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Multiple Stochastic Integrals: Projection and Iteration*

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Summary. Multiple stochastic integrals are defined relative to a class of sets. The classic cases of multiple Wiener integral and Ito integral (as well as its generalization by Wong-Zakai-Yor) are recovered by specializing the class of sets appropriately. Any square-integrable functional of the Wiener process has a canonical representation in terms of the integrals.

Formulas are given for projecting a stochastic integral onto the space of Wiener functionals and for representing multiple stochastic integrals as iterated integrals. Applications to a change in probability measure arising in a signal detection problem are given.

1. Introduction

Let $\mathscr{R}$ denote the collection of all Borel sets in $\mathbb{R}^n$ with finite Lebesque measure (denoted by μ). Define a *Wiener process* $\{W(A), A \in \mathscr{R}\}$ as a family of Gaussian random variables with zero mean and

$$E\,W(A)\,W(B) = \mu(A \cap B). \tag{1.1}$$

As a set-parameter process, $W(A)$ is additive, i.e.,

$$W(A+B) = W(A) + W(B), \qquad \text{a.s.} \tag{1.2}$$

where $A+B$ denotes the union of disjoint sets, and intuitively, we can view $W(A)$ as the integral over A of a Gaussian white noise.

The connection with white noise renders the Wiener process important in applications as well as theory. Consider for example, the following signal detection problem.

A process ξ_t is observed on $t \in T$ where T is a fixed rectangle in $\mathbb{R}^n$, and we

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have to decide between the possibilities: (a) ξ_t contains a random signal Z_t plus an additive Gaussian white noise and (b) ξ_t contains only noise.

Formulated so as to avoid the pathologies of "white noise", the problem can be stated as follows: Let $\{W(A), A\in\mathscr{R}(T)\}$ be a set-parameter process, with parameter space $\mathscr{R}(T)=\{\text{Borel subsets of } T\}$, and defined on a fixed measurable space $(\Omega, \mathscr{F})$. Let $\mathscr{P}'$ and $\mathscr{P}$ be two probability measures such that (a) under $\mathscr{P}'$ $W(A)-\int_A Z_t\,dt$ is a Wiener process independent of $\{Z_t, t\in T\}$, (b) under $\mathscr{P}$ $W(A)$ is a Wiener process.

Now, let $\mathscr{F}_W$ denote the σ-algebra generated by the process W, and let $\mathscr{P}_W$ and $\mathscr{P}'_W$ denote the respective probability measures restricted to $\mathscr{F}_W$. If $\int_T Z_t^2\,dt<\infty$; a.s., then $\mathscr{P}'_W \ll \mathscr{P}_W$ and the detection problem in most cases reduces to one of computing the likelihood ratio

$$\Lambda=\frac{d\mathscr{P}'_W}{d\mathscr{P}_W} \tag{1.3}$$

in terms of the observed process W.

With respect to the probability space $(\Omega, \mathscr{F}, \mathscr{P})$ $\{W(A), A\in\mathscr{R}(T)\}$ is a Wiener process. Hence, Λ is a positive integrable functional of a Wiener process. Computing Λ in terms of W is a problem that can be embedded in a more general one of finding representations of a Wiener functional, which in turn can be embedded (and illuminated in the process) in a still more general problem of representing martingales generated by a Wiener process.

For a random variable Y that is a square-integrable functional of a Wiener process $\{W(A), A\in\mathscr{R}(T)\}$, several representations already exist. The first is the Hermite-Wiener series of Cameron and Martin [1]. The second is in terms of the multiple Wiener integrals as defined by Ito [6]. The third is in terms of the Ito integral [5], and it generalization as defined by Wong and Zakai [9] and Yor [11]. In the last representation the concept of martingales plays a crucial role.

For processes with a multidimensional parameter, it is both more natural and more general to define *martingales* for processes parameterized by sets rather than by points in $\mathbb{R}^n$. Let $\mathscr{C}\subset\mathscr{R}(T)$ be a collection of closed sets, let $\{\mathscr{F}(A), A\in\mathscr{C}\}$ be a family of σ-algebras such that $A\supseteq B \Rightarrow \mathscr{F}(A)\supseteq\mathscr{F}(B)$, and let $\{M(A), A\in\mathscr{C}\}$ be a set-parameter process. We say that $\{M(A), \mathscr{F}(A), A\in\mathscr{C}\}$ is a *martingale* if

$$E(M(A)|\mathscr{F}(B))=M(B) \qquad \text{a.s.}$$

whenever $A\supset B$. Let $\{W(A), A\in\mathscr{R}(T)\}$ be a Wiener process and denote

$$\mathscr{F}_W(A)=\sigma(\{W(B), B\subset A \text{ and } B\in\mathscr{R}(T)\}) \tag{1.4}$$

One of the main objects of this paper is to show that under very general conditions on $\mathscr{C}$, there is a canonical representation of all square-integrable martingales with respect to $\{\mathscr{F}_W(A), A\in\mathscr{C}\}$, and hence representation for square integrable Wiener functionals. For $\mathscr{C}=\{\text{all closed sets}\}$ the representation reduces to that of multiple Wiener integrals. For $\mathscr{C}=\{$all closed rectangles in

$\mathbb{R}_+^n$ with the origin as one corner} the representations of Ito, Wong-Zakai, and Yor are recovered. These two are in a sense limiting cases, and between them lies a vast spectrum of choices for $\mathscr{C}$, giving rise to an equally large array of representations for $\mathscr{C}$-martingales and Wiener functionals.

The key to these representations is to define multiple stochastic integrals of the form

$$\int_{T^m} \phi(t_1, t_2, \ldots, t_m) W(dt_1) \ldots W(dt_m)$$

where ϕ is (in general) a random integrand $\mathscr{C}$-adapted in a suitable sense to be defined later. The integrand in such a stochastic integral is then identified as a certain density of conditional moments.

Next, formulae are found for transformation of multiple stochastic integrals under two operations. The first is a projection formula for the projection of a multiple stochastic integral onto $L^2(\Omega, \mathscr{F}_W, \mathscr{P})$ (equivalently, this is a formula for the conditional expectation of a multiple stochastic integral given $\mathscr{F}_W$). The second is an iterated integral formula for expressing multiple stochastic integrals defined relative to $\mathscr{C}$ in terms of stochastic integrals defined relative to another class of sets $\mathscr{C}'$.

Finally, the transformation formulae are applied to the signal detection problem noted above. The projection formula is relevant since the likelihood ratio is the projection of the Radon-Nikodym derivative $d\mathscr{P}'/d\mathscr{P}$, and the iterated integral formula is relevant as a first step towards a stochastic calculus in a general framework.

Portions of this paper appear in [4]. This work is an outgrowth of ideas first introduced in the dissertation [3]. The present paper is self-contained except for the ommission of two proofs for which the reader is referred to [4].

2. Multiple Stochastic Integrals

Let $\mathscr{C}$ be a collection of closed subsets of a fixed rectangle T in $\mathbb{R}^n$. Given sets $A_1, A_2, \ldots, A_m \in \mathscr{R}(T)$, we shall define their *support* relative to $\mathscr{C}$ to be the following subset of $\mathscr{C}$:

$$S_{A_1 A_2 \ldots A_m} = \bigcap \{B: B \in \mathscr{C} \text{ and } B \cap A_i \neq \emptyset \text{ for } 1 \leqq i \leqq m\} \tag{2.1}$$

with the convention that if no such sets B exist then the support is taken to be all of T. Also, the support of the empty collection of sets (i.e. $m=0$) is simply the intersection of all sets in $\mathscr{C}$ and is denoted by S. (Note that $S = S_T$). It will be assumed that the support of any collection of sets $A_1, \ldots, A_m \in \mathscr{R}(T)$ is contained in $\mathscr{C}$. This assumption can be met by enlarging a given collection of sets $\mathscr{C}$.

If $t_1, t_2, \ldots, t_m$ are points in T, their support will be written as $S_{t_1 t_2 \ldots t_m}$. We say $t_1, t_2, \ldots, t_m$ are *$\mathscr{C}$-independent* if no point is contained in the support of the remaining ones.

For $\mathscr{C} = \{$all closed sets in $T\}$, $S_{t_1 t_2 \ldots t_m}$ is just $\{t_1, \ldots, t_m\}$ so that $\mathscr{C}$-independent means distinct. For $\mathscr{C} = \{$all convex sets in $T\}$, the support of m points is their convex hull and the points are $\mathscr{C}$-independent if and only if they are extreme points of their convex hull. When $T \subset \mathbb{R}_+^n$ and $\mathscr{C} = \{R_t : t \in T\}$ where R_t denotes the closed rectangle bounded by the origin and t, then $S_{t_1 t_2 \ldots t_m}$ is the smallest set in $\mathscr{C}$ which contains $t_1, t_2, \ldots, t_m$.

Another example is when $T \subset \mathbb{R}_+^n$ and $\mathscr{C}$ is generated by $\{Q_t : t \in T\}$ where $Q_t = \{s \in T: \ s_i \leqq t_i \ \text{for some } i\}$. Then for $t_1, t_2, \ldots, t_m \in T, S_{t_1 t_2 \ldots t_m} = \bigcup_i R_{t_i}$. Moreover

$$\mathscr{C} = \left\{ \bigcup_{i=1}^{m} R_{t_i} : m < +\infty \text{ and } t_1, t_2, \ldots, t_m \in T \right\}$$

For this example, m points are unordered if and only if they are pairwise unordered.

Let $\hat{T}^m$ denote the subset of $\mathscr{C}$-independent points in T^m. For a given collection $\mathscr{C}$, $\hat{T}^m$ may be vacuous for sufficiently large m. For example, if $\mathscr{C} = \{R_t\}$ is the collection of rectangles bounded by the origin and $t \in T \subset \mathbb{R}_+^n$, then $\hat{T}^m$ is empty for $m > n$. That is, no more than n points can be $\mathscr{C}$-independent. In the extreme case $\mathscr{C} = \{T\}$, $\hat{T}^m$ is empty for all $m \geqq 1$.

For a subset A of T define $B(\varepsilon, A)$ to be the set of points in T of Euclidean distance at most ε from A. For $\varepsilon > 0$ define the ε-support relative to $\mathscr{C}$ of $A_1, A_2, \ldots, A_m \in \mathscr{R}^n(T)$ by

$$S^{\varepsilon}_{A_1 A_2 \ldots A_m} = S_{B(\varepsilon, A_1) B(\varepsilon, A_2) \ldots B(\varepsilon, A_m)},$$

and let $S^{(-)}_{A_1 A_2 \ldots A_m}$ denote the union over all $\varepsilon > 0$ of the ε-support of $A_1, A_2, \ldots, A_m$. Note that the ε-support of $A_1, A_2, \ldots, A_m$ increases to $S^{(-)}_{A_1 A_2 \ldots A_m}$ as ε decreases to zero and $S^{(-)}_{A_1 A_2 \ldots A_m}$ is contained in the support of $A_1, A_2, \ldots, A_m$.

Let $(\Omega, \mathscr{F}, \mathscr{P})$ be a fixed probability space, let $\{\mathscr{F}(A): A \in \mathscr{R}(T)\}$ be a family of sub-σ-algebras of $\mathscr{F}$ which is increasing in the sense that $A \subset B$ implies that $\mathscr{F}(A) \subset \mathscr{F}(B)$, and let $\{W(A): A \in \mathscr{R}(T)\}$ be a Wiener process such that $\mathscr{F}_W(A) \subset \mathscr{F}(A)$ and $\mathscr{F}_W(A^c)$ is independent of $\mathscr{F}(A)$ for all A in $\mathscr{R}(T)$, where $\mathscr{F}_W(A)$ is defined by Eq. (1.4). These conditions are true, for example, if $\mathscr{F}(A) = \mathscr{F}_W(A)$ for all A.

We shall assume the following conditions on $\mathscr{C}$ and $\{\mathscr{F}(A): A \in \mathscr{R}^n(T)\}$:

(c_1) For every collection of rectangles $A_1, A_2, \ldots, A_m$ such that $\prod_{i=1}^{m} A_i \subset \hat{T}^m$,

$$\mu(A_i \cap S_{A_1 A_2 \ldots A_m}) = 0;$$

(c_2) For each $m \geqq 1$, the mapping

$$t = (t_1, t_2, \ldots, t_m) \rightsquigarrow S_t$$

is a continuous map from T^m to the collection of compact sets under the Hausdorff metric:

$$\rho(A, B) = (\max_{x \in A} \min_{y \in B} |x - y| + \max_{x \in B} \min_{y \in A} |x - y|); \tag{2.2}$$

(c_3) For every collection of rectangles $A_1, A_2, \ldots, A_m$ in T,

$$\bigvee_{\varepsilon > 0} \mathscr{F}(S^{\varepsilon}_{A_1 A_2 \ldots A_m}) = \mathscr{F}(S_{A_1 A_2 \ldots A_m}).$$

Since $\mathscr{F}_W(A) \subset \mathscr{F}(A)$ for all A in $\mathscr{R}(T)$, condition (c_3) implies the following condition:

(c_3') For every collection of rectangles $A_1, A_2, \ldots, A_m$ in T,

$$\mu(S_{A_1 A_2 \ldots A_m} - S^{(-)}_{A_1 A_2 \ldots A_m}) = 0, \tag{2.3}$$

and if $\mathscr{F}_W(A) = \mathscr{F}(A)$ for all A then conditions (c_3) and (c_3') are equivalent. Condition (c_3), as well as condition (c_2), is a continuity condition. Note that since the sets in $\mathscr{C}$ are closed, condition (c_2) insures that $\hat{T}^m$ is an open subset of T^m in the relative topology on T^m.

For a $\mathscr{C}$ satisfying conditions c_1–c_3, we shall define multiple stochastic integrals of order m

$$\phi \circ W^m = \int_{\hat{T}^m} \phi_t W(dt_1) \ldots W(dt_m)$$

for integrands $\phi(\omega, t)$, $(\omega, t) \in \Omega \times \hat{T}^m$, satisfying

(h_1) ϕ is $\mathscr{F} \times \mu^m$-measurable.

(h_2) For each $t \in \hat{T}^m$ ϕ_t is $\mathscr{F}(S_t)$-measurable.

(h_3) $\int_{\hat{T}^m} E\phi_t^2 \, dt < \infty$.

The space of functions satisfying h_1–h_3 will be denoted by $L_a^2(\Omega \times \hat{T}^m)$. For ϕ and θ in $L_a^2(\Omega \times \hat{T}^m)$ define

$$\langle \phi, \theta \rangle = E \int_{\hat{T}^m} \phi_t \theta_t \, dt$$

and let $\tilde{\phi}$ denote the symmetrization of ϕ, i.e., $\tilde{\phi}_t = \frac{1}{m!} \sum_\pi \phi_{\pi(t)}$, $\pi(t)$ = permutation. Call ϕ *atomic* if $\phi(\omega, t) = \alpha(\omega) I_A(t)$ where I_A is the indicator function of a product of rectangles $A = \prod_{i=1}^m A_i$ such that $A \subset \hat{T}^m$. A proof of the following theorem is given in [4]. The proof parallels Ito's original construction [5] of the stochastic integral.

Theorem 2.1. *There is a unique linear map denoted by $\phi \circ W^m$ of $\phi \in L_a^2(\Omega \times \hat{T}^m)$ into the space of square-integrable random variables such that*

(a) *For an atomic function $\phi = \alpha I_A$*

$$\phi \circ W^m = \alpha \prod_i W(A_i).$$

(b) *Symmetry:*

$$\phi \circ W^m = \tilde{\phi} \circ W^m.$$

(c) *Isometry:*

$$E(\phi \circ W^m)(\theta \circ W^p) = \langle \tilde{\phi}, \tilde{\theta} \rangle \delta_{mp}.$$

Remark. Observe that the isometry property of the multiple stochastic integral implies uniqueness up to equivalence of the integrand. That is, if $\phi \circ W^m = \theta \circ W^m$ then

$$\|\tilde{\phi} - \tilde{\theta}\|^2 = \int_{\hat{T}^m} E(\tilde{\phi}_t - \tilde{\theta}_t)^2 \, dt = 0.$$

Let $\{(\phi \circ W^m)_B, B \in \mathscr{C}\}$ be the set-parameterized process defined by

$$(\phi \circ W^m)_B = \phi I_{B^m} \circ W^m.$$

We shall call $(\phi \circ W^m)_B$ the *indefinite integral* of $\phi \circ W^m$.

Proposition 2.2. *The process* $\{(\phi \circ W^m)_B, \mathscr{F}(B): B \subset \mathscr{C}\}$ *is a martingale.*

Proof. It is enough to establish the proposition when ϕ is atomic. Let $\phi = \alpha I_{A_1 \times A_2 \times \ldots \times A_m}$. Then for $B \in \mathscr{C}$,

$$\begin{aligned} E(\phi \circ W^m | \mathscr{F}(B)) &= E\left(\alpha \prod_{i=1}^{m} W(A_i) | \mathscr{F}(B)\right) \\ &= E\left(\alpha E\left[\prod_{i=1}^{m} W(A_i) | \mathscr{F}(B \cup S_{A_1 A_2 \ldots A_m})\right] \middle| \mathscr{F}(B)\right) \\ &= E\left(\alpha \prod_{i=1}^{m} W(A_i \cap B) | \mathscr{F}(B)\right) \\ &= E(\alpha | \mathscr{F}(B)) \prod_{i=1}^{m} W(A_i \cap B). \end{aligned}$$

Now since $B \in \mathscr{C}$, if $A_i \cap B \neq \emptyset$ for $1 \leqq i \leqq m$ then $B \supset S_{A_1 A_2 \ldots A_m}$ and in that case $E(\alpha | \mathscr{F}(B)) = \alpha$ a.s. On the other hand, if $A_i \cap B = \emptyset$ for some i, then $\prod_{i=1}^{n} W(A_i \cap B) = 0$. Hence in either case

$$E(\alpha | \mathscr{F}(B)) \prod_{i=1}^{m} W(A_i \cap B) = \alpha \prod_{i=1}^{m} W(A_i \cap B) = (\psi \circ W^m)_B.$$

Thus, $\{(\phi \circ W^m)_B, \mathscr{F}(B): B \subset \mathscr{C}\}$ is indeed a martingale. □

3. Integrands as Moment Densities and a Projection Formula

The isometry property of the multiple stochastic integrals can be given the following interpretation. Suppose that for each $m \geqq 1$ and $t \in T^m$ that $\{\phi_{m,k}(t): k \geqq 1\}$ is a complete orthogonal basis for the space of square integrable, $\mathscr{F}(S_t)$-measurable random variables, and suppose that $\phi_{m,k}(t)$ is a symmetric function of t. Then, formally, the isometry property of multiple stochastic integrals means that the set of "incremental" random variables

$$\{\phi_{m,k}(t) W(dt_1) W(dt_2) \ldots W(dt_m): m \geqq 0, k \geqq 1, t \in \hat{T}^m\} \tag{3.1}$$

is an orthogonal collection of random variables which are also orthogonal to the $\mathscr{F}(S)$-measurable random variables. (Of course, the increments dt_i in (3.1) are "outward" from S_t.) This fact is reflected in the next proposition which states that the symmetrized integrands are uniquely determined as moment densities. The completeness property proven in Sect. 5 formally means that the collection of variables in (3.1) together with the $\mathscr{F}(S)$-measurable variables are complete in $L^2(\Omega, \mathscr{F}_W(T), \mathscr{P})$ if $\mathscr{F}(A) = \mathscr{F}_W(A)$ for all A in $\mathscr{R}(T)$.

Proposition 3.1. *Let $\gamma \in L_a^2(\Omega \times \hat{T}^m)$. Then for $t \in \hat{T}^k$,*

$$E[W(dt_1)W(dt_2)\ldots W(dt_k)\gamma \circ W^m | \mathscr{F}(S_t)]/dt_1 dt_2 \ldots dt_k = m!\tilde{\gamma}(t)\delta_{mk} \tag{3.2}$$

in the sense that the linear functional

$$f \rightsquigarrow E \int_{\hat{T}^k} f(t) W(dt_1) W(dt_2) \ldots W(dt_k) \gamma \circ W^m$$

defines a symmetric finite signed measure on the σ-algebra of subsets of $\Omega \times \hat{T}^k$ generated by $\mathscr{C}$-adapted atomic functions, the measure is absolutely continuous with respect to $\mathscr{P} \times \mu^k$ measure, and the Radon-Nikodym derivative is $m!\gamma\delta_{mk}$.

Proof. In view of the definition of Radon-Nikodym derivatives, Proposition 3.1 is simply a restatement of the isometry property of the multiple stochastic integrals. □

In the following proposition, $L_a^2(\Omega \times \hat{T}^m, \mathscr{F}_W(\cdot))$ is defined in the same way as $L_a^2(\Omega \times \hat{T}^m)$ except with the σ-algebras $\mathscr{F}(A)$ replaced by $\mathscr{F}_W(A)$ for all $A \in \mathscr{R}(T)$.

Proposition 3.2. (*Projection formula.*) *For each $\gamma \in L_a^2(\Omega \times \hat{T}^m)$ there is a $\hat{\gamma} \in L_a^2(\Omega \times \hat{T}^m, \mathscr{F}_W(\cdot))$ such that*

$$\hat{\gamma}(t) = E[\tilde{\gamma}(t) | \mathscr{F}_W(S_t)] \qquad \text{for a.e. } t \in \hat{T}^m \tag{3.3}$$

and for such $\hat{\gamma}$ and all $A \in \mathscr{C}$,

$$E[\gamma \circ W^m | \mathscr{F}_W(A)] = (\hat{\gamma}_m \circ W^m)(A) \tag{3.4}$$

Proof. By the completeness of multiple stochastic integrals in $L^2(\Omega, \mathscr{F}_W(T), \mathscr{P})$ (see Proposition 5.1 below) and the fact that $E[\gamma \circ W^m | \mathscr{F}_W(S)] = 0$, there exists a collection $\{\phi_k : k \geqq 1\}$ with $\phi_k \in L_a^2(\Omega \times \hat{T}^m, \mathscr{F}_W(\cdot))$ such that

$$E[\gamma \circ W^m | \mathscr{F}_W(T)] = \sum_{k=1}^{\infty} \phi_k \circ W^k.$$

Now by Proposition 3.1 with $\mathscr{F}$ replaced by $\mathscr{F}_W$,

$$E[W(dt_1)W(dt_2)\ldots W(dt_k)E[\gamma \circ W^m | \mathscr{F}_W(T)] | \mathscr{F}_W(S_t)]/dt_1 dt_2 \ldots dt_k = k!\tilde{\phi}_k(t)$$

so that

$$E[W(dt_1)W(dt_2)\ldots W(dt_k)\gamma \circ W^m | \mathscr{F}_W(S_t)]/dt_1 dt_2 \ldots dt_k = k!\tilde{\phi}_k(t) \tag{3.5}$$

on $\hat{T}^k$. Comparison of Eqs. (3.2) and (3.5) reveals that

$$\tilde{\phi}_k(t) = E[\tilde{\gamma}(t)\delta_{mk} | \mathscr{F}_W(S_t)] \qquad \text{a.e. } t \in \hat{T}^k.$$

Thus $\tilde{\phi}_k(t) = 0$ for a.e. $t \in \hat{T}^k$ unless $k = m$. So if $\hat{\gamma}$ is defined by $\hat{\gamma} = \phi_m$ then $\hat{\gamma}$ satisfies Eqs. (3.3) and (3.4) is true for $A = T$. Since each side of Eq. (3.4) is a martingale relative to $\{\mathscr{F}_W(A): A \in \mathscr{C}\}$, (3.4) is true for all $A \in \mathscr{C}$. Finally, since $\hat{\gamma}$ is uniquely determined on $\hat{T}^m$ up to a set of $\mathscr{P} \times \mu^m$ measure zero by Eq. (3.3), any $\hat{\gamma} \in L_a^2(\Omega \times \hat{T}^m, \mathscr{F}_W(\cdot))$ satisfying (3.3) also satisfies Eq. (3.4). □

4. Nested Classes of Sets $\mathscr{C}$ and the Iterated Integration Formula

Let $\mathscr{C}$ and $\dot{\mathscr{C}}$ with $\mathscr{C} \supset \dot{\mathscr{C}}$ be two classes of sets which each satisfy conditions c_1–c_3 for a Wiener process $\{W(A): A \in \mathscr{R}(T)\}$ and a collection of σ-algebras $\{\mathscr{F}(A): A \in \mathscr{R}(T)\}$ as in Sect. 2. A dot above (or above and to the right) denotes definition relative to $\dot{\mathscr{C}}$ so that, for example, $\dot{S}_t$ denotes the $\dot{\mathscr{C}}$-support of t and $\hat{T}^{m\cdot}$ denotes the collection of $\dot{\mathscr{C}}$-independent points in T^m.

An important example which is exploited in the next section is when $\dot{\mathscr{C}}$ is any class satisfying conditions c_1–c_3 with $\mathscr{F}(A) = \mathscr{F}_W(A)$ for all $A \in \mathscr{R}(T)$, and $\mathscr{C}$ is the collection of all closed sets. Another natural way in which nested collections of sets $\mathscr{C}$ arise is given by the following propositions.

Proposition 4.1. *Let $\mathscr{C}$ satisfy conditions c_1–c_3. Suppose that $t = (t_1, t_2, \ldots, t_k) \in T^k$ is fixed and define a subcollection $\mathscr{C}_t$ of $\mathscr{C}$ by*

$$\mathscr{C}_t = \{C \in \mathscr{C} : \{t_1, t_2, \ldots, t_k\} \subset C\}.$$

Then $\mathscr{C}_t$ also satisfies conditions c_1–c_3.

Proof. See Appendix A.

Theorem 4.2 (Iterated integration formula). *Suppose that $\mathscr{C}$ and $\dot{\mathscr{C}}$ each satisfy conditions c_1–c_3 and that $\dot{\mathscr{C}} \subset \mathscr{C}$. Then for $\theta \in L_a^2(\Omega \times \hat{T}^m)$ the class-$\mathscr{C}$ stochastic integral $\theta \circ W^m$ can be represented as a sum of class-$\dot{\mathscr{C}}$ integrals:*

$$\theta \circ W^m = E[\theta \circ W^m | \mathscr{F}(\dot{S})] + \sum_{k=1}^{m} \binom{m}{k} \phi_k \mathbin{\dot{\circ}} W^k \tag{4.1}$$

where the integrands $\phi_k \in \dot{L}_a^2(\Omega \times \hat{T}^{k\cdot})$ satisfy

$$\phi_k(t) = (\tilde{\theta}(t \times \cdot) I_{\dot{S}_t^{m-k}}) \overset{t}{\circ} W^{m-k} \qquad \text{for a.e. } t \in \hat{T}^{k\cdot}. \tag{4.2}$$

For each fixed t the integral on the right side of Eq. (4.2) is defined relative to the collection of sets $\mathscr{C}_t$.

Proof. Let $\Pi\theta$ denote transformation of θ by a permutation of its arguments. Suppose for some permutation Π that

$$\Pi\theta = \alpha I_{A_1 \times \ldots \times A_m}$$

where $A_i \subset \mathscr{R}(T)$, $A_1 \times \ldots \times A_m \subset \hat{T}^m$, α is a bounded $\mathscr{F}(S_{A_1 \ldots A_m})$ measurable random variable, $A_1 \times \ldots \times A_k \subset \hat{T}^{k\cdot}$ and $A_{k+1}, A_{k+2}, \ldots, A_m \subset S_{A_1 A_2 \ldots A_k}$. Then, symmetry implies that

$$\begin{aligned} \theta \circ W^m = \Pi\theta \circ W^m &= \left[\alpha \prod_{i=k+1}^{m} W(A_i)\right] \prod_{i=1}^{k} W(A_i) \\ &= h_k \mathbin{\dot{\circ}} W^k \end{aligned}$$

where for $t \in \hat{T}^{k\cdot}$

$$\begin{aligned} h_k(t) &= I_{A_1 \times \ldots \times A_k}(t)[\alpha I_{A_{k+1} \times \ldots \times A_m} \overset{t}{\circ} W^{m-k}] \\ &= [\Pi\theta(t \times \cdot) I_{\dot{S}_t^{m-k}}] \overset{t}{\circ} W^{m-k}. \end{aligned}$$

The isometry property of multiple stochastic integrals relative to $\dot{\mathscr{C}}$ implies that both k and the two sets $\{A_1, A_2, \ldots, A_k\}$ and $\{A_{k+1}, A_{k+2}, \ldots, A_m\}$ are unique. The integer k is unique because otherwise we would have

$$E(\theta \circ W^m)^2 = E(h_k \dot{\circ} W^k)(h_{k'} \dot{\circ} W^{k'}) = 0.$$

The collection $\{A_1, A_2, \ldots, A_k\}$ is unique because otherwise we would have

$$\theta \circ W^m = h_k \dot{\circ} W^k = g_k \dot{\circ} W^k$$

and $\tilde{h}_k \tilde{g}_k \equiv 0$. It follows that

$$\sum_{\text{all } \Pi} [(\Pi\theta)(t \times \cdot) I_{\tilde{S}_t^{m-k}}(\cdot) \overset{t}{\circ} W^{m-k}] \dot{\circ} W^k = k!(m-k)!\, \theta \circ W^m$$
$$= m!\, \phi_k \dot{\circ} W^k$$

where ϕ_k is given by Eq. (4.2). Hence

$$\theta \circ W^m = \binom{m}{k} \phi_k \dot{\circ} W^k$$

which is just Eq. (4.2) for the given θ. In Appendix B it is proved that linear combinations of such θ's are dense in $L_a^2(\Omega \times \hat{T}^m)$. The proof of the theorem is then completed by an application of the isometric property of the stochastic integrals. □

5. Completeness of Multiple Stochastic Integrals and an Exponential Formula

The iterated integration formula is applied in this section when one of the classes of sets $\mathscr{C}$ consists of all closed subsets of T and $\mathscr{F}(A) = \mathscr{F}_W(A)$ for all $A \in \mathscr{R}(T)$. The associated integrals are then multiple Wiener integrals. Let $\tilde{T}^m$ denote the set of m-tuples of distinct points in T and for θ in $L^2(\tilde{T}^m)$ let $\theta \square W^m$ denote a multiple Wiener integral of order m.

Proposition 5.1 (Completeness of multiple stochastic integrals). *Let $\mathscr{C}$ be a collection of sets such that $\mathscr{C}$ and $\{\mathscr{F}_W(A)\}$ satisfy conditions c_1–c_3. Then every square-integrable $\mathscr{F}_W(T)$-measurable random variable Z has a representation of the form*

$$Z = E[Z \mid \mathscr{F}(S)] + \sum_{m=1}^{\infty} Z_m \circ W^m \tag{5.1}$$

where $Z_m \circ W^m$ are stochastic integrals defined relative to $\mathscr{C}$ and $S = \bigcap \{C: C \in \mathscr{C}\}$.

Proof. The proposition is well known [6] in case $\mathscr{C}$ consists of all closed subsets of T, for then the integrals are multiple Wiener integrals. Since by the iterated integration formula any multiple Wiener integral can be represented as a sum of multiple stochastic integrals relative to the smaller class of sets $\mathscr{C}$, Proposition 5.1 is true in general. □

Proposition 5.2. *For f in $L^2(T)$, define*

$$\overset{*}{f}{}^m(t_1, t_2, \ldots, t_m) = \prod_{i=1}^{m} f(t_i) \tag{5.2}$$

and set

$$W_m(f, A) = (\overset{*}{f}{}^m \square W^m)_A. \tag{5.3}$$

If $\mathscr{C}$ and $\{\mathscr{F}_W(A)\}$ satisfy conditions c_1–c_3 then for $A \in \mathscr{C}$,

$$W_m(f, A) = W_m(f, S \cap A) + \sum_{k=1}^{n} \binom{m}{k} [\overset{*}{f} k(\cdot) W_{m-k}(f, S.) \circ W^k]_A \tag{5.4}$$

Proof. Observe that $\overset{*}{f}{}^k$ is symmetric and

$$\overset{*}{f}{}^m(t_1, t_2, \ldots, t_m) = \overset{*}{f}{}^k(t_1, t_2, \ldots, t_k)\, \overset{*}{f}{}^{m-k}(t_{k+1}, \ldots, t_m)$$

Hence. Eq. (5.4) for $A = T$ is obtained by applying the iterated integration formula to express the multiple Wiener integral $W_m(f, T)$ in terms of stochastic integrals relative to $\mathscr{C}$. Then Eq. (5.4) is true in general since each side is a martingale relative to $\{\mathscr{F}_W(A)\colon A \in \mathscr{C}\}$.

Proposition 5.3. *Let $\mathscr{C}$ and $\{\mathscr{F}(A)\colon A \in \mathscr{R}(T)\}$ satisfy the conditions of Sect. 2. Then if either $f \in L^2(T)$ or if f is a bounded function in $L_a^2(\Omega \times T)$ define*

$$L(f, A) = \exp\left((f \circ W)_A - \tfrac{1}{2}(f^2 \circ \mu)_A\right) \tag{5.5}$$

where $(f^2 \circ \mu)_A$ denotes the Lebesgue integral of f^2 over A. Then for $A \in \mathscr{C}$,

$$L(f, A) = L(f, S \cap A) + \sum_{m=1}^{\infty} \frac{1}{m!} [\overset{*}{f} m(\cdot) L(f, S.) \circ W^m]_A. \tag{5.6}$$

Proof. Suppose first that $f \in L^2(T)$. For multiple Wiener integrals ($\mathscr{C} = \{$all closed sets$\}$) Eq. (5.6) reduces to

$$L(f, A) = 1 + \sum_{m=1}^{\infty} \frac{1}{m!} W_m(f, A) \tag{5.7}$$

which is well known [6]. For the case of general $\mathscr{C}$, we use (5.4) in (5.7) and write

$$\begin{aligned} L(f, A) &= 1 + \sum_{m=1}^{\infty} \frac{1}{m!} \left(W_m(f, S \cap A) + \sum_{k=1}^{m} \binom{m}{k} [\overset{*}{f}{}^k W_{m-k}(f, S.) \circ W^k]_A \right) \\ &= L(f, S \cap A) + \sum_{k=1}^{\infty} \frac{1}{k!} \left[\overset{*}{f}{}^k \sum_{j=0}^{\infty} \frac{1}{j!} W_j(f, S.) \circ W^k \right]_A \\ &= L(f, S \cap A) + \sum_{k=1}^{\infty} \frac{1}{k!} [\overset{*}{f}{}^k L(f, S.) \circ W^k]_A \end{aligned}$$

which establishes (5.6) for f in $L^2(T)$. The Eq. (5.6) can then be extended to bounded f in $L_a^2(\Omega \times \hat{T})$ by an approximation argument (see [4], Proposition 3.2). $\square$

6. Change of Measure and Likelihood Ratio Formulas

Let $\{W(A), \mathscr{F}(A): A \in \mathscr{R}(T)\}$ on $(\Omega, \mathscr{F}, \mathscr{P})$ and a collection of sets $\mathscr{C}$ satisfy the assumptions of Sect. 2. Suppose that $\mathscr{P}'$ is another probability measure on $(\Omega, \mathscr{F})$ which is mutually absolutely continuous relative to $\mathscr{P}$ and is such that the Radon-Nikodym derivative $d\mathscr{P}'/d\mathscr{P}$ is $\mathscr{P}$-square-integrable and has the representation

$$\frac{d\mathscr{P}'}{d\mathscr{P}} = E\left[\frac{d\mathscr{P}'}{d\mathscr{P}}\middle|\mathscr{F}(S)\right] + \sum_{m=1}^{\infty} \gamma_m \circ W^m \tag{6.1}$$

in terms of $\mathscr{C}$-stochastic integrals. If $L(A)$ denotes the Radon-Nikodym derivative of ($\mathscr{P}'$ restricted to the σ-algebra $\mathscr{F}(A)$) relative to ($\mathscr{P}$ restricted to $\mathscr{F}(A)$) then $\{L(A), \mathscr{F}_W(A): A \in \mathscr{C}\}$ is a martingale with the representation

$$L(A) = E\left[\frac{d\mathscr{P}'}{d\mathscr{P}}\middle|\mathscr{F}(S)\right] + \sum_{m=1}^{\infty} (\gamma_m \circ W^m)(A).$$

Now replacing γ by γ_m in each side of Eq. (3.2) and summing over m yields that for each $m \geqq 1$ and for $t \in \hat{T}^m$,

$$E[W(dt_1)W(dt_2)\dots W(dt_m)L(T)|\mathscr{F}(S_t)]/dt_1 dt_2 \dots dt_m = m!\,\tilde{\gamma}_m(t).$$

Dividing each side of this equation by $L(S_t)$ and defining $r_m(t) = m!\,\tilde{\gamma}_m(t)/L(S_t)$ yields that

$$E'[W(dt_1)W(dt_2)\dots W(dt_m)|\mathscr{F}(S_t)]/dt_1 \dots dt_m = r_m(t) \tag{6.2}$$

where E' denotes (conditional) expectation relative to measure $\mathscr{P}'$. Thus, the Radon-Nikodym derivative $L(A)$ for $A \in \mathscr{C}$ has the representation

$$L(A) = E\left[\frac{d\mathscr{P}'}{d\mathscr{P}}\middle|\mathscr{F}(S)\right] + \sum_{m=1} \frac{1}{m!}[(r_m(\cdot)L(S_\cdot)) \circ W^m]_A \tag{6.3}$$

where the functions r_m have been identified in Eq. (6.2) as the density of conditional m-th moments of W under measure $\mathscr{P}'$.

Next, define $\Lambda(A) = E[L(A)|\mathscr{F}_W(A)]$. $\Lambda(A)$ is called a likelihood ratio. By an application of the projection formula to each term on the right side of Eq. (6.1),

$$\Lambda(A) = E\left[\frac{d\mathscr{P}'}{d\mathscr{P}}\middle|\mathscr{F}_W(S)\right] + \sum_{m=1}^{\infty} (\hat{\gamma}_m \circ W^m)(A)$$

where the integrands $\hat{\gamma}_m \in L_a^2(\Omega \times \hat{T}^m, \mathscr{F}_W(\cdot))$ satisfy

$$\hat{\gamma}_m(t) = E[\tilde{\gamma}_m(t)|\mathscr{F}_W(t)] \qquad \text{a.e. } t \in \hat{T}^m. \tag{6.4}$$

Now $\Lambda(A)$ is the Radon-Nikodym derivative of ($\mathscr{P}'$ restricted to $\mathscr{F}_W(A)$) relative to ($\mathscr{P}$ restricted to $\mathscr{F}_W(A)$) and thus $(\Lambda(A), \mathscr{F}_W(A))$ has the same structure assumed of $(L(A), \mathscr{F}(A))$. Thus, if $\hat{r}_m$ is defined by $\hat{r}_m(t) = m!\,\hat{\gamma}_m(t)/\Lambda(S_t)$ for $t = (t_1, t_2, \dots, t_m) \in \hat{T}^m$ then

$$\hat{r}_m(t) = E'[W(dt_1)W(dt_2)\dots W(dt_m)|\mathscr{F}_W(S_t)]/dt_1 dt_2 \dots dt_m \tag{6.5}$$

and the likelihood ratio $\Lambda(A)$ for $A\in\mathscr{C}$ has the representation

$$\Lambda(A)=E\left[\frac{d\mathscr{P}'}{d\mathscr{P}}\middle|\mathscr{F}_W(S)\right]+\sum_{m=1}^{\infty}\frac{1}{m!}\left[(\hat{r}(\cdot)\Lambda(S.))\circ W^m)\right]_A. \tag{6.6}$$

Also, comparing (6.2) and (6.5) (or using Eq. (6.4)) yields that

$$\hat{r}_m(t)=E'\left[r_m(t)\,|\,\mathscr{F}_W(S_t)\right] \qquad \text{a.e. } t\in\hat{T}^m. \tag{6.7}$$

Remark. Equation (6.3) (resp. Eq. (6.6)) can be viewed as an integral equation for $L(\cdot)$ (resp. $\Lambda(\cdot)$) in terms of the moment densities r_m (resp. $\hat{r}_m$) and the Wiener process. As shown in examples below, it is sometimes possible to explicitly solve these equations (also see [2], [7] and [8]).

Since the measure $\mathscr{P}'$ is thus at least formally determined by the functions $\{r_m\}$, it should be possible to express other moments under measure $\mathscr{P}'$ in terms of $\{r_m\}$. In this direction, we consider next moments as in Eq. (6.2) but with S_t replaced by a larger set.

Let $\dot{\mathscr{C}}$ be another class of sets with $\dot{\mathscr{C}}\subset\mathscr{C}$ so that the assumptions of Sect. 2 are also satisfied by $\{W(A), \mathscr{F}(A)\colon A\in\mathscr{R}(T)\}$ and $\dot{\mathscr{C}}$. The notation introduced in Sect. 4 will be used in what follows.

By the iterated integration formula for $\mathscr{C}$-stochastic integrals in terms of $\dot{\mathscr{C}}$-stochastic integrals, Eq. (6.1) yields that

$$\frac{d\mathscr{P}'}{d\mathscr{P}}=E\left[\frac{d\mathscr{P}'}{d\mathscr{P}}\middle|\mathscr{F}(\dot{S})\right]+\sum_{k=1}^{\infty}\dot{\gamma}_k\,\dot{\circ}\,W^k \tag{6.8}$$

where the $\dot{\mathscr{C}}$-adapted integrands $\dot{\gamma}_k$ satisfy

$$\dot{\gamma}_k(t)=\sum_{m=k}^{\infty}\binom{m}{k}(\tilde{\gamma}_m(t\times\cdot)I_{\dot{S}_t^{m-k}})\,\dot{\circ}\,W^{m-k} \qquad \text{for a.e. } t\in\hat{T}^{k\cdot}. \tag{6.9}$$

Now Eq. (6.8) is the same as Eq. (6.1) with $\mathscr{C}$ and $\{\gamma_m\colon m\geqq 1\}$ replaced by $\dot{\mathscr{C}}$ and $\{\dot{\gamma}_m\colon m\geqq 1\}$. Thus, $\{\dot{L}(A)\colon A\in\dot{\mathscr{C}}\}$ and $\dot{r}_m$ can be defined relative to $\dot{\mathscr{C}}$ in the same way as the corresponding quantities were defined relative to $\mathscr{C}$. In particular, Eq. (6.2) yields that for $t\in\hat{T}^{m\cdot}$,

$$E'\left[W(dt_1)\,W(dt_2)\ldots W(dt_m)\,|\,\mathscr{F}(\dot{S}_t)\right]/dt_1\,dt_2\ldots dt_m=\dot{r}_m(t).$$

That is, $\dot{r}_m$ is a conditional moment density for W under $\mathscr{P}'$ just as r_m is, except that S_t is replaced by the larger set $\dot{S}_t$.

Multiplying each side of Eq. (6.9) by $k!/L(\dot{S}_t)$ yields that for a.e. $t\in\hat{T}^{k\cdot}$,

$$\dot{r}_k(t)=\frac{1}{L(\dot{S}_t)}\sum_{m=k}^{\infty}\frac{1}{(m-k)!}(L(S_{t\times\cdot})\,r_m(t\times\cdot)\,I_{\dot{S}_t^{m-k}})\,\dot{\circ}\,W^{m-k} \tag{6.10}$$

This equation represents the moment density $\dot{r}_k$ in terms of the moment densities $\{r_m\colon m\geqq k\}$, the Wiener process, and L. Of course a similar representation holds for $\dot{\hat{r}}_k$ in terms of $\{\hat{r}_k\colon k\geqq m\}$, the Wiener process, and Λ.

At this point more structure will be assumed on the Radon-Nikodym derivative $d\mathscr{P}'/d\mathscr{P}$. Suppose that $\{Z(t)\colon t\in T\}$ is a bounded, measurable process such that $Z(t)$ is $\mathscr{F}(S_t)$-measurable for each $t \in T$. Then it will be assumed that, in the notation of Propositions 5.3,

$$\frac{d\mathscr{P}'}{d\mathscr{P}} = L(Z, T) \tag{6.11}$$

Thus $L(A) = L(Z, A)$ and $r_m = \dot{Z}^m$. By Eq. (6.7), the moment density $\hat{r}_m$ in the likelihood ratio representation (6.6) satisfies

$$\hat{r}_m(t) = E'[\dot{Z}^m(t) | \mathscr{F}(S_t)] \qquad \text{a.e. } t \in T^m. \tag{6.12}$$

Therefore $\hat{r}_m$ is now actually a conditional m-th moment rather than just a moment density as in (6.5).

The assumption (6.11) arises in a detection problem for which a signal is observed in white Gaussian noise. Indeed, define a process $\{X(A) \colon A \in \mathscr{R}(T)\}$ by

$$X(A) = W(A) - (Z \circ \mu)(A)$$

Then trivially $W(A) = X(A) + (Z \circ \mu)(A)$, and the following proposition is true:

Proposition 6.1. *$\{X(A) \colon A \in \mathscr{R}(T)\}$ is a Wiener process under $\mathscr{P}'$ and for each $t \in \hat{T}^m$, the collection of random variables $\{X(A) \colon A \cap S_t = \emptyset\}$ is $\mathscr{P}'$-independent of $\mathscr{F}(S_t)$.*

Proof. It suffices to prove that for $t \in T^m$, if $A_1, A_2, \ldots, A_k$ are disjoint rectangles contained in $T - S_t$ and if $\alpha_1, \alpha_2, \ldots, \alpha_k$ are bounded, $\mathscr{F}(S_t)$-measurable random variables, then $E'\Phi = 1$ where

$$\Phi = \exp\left(\sum_i \alpha_i X(A_i) - \tfrac{1}{2} \sum_i \alpha_i^2 \mu(A_i)\right).$$

Define a function h on $\Omega \times T$ by $h = \sum \alpha_i I_{A_i}$. Then

$$\Phi = \exp\left(h \overset{t}{\circ} W + (hZ - \tfrac{1}{2} h^2) \circ \mu\right) \tag{6.13}$$

where $h \overset{t}{\circ} W$ is a stochastic integral defined relative to the class of sets $\mathscr{C}_t$. By the fact that $L(Z, S_t) = E\left[\frac{d\mathscr{P}'}{d\mathscr{P}} \middle| \mathscr{F}(S_t)\right]$ we have

$$E'[\Phi | \mathscr{F}(S_t)] = E[\Phi L(Z, T) | \mathscr{F}(S_t)] / L(Z, S_t) = E[\Phi L(Z, T - S_t) | \mathscr{F}(S_t)]. \tag{6.14}$$

Now the integral $(Z I_{S_t^c}) \circ W$ in the definition of $L(Z, T - S_t)$ can be defined relative to $\mathscr{C}_t$ with the same result as its definition relative to $\mathscr{C}$. Thus, using Eq. (6.13), Eq. (6.14) becomes

$$E'[\Phi | \mathscr{F}(S_t)] = E[L^t(h + Z I_{S_t^c}, T) | \mathscr{F}(S_t)] \tag{6.15}$$

where L^t is defined in the same way as L except relative to the class of sets $\mathscr{C}_t$ instead of $\mathscr{C}$. Finally, by the martingale property of L^t relative to the class of sets $\mathscr{C}_t$ (which contains S_t), the right side of Eq. (6.15) is equal to $L^t(h + Z I_{S_t^c}, S_t) = 1$. Thus $E'\Phi = 1$. $\square$

Four examples are considered in the remainder of this section. First, let $a\in\mathbb{R}^n$ be a fixed unit vector (i.e., $\|a\|=1$) and let H_α denote the half space $\{t\in\mathbb{R}^n: (t,a)\geqq\alpha\}$. Then the collection $\mathscr{C}=\{H_\alpha\cap T\}$ is a one-parameter family of sets such that $\hat{T}^m$ is vacuous for $m>1$. That is, two or more points are always $\mathscr{C}$-dependent. For this choice of $\mathscr{C}$ if $d\mathscr{P}'/d\mathscr{P}$ has the form (6.1) then $\gamma_m=0$ for $m\geqq 2$ so that the structure assumption (6.11) is then also satisfied for $Z=\gamma_1$. In this case the likelihood ratio formula given by Eqs. (6.6) and (6.12) reduces to

$$\Lambda(A)=1+[\hat{Z}(\cdot)\Lambda(S.)\circ W]_A, \qquad A\in\mathscr{C}$$

and an application of (5.6) yields

$$\Lambda(A)=L(\hat{Z},A)=\exp\{(\hat{Z}\circ W-\tfrac{1}{2}\hat{Z}^2\circ\mu)_A\}, \qquad A\in\mathscr{C} \tag{6.16}$$

where

$$\begin{aligned}\hat{Z}(t)&=E'(Z(t)|\mathscr{F}_W(S_t))\\ &=E'(Z(t)|\mathscr{F}_W(H_{(t,a)}\cap T))\end{aligned}$$

In this case we see that the likelihood ratio is expressible as an exponential of the conditional mean.

For the second example take $\mathscr{C}=\{\text{all closed sets}\}$ and assume that Eq. (6.11) holds for some bounded $\mathscr{C}$-adapted process Z. Then for $t=(t_1,t_2,\ldots,t_m)\in T^m$,

$$(Z_{t_1},Z_{t_2},\ldots,Z_{t_m}) \quad \text{is } \mathscr{F}(S_t)\text{-measurable.} \tag{6.17}$$

By our standing assumptions on the σ-algebras, $\mathscr{F}(S_t)$ is independent of $\mathscr{F}_W(S_t^c)$. But for this example

$$S_t=\{t_1,t_2,\ldots,t_m\} \tag{6.18}$$

so that, up to events of $\mathscr{P}$-measure zero, $\mathscr{F}_W(S_t^c)=\mathscr{F}_W(T)$. Thus the processes Z and W are $\mathscr{P}$-independent. This implies that Z is identically distributed under $\mathscr{P}$ and $\mathscr{P}'$. Then, again using (6.18), $\Lambda(S_t)=1$ $\mathscr{P}$-almost surely and $\hat{r}_m=\rho_m$ where ρ_m is the m-th moment

$$\rho_m(t_1,t_2,\ldots,t_m)=E[Z(t_1)Z(t_2)\ldots Z(t_m)]. \tag{6.19}$$

Thus, Eq. (6.6) becomes

$$\Lambda(A)=1+\sum_{m=1}^{\infty}\frac{1}{m!}(\rho_m\,\square\,W^m)(A). \tag{6.20}$$

Equation (6.20) provides a martingale representation of the likelihood ratio for the "additive white Gaussian noise" model under very general conditions. In the one-dimensional case, it was recently obtained in [8].

For the next example let T be the unit interval $[0,1]$ and, for $k\geqq 1$, consider the class of sets

$$\mathscr{C}^k=\{[0,t_1]\cup\{t_2,t_3,\ldots,t_k\}: 0\leqq t_1\leqq\ldots\leqq t_k\leqq 1\}.$$

Relative to $\mathscr{C}^k$, if $0 \leqq t_1 \leqq t_2 \leqq \ldots \leqq t_m \leqq 1$ then

$$S_{t_1 t_2 \ldots t_m} = \begin{cases} \{t_1, t_2, \ldots, t_m\} & \text{if } m < k \\ \{t_{m-k+1}, \ldots, t_m\} \cup [0, t_{m-k+1}] & \text{if } m \geqq k \end{cases} \tag{6.21}$$

and $\hat{T}^m$ is vacuous for $m > k$. Now suppose that for each $k \geqq 1$, $d\mathscr{P}'/d\mathscr{P}$ has a representation of the form (6.1) relative to $\mathscr{C} = \mathscr{C}^k$. Let r_m^k denote the m^{th} order moment densities when $\mathscr{C}$ is equal to $\mathscr{C}^k$. Then by (6.21), it is clear that $r_m^k = r_m^j$ $\mathscr{P} \times \mu^m$ a.e. on $[0,1]^m$ if k and j are each larger than m. Thus, we can define functions r_m by

$$r_m(t_1, t_2, \ldots, t_m) = r_m^k(t_1, t_2, \ldots, t_m) \text{ for some } k > m$$

and

$$r_m(t_1, t_2, \ldots, t_m | [0, t_1]) = r_m^m(t_1, \ldots, t_m).$$

Moment densities $\hat{r}_m$ are defined analogously. Thus

$$\hat{r}_m(t_1, t_2, \ldots, t_m) = E'[W(dt_1) W(dt_2) \ldots W(dt_m)]/dt_1\, dt_2 \ldots dt_m$$

and

$$\begin{aligned} &\hat{r}_m(t_1, t_2, \ldots, t_m | [0, t_1]) \\ &\quad = E'[W(dt_1) W(dt_2) \ldots W(dt_m) | \mathscr{F}_W([0, t_1])]/dt_1\, dt_2 \ldots dt_m. \end{aligned}$$

Let $\Lambda_{t_1} = E\left[\frac{d\mathscr{P}'}{d\mathscr{P}} \middle| \mathscr{F}_W([0, t_1])\right]$ for $t_1 \in T$. Then by (6.21),

$$\Lambda(S^k_{t_1 t_2 \ldots t_m}) = \begin{cases} 1 & \text{if } k < m \\ \Lambda_{t_{m-k+1}} & \text{if } k \geqq m \end{cases} \quad \mathscr{P} \quad \text{a.s.}$$

if $t_1 < t_2 < \ldots < t_m$.

Thus, Eq. (6.6) for the likelihood ratio when $\mathscr{C} = \mathscr{C}^m$ becomes

$$\begin{aligned} \Lambda_t = 1 &+ \int_0^t \hat{r}_1(t_1) W(dt_1) + \int_0^t \int_0^{t_2} \hat{r}_2(t_1, t_2) W(dt_1) W(dt_2) + \ldots \\ &+ \int_0^t \int_0^{t_{m-1}} \ldots \int_0^{t_2} \hat{r}_{m-1}(t_1, \ldots, t_{m-1}) W(dt_1) \ldots W(dt_{m-1}) \\ &+ \int_0^t \int_0^{t_m} \ldots \int_0^{t_2} \Lambda_{t_1} \hat{r}_m(t_1, \ldots, t_m | [0, t_1]) W(dt_1) W(dt_2) \ldots W(dt_m) \end{aligned} \tag{6.22}$$

and the moment Eq. (6.10) when $\hat{\mathscr{C}} = \mathscr{C}_m$ and $\mathscr{C} = \mathscr{C}_{m+1}$ becomes

$$\Lambda_{t_1} \hat{r}_m(t_1, \ldots, t_m | [0, t_1]) = \hat{r}_m(t_1, \ldots, t_m) + \int_0^t \Lambda_\tau \hat{r}_{m+1}(\tau, t_1, \ldots, t_m | [0, \tau]) W(d\tau). \tag{6.23}$$

Now suppose that $d\mathscr{P}'/d\mathscr{P}$ satisfies (6.11) where $\{Z_t : t \in T\}$ is bounded and $\mathscr{C}^k$-adapted for all $k \geqq 1$. By reasoning similar to that in the second example, the processes Z and W must be independent under $\mathscr{P}'$. Thus

$$\hat{r}_m(t_1, \ldots, t_m | [0, t_1]) = E'[Z_{t_1} Z_{t_2} \ldots Z_{t_m} | \mathscr{F}_W([0, t])]$$

and

$$r_m(t_1, \ldots, t_m) = \rho_m(t_1, \ldots, t_m)$$

where ρ_m is the m-th moment defined in Eq. (6.19). Defining $\hat{Z}_t = E'[Z_t | \mathscr{F}_W([0, t])]$, Eqs. (6.22) and (6.23) for $m=1$ become

$$\Lambda_t = 1 + \int_0^t \Lambda_\tau \hat{Z}_\tau W(d\tau) \tag{6.24}$$

and

$$\Lambda_t \hat{Z}_t = E[Z_t] + \int_0^t \Lambda_\tau E'[Z_t Z_\tau | \mathscr{F}^W([0, \tau])] W(d\tau) \tag{6.25}$$

respectively.

Note that (6.25) is a well-known representation for the first "unnormalized" conditional moment $\hat{Z}_t \Lambda_t$. Using Ito's formula for one-parameter process, Eq. (6.24) and (6.25) together yield a representation for the conditional moment $\hat{Z}_t = (\Lambda_t \hat{Z}_t)/\Lambda_t$ itself. As an intermediate step, an integral representation for Λ_t^{-1} would also be derived via Ito's formula. In constrast, in the setting of a general class of sets $\mathscr{C}$, a suitable analogue of Ito's formula is not yet available. Such a formula might provide, for example, an analogue of the moment Eq. (6.10) for $\dot{r}_k$ which does not contain the factor $1/L(\dot{S}_t)$. (An instance of such equations is provided in [10].)

For the final example suppose that T is a rectangle in $\mathbb{R}^n$ with $\mu(T)=1$, suppose that $\{W(A), \mathscr{F}(A): A \in \mathscr{R}(T)\}$ is as in Sect. 2, and consider $\mathscr{P}'$ defined by

$$\frac{d\mathscr{P}'}{d\mathscr{P}} = W(T)^2.$$

In terms of multiple Wiener integrals,

$$\frac{d\mathscr{P}'}{d\mathscr{P}} = 1 + 1 \,\square\, W^2$$

so that the moment densities r_m corresponding to $\mathscr{C} = \{\text{all closed sets}\}$ are given by $r_m \equiv \delta_{m2}$. Also,

$$L(A) = 1 + [W(A)^2 - \mu(A)].$$

Now let $\dot{\mathscr{C}}$ be a class of sets satisfying the conditions c_1-c_3. Then by the moment Eq. (6.10), the moment densities $\dot{r}_m$ satisfy

$$\dot{r}_1(t) = W(\dot{S}_t)/L(\dot{S}_t)$$
$$\dot{r}_2(t_1, t_2) = 1/L(\dot{S}_t)$$

and $\dot{r}_k \equiv 0$ for $k \geqq 2$. Thus, in terms of $\dot{\mathscr{C}}$-stochastic integrals,

$$\frac{d\mathscr{P}'}{d\mathscr{P}} = L(\dot{S}) + (L(S_\cdot)\dot{r}_1(\cdot)) \dot{\circ} W + (L(S_\cdot)\dot{r}_2(\cdot)) \dot{\circ} W^2.$$

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Appendix A

Proposition 4.1 is proved in this appendix. For the proof fix $t\in T^k$ and let $\dot{\mathscr{C}}=\mathscr{C}_t$. Clearly for subsets $A_1, A_2, \ldots, A_m$ of T,

$$\dot{S}_{A_1 A_2 \ldots A_m} = S_{A_1 A_2 \ldots A_m t_1 t_2 \ldots t_k},$$

so that conditions c_2 and c_3 for $\dot{\mathscr{C}}$ are immediate consequences of these conditions for $\mathscr{C}$. It remains to establish condition c_1 for $\dot{\mathscr{C}}$.

To begin we will establish that

$$\hat{T}^{m\cdot} \subset \bigcup_{\tau} \{s\in T^m\colon s\times\tau\in\hat{T}^{m+j} \text{ and } t\subset(S_{s\times\tau})^k\} \tag{A.1}$$

where the union is over all $\tau=(\tau_1, \tau_2, \ldots, \tau_j)$ such that $\{\tau_1, \tau_2, \ldots, \tau_j\}\subset\{t_1, t_2, \ldots, t_k\}$ and $s\times\tau$ denotes the point $(s_1, s_2, \ldots, s_m, \tau_1, \tau_2, \ldots, \tau_k)$ in T^{m+k}. To see (A.1) suppose that $s\in\hat{T}^m$. Then s is contained in any of the sets on the right side of relation (A.1) which correspond to a minimal subset $\{\tau_1, \tau_2, \ldots, \tau_j\}$ with satisfies $S_{s\times\tau}=S_{s\times t}$. This establishes relation (A.1).

Now by relation (A.1) if $\varepsilon>0$ then

$$\hat{T}^{m\cdot} \subset \bigcup_{\tau} \{s\in T^m\colon s\times\tau\in\hat{T}^{m+j} \text{ and } d(t_i, S_{s\times\tau})<\varepsilon \text{ for } 1\leqq i\leqq k\} \tag{A.2}$$

where d denotes the usual Euclidean distance between subsets of T, and the union is over τ as in (A.1).

Now choose any product of rectangles $A=A_1\times\ldots\times A_m$ so that $A\in\hat{T}^{m\cdot}$. Since each of the sets in the union on the right side of relation (A.2) is open (by condition c_1), the set A can be expressed as a countable union of sets of the form $B=B_1\times\ldots\times B_m$ where $B_1, B_2, \ldots, B_m$ are rectangles such that

$$B\subset\{s\in T^m\colon s\times\tau\in\hat{T}^{m+j} \text{ and } d(t_i, S_{s\times\tau})<\varepsilon \text{ for } 1\leqq i\leqq k\} \tag{A.3}$$

for some τ as in relation (A.2), and where τ depends on B. Condition c_1 for $\mathscr{C}$ applied to such B implies that

$$\mu(B_i\cap S_{B\times\tau})=0 \quad \text{for } 1\leqq i\leqq m. \tag{A.4}$$

By relation (A.3) it is clear that $S_{B\times\tau}\supset S^{\varepsilon}_{B\times t}$, and since $B\subset A$ we also have that $S^{\varepsilon}_{B\times t}\supset S^{\varepsilon}_{A\times t}$. Thus (A.4) implies that $\mu(B_i\cap S^{\varepsilon}_{A\times t})=0$ for $1\leqq i\leqq m$, and since A_i is a countable union of such sets B_i,

$$\mu(A_i\cap S^{\varepsilon}_{A\times t})=0 \quad \text{for } 1\leqq i\leqq m.$$

Now sending ε to zero and applying condition c_3 for $\mathscr{C}$ shows that $\mu(A_i\cap S_{A\times t})=0$. Finally, since $S_{A\times t}=\dot{S}_A$, this establishes condition c_1 for $\dot{\mathscr{C}}=\mathscr{C}_t$. □

Appendix B

Let $\mathscr{C}$ and $\dot{\mathscr{C}}$ with $\mathscr{C}\supset\dot{\mathscr{C}}$ each satisfy conditions c_1–c_3. Let $\mathscr{I}_m$ denote the collection of subsets of T^m of the form $A_1\times\ldots\times A_m$ such that each $A_i\in\mathscr{R}^n(T)$ and for some permutation Π,

1) $A_{\Pi(1)}, \ldots, A_{\Pi(k)}$ are $\mathscr{C}$-independent, and

2) $A_{\Pi(k+1)}, \ldots, A_{\Pi(m)} \subset S_{A_{\Pi(1)} A_{\Pi(2)} \ldots A_{\Pi(k)}}$.

Define $\dot{\mathscr{I}}_m$ relative to $\mathscr{C}$ similarly. The purpose of this appendix is to prove the following proposition:

Proposition B. *The linear span of* $\{\alpha I_B\colon B \in \dot{I}_m, B \in \hat{T}^m, \alpha$ *is bounded,* $\mathscr{F}(\dot{S}_{A_1 A_2 \ldots A_m})$ *meas.*$\}$ *is dense in* $L^2_a(\Omega \times \hat{T}^m)$ *for each* $m \geqq 1$.

Proof. Consider the following two conditions:

(b_1) There is a countable subcollection of $\mathscr{I}_m$ which covers T^m a.e.

(b_2) There is a countable subcollection $\mathscr{I}^d_m$ of disjoint sets in $\mathscr{I}_m$ which covers T^m a.e.

By a sequence of lemmas it is shown below that condition b_1 is satisfied and then that condition $b_1 \Rightarrow$ condition b_2 and finally that condition b_2 (but with $\mathscr{I}^d_m$ replaced by $\dot{\mathscr{I}}^d_m$) implies Proposition *B*. □

Lemma B.1.

$$\bigcup_{l=1}^{m} \bigcup_{\Pi \in \mathscr{P}(m)} \Pi \circ \{(\mathbf{x}, \mathbf{y})\colon \mathbf{x} \in \hat{T}^l, \mathbf{y} \in (S_{\mathbf{x}})^{m-l}\} = T^m \qquad (*)$$

Proof. Let $\mathbf{q} = (q_1, \ldots, q_m) \in T^m$. Choose a permutation $\mathbf{p} = (p_1, \ldots, p_m) = \Pi(q_1, \ldots, q_m)$ so that for some l with $1 \leqq l \leqq m$,

$$S_{\mathbf{q}} = S_{p_1, \ldots, p_l} \neq S_{p_1, \ldots, \hat{p}_i, \ldots, p_l} \qquad \text{for } 1 \leqq i \leqq l$$

where "$\hat{p}_i$" denotes that p_i is to be omitted. That is, the permutation is choosen so that $p_1, \ldots, p_l$ is a minimal set from $q_1, \ldots, q_m$ with the same support as $q_1, \ldots, q_m$. Now $p_{l+1}, \ldots, p_m \in S_{p_1, \ldots, p_l}$ since $q_1, \ldots, q_m \in S_{\mathbf{q}} = S_{p_1, \ldots, p_l}$.

To show that $\mathbf{q}$ is contained in the left side of $(*)$, it remains to show that $p_1, \ldots, p_l$ are $\mathscr{C}$-independent. Now, if $p_1, \ldots, p_l$ were not $\mathscr{C}$-independent, then $p_i \in S_{p_1, \ldots, \hat{p}_i, \ldots, p_l}$ for some i. Then

$$\{A \in \mathscr{C}\colon p_1, \ldots, p_l \in A\} = \{A \in \mathscr{C}\colon p_1, \ldots, \hat{p}_i, \ldots, p_l \in A\}.$$

Intersecting all the sets contained in this collection of sets yields that

$$S_{p_1, \ldots, p_l} = S_{p_1, \ldots, \hat{p}_i, \ldots, p_l}$$

which contradicts our choice of $p_1, \ldots, p_l$. Thus $p_1, \ldots, p_l$ are $\mathscr{C}$-independent so that $\mathbf{p}$, and hence $\mathbf{q}$, is contained in the left side of $(*)$. □

Lemma B.2. Condition b_1 is satisfied.

Proof. Let $\mathscr{I}^0_m$ denote the subsets of T^m of the form $A_1 \times \ldots \times A_m$ such that, for some $\Pi \in \mathscr{P}(m)$ and some $l > 0$,

a) $A_{\Pi(1)}, \ldots, A_{\Pi(l)}$ are $\mathscr{C}$-independent, closed rectangles whose vertices have rational coordinates in $T \subset \mathbb{R}^n$, and

b) $A_{\Pi(l+1)} = \ldots = A_{\Pi(m)} = S_{A_{\Pi(1)} A_{\Pi(2)} \ldots A_{\Pi(l)}}$.

Then $\mathscr{I}^0_m$ is a countable subset of $\mathscr{I}_m$ and

$$\bigcup_{A\in\mathscr{I}_m} A \supset \bigcup_{l=1}^{m} \bigcup_{\Pi\in\mathscr{P}(m)} \Pi\circ\{(\mathbf{x},\mathbf{y})\colon \mathbf{x}\in\hat{T}^l,\ \mathbf{y}\in(S_{\mathbf{x}}^{(-)})^{m-l}\}$$
$$=\bigcup_{l=1}^{m} \bigcup_{\Pi\in\mathscr{P}(m)} \Pi\circ\{(\mathbf{x},\mathbf{y})\colon \mathbf{x}\in\hat{T}^l,\ y\in(S_{\mathbf{x}})^{m-l}\}$$
$$-\bigcup_{l=1}^{m} \bigcup_{\Pi\in\mathscr{P}(m)} \Pi\circ S_{m,l} \tag{B.1}$$

where

$$S_{m,l}=\{(\mathbf{x},\mathbf{y})\colon \mathbf{x}\in\hat{T}^l,\ \mathbf{y}\in(S_{\mathbf{x}})^{m-l}-(S_{\mathbf{x}}^{(-)})^{m-l}\}.$$

The first term on the right side of (B.1) is equal to T^m by Lemma B.1. Thus, to complete the proof it must be shown that $\mu^m(S_{m,l})=0$ for all $m\geqq 1$ and $1\leqq l\leqq m$.

By condition c_2,

$$F_\varepsilon=\{(\mathbf{x},\mathbf{y})\colon \mathbf{x}\in\hat{T}^l,\ \mathbf{y}\in(S_{\mathbf{x}}^{\varepsilon})^{m-l}\}$$

is a closed subset of $\hat{T}^l\times T^{m-l}$ which increases as ε decreases to zero. Since $S_{m,l}=F_0-\bigcup_{\varepsilon>0} F_\varepsilon$, it follows that $S_{m,l}$ is a Borel subset of T^m. By condition c_3, the section

$$\{\mathbf{y}\colon (\mathbf{x},\mathbf{y})\in S_{m,l}\}\subset T^{m-l}$$

of $S_{m,l}$ at $\mathbf{x}$ has Lebesgue measure zero for a.e. $\mathbf{y}\in\hat{T}^m$. Hence, by Fubini's theorem, $\mu^m(S_{m,l})=0$ for $1\leqq l\leqq m$. $\square$

Lemma B.3. *Condition* b_1 *implies condition* b_2.

Proof. Let $F_1, F_2, \ldots$ be a countable subcollection of $\mathscr{I}_m$ which couvers T^m a.e. Then the disjoint sets $D_i=F_i-\bigcup_{j=1}^{i-1} F_j i\geqq 1$ cover T^m a.e. We claim that for each $i\geqq 1$ there is a finite collection of disjoint sets $D_{i1},\ldots,D_{in}$ in $\mathscr{I}_m$ such that $D_i=\bigcup_{j=1}^{n_i} D_{ij}$. Condition b_2 is then satisfied with $\mathscr{I}_m^d=\{D_{ij}\colon i\geqq 1,\ 1\leqq j\leqq n_i\}$. It remains to prove the claim.

By induction, it suffices to establish the claim for $i=2$. Now $F_1=A_1\times\ldots\times A_m$ for some Borel sets $A_1,\ldots,A_m\subset T$. Thus, $F_1^c=\bigcup_{j=1}^{r} K_j$ where $K_1,\ldots,K_r$ are disjoint and each K_j is the product of m Borel subsets of T. In fact, F_1^c is the union of all sets of the form $B_1\times\ldots\times B_m$ such that $B_i=A_i$ or $B_i=A_i^c$ for each i and such that $B_i=A_i^c$ for at least one i, and these sets are disjoint. So $D_2=\bigcup_{j=1}^{k} K_j\cap F_2$. The sets $K_j\cap F_2$ are disjoint sets in $\mathscr{I}_m$ as required so the claim is established. $\square$

Lemma B.4. *Condition* b_2 *with* $\mathscr{I}_m^d$ *replaced by* $\dot{\mathscr{I}}_m^d$ *implies that the linear span of* $\{\alpha 1_B\colon B\in\dot{\mathscr{I}}_m,\ B\subset\hat{T}^m,\ \alpha$ *is bounded,* $\mathscr{F}(\dot{S}_{A_1\ldots A_m})$ *measurable*$\}$ *is dense in* $L_a^2(\Omega\times\hat{T}^m)$.

Proof. Let $F=F_1\times\ldots\times F_m$ where each $F_i\in\mathscr{R}^n(T)$ such that $F\in\hat{T}^m$ and let α be a bounded, $\mathscr{F}(S_{F_1F_2\ldots F_m})$-measurable random variable. If $A=A_1\times\ldots\times A_m\in\dot{\mathscr{I}}_m^d$ then $B=A\cap F$ satisfies $B\in\dot{\mathscr{I}}_m$ and $B\subset\hat{T}^m$, and α is $\mathscr{F}(\dot{S}_{A_1\cap F_1\ldots A_m\cap F_m})$-

measurable since $\hat{\mathscr{C}}$-supports are no smaller than $\mathscr{C}$-supports. By condition b_2 with $\mathscr{I}_m^d$ replaced by $\hat{\mathscr{I}}_m^d$,

$$\alpha 1_F = \sum_{A \in \hat{\mathscr{I}}_m^d} \alpha 1_{A \cap F} \quad \text{a.e. in } T^m.$$

Since the linear span of functions of the form $\alpha 1_F$ is dense in $L_a^2(\Omega \times \hat{T}^m)$, the lemma is established by considering sets of the form $B = A \cap F$. □

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