamma, X_{\gamma_0}, X_{\gamma_1}, \gamma \subset L\}$ splits $\sigma\{W(A(\gamma)), \gamma \in \tilde{\Gamma}(L^+)\}$ and $\sigma\{Y_\gamma, X_{\gamma_0}, X_{\gamma_1}, \gamma \in \tilde{\Gamma}(L^-)\}$. Applying part (b) of Knight's lemma and (26) yields that $\sigma\{Y_\gamma, X_{\gamma_0}, X_{\gamma_1}, \gamma \subset L\}$ splits $\sigma\{Y_\gamma, X_{\gamma_0}, X_{\gamma_1}, \gamma \in \tilde{\Gamma}(L^-)\}$ and $\sigma\{Y_\gamma, X_{\gamma_0}, X_{\gamma_1}, \gamma \in \tilde{\Gamma}(L^+)\}$. Hence $(Y_\gamma, X_{\gamma_0}, X_{\gamma_1})$ is Markov with respect to separating lines.

The notion of a Markov process with respect to separating lines leads directly to the notion of a strong Markov process as follows (cf. Chapter 5 of [5] for a discussion of the strong Markov property for non-stationary Markov processes on $\mathbb{R}_+$). Let $(\Omega, \mathscr{F}, P)$ be a probability space and $\mathscr{F}_z$, $z \in \mathbb{R}_+^2$ a collection of subsigma fields of $\mathscr{F}$ that satisfy the conditions: (a) $\mathscr{F}_{z_1} \subseteqq \mathscr{F}_{z_2}$ whenever $z_1 \leqq z_2$; (b) $\cap \mathscr{F}_\xi = \mathscr{F}_z$ where the intersection is over all $\xi > z$; and (c) $\mathscr{F}_0$ contains all the null events of $\mathscr{F}$. Let S denote the collection of all separating lines including the separating line "∞". A stopping line $L(\omega)$ is a function from Ω to S satisfying for every $z \in \mathbb{R}_+^2$

$$\{\omega : z \in L^-(\omega)\} \in \mathscr{F}_z.$$

For every stopping line $L(\omega)$ set

$$\sigma_L = \sigma\{Y_\gamma, X_{\gamma_0}, X_{\gamma_1}, \gamma \subset L(\omega)\}.$$

For every $\gamma \in \tilde{\Gamma}(\mathbb{R}^2_+)$ set:

$$L^-(\gamma) = \gamma \cap L^-(\omega), \quad L^+(\gamma) = \gamma \cap L^+(\omega)$$

$$\sigma_{L^-} = \sigma\{Y_{L^-(\gamma)}, X_{(L^-(\gamma))_0}, X_{(L^-(\gamma))_1}, \gamma \in \tilde{\Gamma}(\mathbb{R}^2_+)\}$$

$$\sigma_{L^+} = \sigma\{Y_{L^-(\gamma)}, X_{(L^+(\gamma))_0}, X_{(L^+(\gamma))_1}, \gamma \in \tilde{\Gamma}(\mathbb{R}^2_+)\}.$$

The process $\{Y_\gamma, X_{\gamma_0}, X_{\gamma_1}, \gamma \in \tilde{\Gamma}(\mathbb{R}^2_+)\}$ will be said to be strongly Markov if for every stopping line L, σ_L splits σ_{L^-} and σ_{L^+}.

Remark A strong Markov property for random fields has already been introduced by Evstingneev ([7], p. 85 of [19]) but it is different from the one introduced here.

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