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MARKOVIAN RANDOM FIELDS

Eugene Wong

Department of Electrical Engineering and Computer Sciences
and the Electronics Research Laboratory
University of California, Berkeley, California 94720

Summary

In this paper we examine the Markovian properties of three important random fields: Lévy's Brownian motion, free Euclidean field, and Wiener process. In so doing, we advance the proposition that appropriate candidates for Markov fields are stochastic differential forms and their Markovian property is characterized by being "one derivative" removed from white noise.

1. Introduction

A continuous parameter random field is a stochastic process $\{X_t, t \in T\}$ with a multidimensional parameter set T (say $T \subseteq \mathbb{R}^n$). Let ∂D be a smooth $(n-1)$-surface separating T into two parts D^- and D^+. Following Lévy[1] we say X is Markov if for every such ∂D, the future $\{X_t, t \in D^+\}$ and the past $\{X_t, t \in D^-\}$ are conditionally independent given the present $\{X_t, t \in \partial D\}$.

It is interesting to examine the Markov property in terms of three examples. All three are Gaussian processes and each has played a prominent role in the theory of random fields. These are: Lévy's Brownian motion, the "free Euclidean field," and the Wiener process

2. Three Processes Exhibiting a Markov Character

Lévy's Brownian motion is a Gaussian process $\{B_t, t \in \mathbb{R}^n\}$ with zero mean and a covariance function

$$R(t,s) = EB_tB_s = \frac{1}{2}(|t| + |s| - |t-s|) \tag{2.1}$$

where $|\ |$ denote the Euclidean norm. Lévy conjectured and McKean proved[2] that B had a Markov property (though generalized) in odd dimensions but none in even dimensions What is responsible for this state of affairs is the fact that for $n = 2p-1$

$$\Delta^p R(t,s) = -\frac{1}{2}\Delta^p|t-s| \tag{2.2}$$
$$= K_p\,\delta(t-s)$$

where Δ denotes the Laplacian operator. For a smooth ∂D, $\{B_t, t \in D^+\}$ and $\{B_t, t \in D^-\}$ are conditionally independent given $\{\partial^k B_t, t \in \partial D, 0 \le k \le p-1\}$ where $\partial^k B_t$ denotes the $\{k^{th}$ outward normal derivative$\}$ at a point t on the boundary ∂D.

In [3], the following question was posed and answered. Let $\{X_t, t \in \mathbb{R}^n\}$ be an isotropic and homogeneous Gaussian process with zero mean. What must its covariance

$$R(t,s) = F X_t X_s$$

be in order for the processes to be Markov? It turns out that strictly speaking there is no such process. However, if we allow X to be a generalized process, then a covariance function of the form

$$R(t,s) = K\int_0^\infty \frac{J_{\frac{n}{2}-1}(|t-s|)}{(|t-s|)^{\frac{n}{2}-1}}\frac{\lambda^{n-1}d\lambda}{(\alpha^2+\lambda^2)} \tag{2.3}$$

where J_k is the Bessel function is both necessary and sufficient for X to be Markov. The fact that $R(t,t) = \infty$ renders X a generalized process.

At the same time, X must also be localizable to $(n-1)$ dimensional surfaces in order for its Markov property to be properly defined. This process independently discovered by Nelson[4] is now widely known as the "free Euclidean field."

Let $\mathbb{R}^n_+$ denote $\{t \in \mathbb{R}^n : t_i \ge 0 \text{ for } 0 \le i \le n\}$. A standard Wiener process $\{W_t, t \in \mathbb{R}^n_+\}$ is defined as a zero-mean Gaussian process with covariance function

$$E\,W_t\,W_s = \prod_{i=1}^{n} \min(t_i,s_i)$$

An alternative, and intuitively appealing, way of defining W is in terms of a Gaussian "white noise." Let $\mathcal{R}^n$ denote the σ-field of Borel sets in $\mathbb{R}^n$. We say $\{\eta(A), A \in \mathcal{R}^n\}$ is a "standard Gaussian white noise" if η is a zero-mean Gaussian random function with

$$E\eta(A)\,\eta(B) = \mu(A \cap B)$$

where μ denotes the Lebesgue measure. Now, let A_t, $t \in \mathbb{R}^n_+$, denote the closed rectangle with the origin and t as its two extreme points. Then, $W_t = \eta(A_t)$ is a standard Wiener process.

One might think that if any process is Markov, it should be Wiener process. Surprisingly, it is not. Even for $n=2$, Walsh[5] has shown that W is not Markov. To see that, consider the triangular region

$$D^- = \{(t_1,t_2) : t_1,t_2 \ge 0,\ t_1+t_2 \le 1\}$$

with a boundary

$$\partial D = \{(t_1,t_2) : t_1,t_2 \ge 0 \text{ and } t_1+t_2 = 1\}$$

Now,

$$E[W_{1,1}|W_t,\ t \in D^-] = \eta(D^-)$$

while

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$$F[W_{1,1}|W_t,\ t \in \partial D] = E[\eta(D^-)\Delta W_t,\ t \in \partial D]$$

$$= \int_0^1 W_{s,1-s}\, ds \neq \eta(D^-)$$

Hence, W cannot be Markov. One of our objectives will be to elucidate why this is the case and to discover what Markov properties, if any, are possessed by the Wiener process.

3. Stochastic Differential Forms

The case of the free Euclidean field suggests that Markov fields are not point-parametered processes, but processes parametered by oriented (n-1)-dimensional sets in $\mathbb{R}^n$. As such, they are stochastic versions of co-chains[6], or equivalently, differential forms. We shall use the latter terminology to emphasize the "local" nature of the process.

The foundation of a theory of stochastic differential forms has been laid and the basic results will be reported elsewhere.[7] Here, we limit ourselves to a brief account of the concepts that we need for studying the Markov properties of the three basic Gaussian fields.

Consider an r-dimensional rectangle σ in $\mathbb{R}^n$ with edges parallel to the axes: $t_{i_1}, t_{i_2}, \ldots, t_{i_v}$. Let $\underline{i}$ denote $(i_1, i_2, \ldots, i_v)$ and $[\underline{i}]$ denote a permutation of $\underline{i}$ that puts it in increasing order. We give σ an orientation (+ or -), and call $[\underline{i}]$ its direction. Let $\sigma_1, \sigma_2, \ldots, \sigma_k$ be oriented r-rectangles (not necessarily co-directional), and let $\sigma_1, \sigma_2, \ldots, \sigma_k$ be real numbers. Sums of the form

$$v = \sum_{i=1}^{k} \alpha_i\, \sigma_i \qquad (3.1)$$

form a vector space V^r under the equivalence relations:

(a) $(-\alpha)\sigma = -(\alpha\sigma) = \alpha(-\sigma)$

(b) if σ is subdivided into σ_1 and σ_2 then $\sigma = \sigma_1 + \sigma_2$

We call elements of the vector space V^r rectangular r-chains, or simply r-chains.

We note that the boundary $\partial\sigma$ of an oriented r-rectangle σ is an (r-1)-chain. Hence, the boundary ∂v of an r-chain v is an (r-1)-chain.

Let S denote the space of all random variables defined on a fixed probability space $(\Omega, \mathcal{F}, \mathcal{P})$. We define a stochastic differential r-form X as follows:

(D_1) X is a linear map : $V \to S$

(D_2) Let $|\sigma|$ denote the r-volume of σ. Then,

$$|\sigma_m| \underset{m\to\infty}{\to} \Rightarrow X(\sigma_m) \underset{m\to\infty}{\to} 0 \text{ in probability}$$

(D_3) Suppose that $\{v_m\}$ is a sequence of (r+1)-chain:

$$|v_m| \underset{m\to\infty}{\to} 0 \Rightarrow X(\partial v_m) \underset{m\to\infty}{\to} 0 \text{ in probability}$$

Condition (D_1) implies that it is sufficient to specify X on positively oriented r-rectangles. Conditions (D_2) and (D_3) imply that X can be extended by continuity to any oriented r-dimensional surrace r that is the limit of a sequence of r chains $\{v_m\}$ such that either $|r-v_m| \underset{m\to\infty}{\to} 0$

or $\quad (r-v_m) = \partial D_m$ and $|D_m| \underset{m\to\infty}{\to} 0$

We note that a 0-form is an ordinary point-parametered process, and an n-form in $\mathbb{R}^n$ is a random measure. For example, a standard Gaussian white noise η is a Gaussian n-form in $\mathbb{R}^n$ such that for any n-rectangle σ $E\eta(\sigma) = 0$ and

$$E\eta^2(\sigma) = |\sigma|$$

Stochastic differential r-forms can be thought of as generalized processes with sufficient smoothness as that when integrated on r-dimensional sets they enjoy good regularity properties. Stochastic differential forms are random currents in the sense of Ito,[8] but not all Ito currents are differential forms.

If X is an r-form with $r \leq n-1$, then we define its exterior derivative dX as an (r+1) form with

$$(dX)(\sigma) = X(\partial\sigma) \qquad (3.2)$$

It is easy to verify that dX is indeed a form, i.e., that D_1-D_3 are satisfied. Thus, "forms" are closed under exterior differentiation, a surprising and important result.

Next, we develop a coordinate system for stochastic differential forms. For oriented r-rectangles in $\mathbb{R}^n$, there are $\binom{n}{r}$ possible directions. For a given direction $[\underline{i}]$, define $X_{[\underline{i}]}$ as the $[\underline{i}]$ coordinate of an r-form X as follows: For any r-rectangle σ,

$$X_{[\underline{i}]}(\sigma) = X(\sigma) \quad \text{if } \sigma \text{ is a rectangle with direction } [\underline{i}]$$

$$= 0 \quad \text{otherwise}$$

So defined, $X_{[\underline{i}]}$ is an r-form for each $[\underline{i}]$, and the map $X \to X_{[\underline{i}]}$ is obviously linear. We note that $\{X_{[\underline{i}]}\}$ is a decomposition of X, so that

$$X = \sum_{[\underline{i}]} X_{[\underline{i}]}$$

Finally, we define d_k for $1 \leq k \leq n$ by combining exterior differentiation with decomposition into coordinates as follows: For any (r+1)-rectangle σ

$$(d_k X_{[\underline{i}]})(\sigma) = dX_{[\underline{i}]}(\sigma) \quad \text{if } \sigma \text{ has direction } [\underline{i}, k]$$

$$= 0 \quad \text{otherwise}$$

We define $d_k X$ by

$$d_k X = \sum_{[\underline{i}]} d_k X_{[\underline{i}]}$$

We observe that (3.2) is in effect the Stokes theorem which we are using for a definition. We further note that $\partial\partial D$ is always zero so that

$$dd \equiv 0$$

as is consistent with ordinary forms. Finally, we note that

$$d = \sum_{k=1}^{n} d_k$$

Hence,

$$d^2 = 0 \Rightarrow \sum_{k=1}^{n} d_k^2 + \sum_{k>j} (d_j d_k + d_k d_1) = 0$$

which implies that $d_k^2 = 0$ and $d_j d_k = -d_k d_j$. Finally, we observe that both exterior differentiation and rectangular coordinate system are well defined for Ito random currents. As defined on differential forms, they are fully compatible with the definitions for random currents.

4. White Noise and Markovian Fields

In this section we show that the Markovian properties of the three processes: Lévy's Brownian motion, Euclidean free field and Wiener process, can all be traced to their relationships with the Gaussian white noise.

We begin with the Wiener process since it is the one most closely related to white noise noise. We recall that a standard Gaussian white noise η is defined as a Gaussian stochastic n-form in $\mathbb{R}^n$ such that for a positively oriented rectangle

$$E\eta^2(\sigma) = |\sigma| \tag{4.1}$$

A standard Wiener process $\{W_t,\ t \in \mathbb{R}_+^n\}$ is in turn defined as a 0-form with

$$W_t = \eta(A_t) \tag{4.2}$$

When A_t is the positively oriented rectangle bounded by the origin and t.

We note that (4.2) can be "inverted" to express η in terms of W as follows:

$$\eta = d_1 d_2 \ldots d_n W$$

where d_i was defined in the previous section. We observe that if we denote

$$\Delta_i = d_1 \ldots d_{i-1}\, d_{i+1} \ldots d_n \tag{4.3}$$

then

$$d\Delta_i = (-1)^{i-1} d_1 d_2 \ldots d_n$$

so that we can write

$$\eta = d((-1)^{i-1}\Delta_i W)$$

Now, for each i $(1 \le i \le n)$, $\Delta_i W$ is an (n-1)-form such that for an (n-1) rectangle σ $(\Delta_i W)(\sigma) = 0$ unless σ lies in a hyperplane perpendicular to the t_i-axis. This property together with

$$d(\Delta_i W) = (-1)^{i-1}\eta \tag{4.4}$$

imply that for each i, $\Delta_i W$ is Markov.

It is clear that we only need to prove this for i=1. The rest follows by symmetry. Let $\mathbb{R}_+^n$ be divided into a bounded region D^- and an unbounded one D^+. Let ∂D denote the boundary of D^+, which includes parts of the hyperplanes $H_i = \{t : t_i = 0\}$. Now, let σ be an (n-1) rectangle in D^+ perpendicular to the t_1-axis, and construct a cylindrical volume V bounded by σ, ∂D, and hyperplanes parallel to t_1-axis. With appropriate orientation, we can write

$$\partial V = \sigma - \sigma' + (\text{faces parallel to } t_i\text{-axis})$$

where σ' is a subset of ∂D. Such $\Delta_1 W$ is zero on the faces parallel to t_1-axis, we have

$$(d\Delta_1 W)(V) = (\Delta_1 W)(\partial V)$$
$$= (\Delta_1 W)(\sigma) - (\Delta_1 W)(\sigma')$$
$$= \eta(V)$$

It follows that for $\sigma \subset D^+$, $\Delta_1 W(\sigma)$ can be expressed as the sum of η on a set V in D^+ and $\Delta_1 W$ on a set σ' on the boundary ∂D. Similarly, for a rectangle γ in D^- perpendicular to t_1-axis, $\Delta_1 W(\gamma)$ can be expressed as the sum of η on a set V' contained in D^- and $\Delta_1 W$ on a set γ' on the boundary ∂D. Hence, given $\Delta_1 W$ or ∂D, $\Delta_1 W(\sigma)$ and $\Delta_1 W(\gamma)$ are conditionally independent, and the Markovian property of $\Delta_1 W$ is proved.

The extent to which a Wiener process is Markovian is now clear. It is not W but $\Delta_i W$ that is Markov. The intuitive reason is that $\Delta_i W$ is "one derivative removed," whereas W is "n derivative removed," from white noise.

For the two dimensional case (n=2), it means that $d_1 W$ $(=\Delta_2 W)$ and $d_2 W$ $(=\Delta_1 W)$ are Markov 1-forms. It follows trivially that, taken together, $\{W, d_1 W, d_2 W\}$ is Markov. This implies that if we set $Z = f(W)$, then $\{Z, d_1 Z, d_2 Z\}$ is also Markov. Thus, this vector Markov character (dubbed γ-Markov in [9]) is preserved under nonlinear transformation. Generalization to the n-dimensional is clear.

Next, we show that the free Euclidean field is also "one derivative away" from white noise. To keep the details simple, we shall restrict ourselves to $n \ge 3$, in which case we can take $\alpha=0$ in (2.3) and assume the free Euclidean field to have a spectral density

$$S(\nu) = \frac{1}{|\nu|^2}\ ,\quad \nu \in \mathbb{R}^n \tag{4.5}$$

Now, take n independent and identically distributed free Euclidean fields $\{\xi_i,\ i=1,2,\ldots,n\}$ each with spectral density (4.5), and define an (n-1)-form X by setting

$$X(\sigma) = \int_\sigma \xi_{it}\, dt \quad \text{for } \sigma \perp t_i\text{-axis} \tag{4.6}$$

The integral in (4.6) is formal, and needs to be properly defined (e.g., by using the spectral representation), but it can be done. Equation (4.6) means that, if we denote by i^* the sequence of 1 through n with i deleted, then

$$X_{[i^*]}(\sigma) = \int_\sigma \xi_{it}\, dt \quad \text{for } \sigma \perp t_i\text{-axis} \tag{4.7}$$
$$= 0 \qquad \text{otherwise}$$

We call the (n-1) form X so constructed a "standard free Markov form."

Now it is easy to verify that if we set

$$\eta = dX \tag{4.8}$$

then η is a Gaussian white noise. From this fact the Markovian property of X, and of each $X_{[i^*]}$, can be proved in a direct way.

Finally, we turn to Lévy's Brownian motion B. Equation (2.2) suggests that we deal with two situations

(a) $n = 2p-1,\ p = 2k \Rightarrow n = 4k-1,\ k = 1,2\ldots$

(b) $n = 2p-1,\ p = 2k+1 \Rightarrow n = 4k+1,\ k = 1,2,\ldots$

For (a) define an n-form Y by setting

$$Y(\sigma) = \int_\sigma B_t \, dt$$

Equation (2.2) then implies

$$\Delta^k Y = \eta$$

is a Gaussian white noise. For (b) define an (n-1) form Z by taking n independent Brownian motions B_{it} setting

$$Z(\sigma) = \int_\sigma B_{it} \, dt, \quad \text{for } \sigma \perp t_i\text{-axis}$$

Then,

$$\Delta^k Z = X$$

is a standard free Markov form.

To summarize, we can elucidate the Markovian character of Lévy's Brownian motion B in odd dimensions (n = 4k-1 or 4k+1) by examining $\Delta^k B$. The generalized process $\Delta^k B$ is either a white noise or a component of a free Markov form X whose exterior derivative dX is again a white noise.

5. Conclusion

We have examined three important examples of multiparameter processes: Lévy's Brownian motion, free Euclidean field, and Wiener processes. In all three cases we have related them Markovian properties to white noise using the machinery developed for stochastic differential forms.

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