SET-PARAMETERED MARTINGALES AND MULTIPLE STOCHASTIC INTEGRATION

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Abstract

The starting point of this paper is the problem of representing square-integrable functionals of a multiparameter Wiener process. By embedding the problem in that of representing set-parameter martingales, we show that multiple stochastic integrals of various order arise naturally. Such integrals are defined relative to a collection of sets that satisfies certain regularity conditions. The classic cases of multiple Wiener integral and Ito integral (as well as its generalization by Wong-Zakai-Yor) are recovered by specializing the collection of sets appropriately.

Using the multiple stochastic integrals, we obtain a martingale representation theorem of considerable generality. An exponential formula and its application to the representation of likelihood ratios are also studied.

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1. Introduction

Let R^n denote the collection of all Borel sets in $\mathbb{R}^n$ with finite Lebesgue measure (denoted by μ). Define a Wiener process $\{W(A), A \in R^n\}$ as a family of Gaussian random variables with zero mean and

$$(1.1) \qquad EW(A)W(B) = \mu(A \cap B)$$

As a set-parameter process, W(A) is additive, i.e.,

$$(1.2) \qquad W(A+B) = W(A) + W(B) \quad , \text{a.s.}$$

where A + B denotes the union of disjoint sets, and intuitively, we can view W(A) as the integral over A of a Gaussian white noise.

The connection with white noise renders the Wiener process important in applications as well as theory. Consider for example, the following signal detection problem.

A process ξ_t is observed on $t \in T \subset \mathbb{R}^n$, and we have to decide between the possibilities: (a) ξ_t contains a random signal Z_t plus an additive Gaussian white noise and (b) ξ_t contains only noise.

Formulated so as to avoid the pathologies of "white noise," the problem can be stated as follows: Let $\{W(A), A \in R^n(T)\}$ be a set-parameter process, with parameter space $R^n(T)$ = {Borel subsets of T}, and defined on a fixed measurable space (Ω, F). Let P' and P be two probability measures such that (a) under P' $W(A) - \int_A Z_t dt$ is a Wiener process independent of $\{Z_t, t \in T\}$, (b) under P W(A) is a Wiener process.

Now, let F_W denote the σ-field generated by the process W, and let P_W and P'_W denote the respective probability measures restricted to F_W. If $\int_T Z_t^2 dt < \infty$; a.s., then $P'_W << P_W$ and the detection problem in most cases reduces to one of computing the likelihood ratio

(1.3) $\Lambda = \frac{dP'_W}{dP_W}$

in terms of the observed process W.

With respect to the probability space (Ω, F, P) $\{W(A), A \in R^n(T)\}$ is a Wiener process. Hence, Λ is a positive integrable functional of a Wiener process. Computing Λ in terms of W is a problem that can be embedded in a more general one of finding representations of a Wiener functional, which in turn can be embedded (and illuminated in the process) in a still more general problem of representing martingales generated by a Wiener process.

For a random variable Y that is a square-integrable functional of a Wiener process $\{W(A), A \in R^n(T)\}$, several representations already exist. The first is the Hermite-Wiener series of Cameron and Martin [1]. The second is in terms of the multiple Wiener integrals as defined by Ito [5]. The third is in terms of the Ito integral [4], and its generalization as defined by Wong and Zakai [8] and Yor [10]. In the last representation the concept of martingales plays a crucial role.

For processes with a multidimensional parameter, it is both more natural and more general to define <u>martingales</u> for processes parameterized by sets rather than points in $\mathbb{R}^n$. Let $C \subset R^n(T)$ be a collection of closed sets. Let $\{F(A), A \in C\}$ be a family of σ-fields such that $A \supseteq B \Rightarrow F(A) \supseteq F(B)$. Let $\{M(A), A \in C\}$ be a set-parameter process. We say that $\{M(A), F(A), A \in C\}$ is a <u>martingale</u> if

$$E(M(A)|F(B)) = M(B) \quad \text{a.s.}$$

whenever $A \supset B$. Let $\{W(A),\ A \in R^n(T)\}$ be a Wiener process and denote

$$F_W(A) = \sigma(\{W(B),\ B \subset A\})$$

The main object of this paper is to show that under very general conditions on C, there is a canonical representation of all square-integrable martingales with respect to $\{F_W(A),\ A \in C\}$, and hence representation for square integrable Wiener functionals. For C = {all closed sets} the representation reduces to that of multiple Wiener integrals. For C = {all closed rectangles in $\mathbb{R}^n_+$ with the origin as one corner} the representations of Ito, Wong-Zakai, and Yor are recovered. These two are in a sense limiting cases, and between them lies a vast spectrum of choices for C, giving rise to an equally large array of representations for C-martingales and Wiener functionals.

The key to these representations is to define multiple stochastic integrals of the form

$$\int_{T^m} \phi(t_1,t_2,\ldots,t_m)\ W(dt_1)\ldots\ldots W(dt_m)$$

where ϕ are (in general) random integrands C-adapted in a suitable sense to be defined later.

The basic ideas underlying this paper were first introduced in the dissertation [3].

2. Multiple Stochastic Integrals

Let C be a collection of Borel subsets of a fixed rectangle T. Given sets $A_1,A_2,\ldots,A_m \in R^n(T)$, we shall define their support in C by

(2.1) $$S_{A_1,A_2,\ldots,A_m} = \cap\{B = B \in C \text{ and } B \cap A_i \neq \phi \text{ for every } i\}$$

If $t_1,t_2,\ldots,t_m$ are points in T, their support will be written as $S_{t,t_2..t_m}$. We say $t_1,t_2,\ldots,t_m$ are C-independent if no point is contained in the support of the remaining ones.

For $C = R^n(T)$, $S_{t_1,t_2..t_m}$ is just $\{t_1,t_2,\ldots,t_m\}$ so that C-independent mean "distinct". For $C = \{T_t, t \in T \subset \mathbb{R}^n_+\}$ where T_t denotes the rectangle bounded by the origin and t, $S_{t_1,t_2..t_m}$ is the smallest T_t that contains $t_1,t_2,\ldots,t_m$, and C-independent means "pairwise unordered". For C = {all convex sets in T}, $S_{t_1,t_2..t_m}$ is the convex hull of $\{t_1,t_2,\ldots,t_m\}$, and C-independent means $t_1,t_2,\ldots,t_m$ are extreme points of their convex hull. More examples will be given later.

Let $\hat{T}^m$ denote the subset of C-independent points in $T^m \subset \mathbb{R}^{mn}$. For a fixed n and C, $\hat{T}^m$ may be vacuous for sufficiently large m. For example, if $C = \{T_t\}$ is the collection of rectangles bounded by the origin and $t \in T \subset \mathbb{R}^n_+$, then $\hat{T}^m$ is empty for $m > n$. That is, no more than n points can be C-independent.

Let (Ω,F,P) be a fixed probability space. Let $\{F(A), A \in C\}$ be a family of σ-subfields parameterized by sets in $C \subset R^n(T)$. Let $\{W(A), A \in R^n(T)\}$ be a Wiener process such that: (a) $A \subset B \Rightarrow W(A)$ is F(B)-measurable, and (b) $A \cap B = \phi \Rightarrow \{W(A'), A' \subset A\}$ is F(B)-independent. We shall assume the following conditions on C:

(c_1) For every collection of rectangles $A_1,A_2,\ldots,A_m$ such that

$$\prod_{i=1}^{m} A_i \subset \hat{T}^m$$

$$\mu(A_i \cap S_{A_1A_2..A_m}) = 0, \quad i = 1,2,\ldots,m$$

(c_2) For each $m \geq 1$, the mapping

$$t = (t_1, t_2, \ldots, t_m) \rightsquigarrow S_t$$

is a continuous map from T^m to the collection of sets that are compact under the metric

$$(2.2) \qquad \rho(A,B) = (\max_{x \in A} \min_{y \in B} |x-y| + \max_{x \in B} \min_{y \in A} |x-y|)$$

(c_3) For each $m \geq 1$ and for almost all $t \in T^m$

$$\mu(S_t - \bigcup_{\varepsilon > 0} S_{B(\varepsilon,t_1),B(\varepsilon,t_2)..,B(\varepsilon,t_m)}) = 0$$

when $B(\varepsilon,t_i)$ denotes the ball with radius ε centered at t_i.

For a C satisfying conditions $c_1 - c_3$, we shall define multiple stochastic integrals of order m

$$(2.3) \qquad \phi \circ W^m = \int_{\hat{T}^m} \phi_t \; W(dt_1)..W(dt_m)$$

for integrands $\phi(t,\omega)$, $(t,\omega) \in \hat{T}^m \times \Omega$, statifying

(h_1) ϕ is $F \times \mu^m$-measurable

(h_2) For each $t \in \hat{T}^m$ ϕ_t is $F(S_t)$-measurable.

(h_3) $$\int_{\hat{T}^m} E\phi_t^2 \, dt < \infty$$

The space of functions satisfying $h_1 - h_3$ will be denoted by $L_a^2(\hat{T}^m x\Omega)$.

Call ϕ <u>atomic</u> if $\phi(t,\omega) = \alpha(\omega) \; I_A(t)$ where I_A is the indication function of a product of rectangles $A = \prod_{i=1}^{m} A_i$ such that $A \subset \hat{T}^m$. Two atomic functions

$$(2.4) \qquad \phi(t,\omega) = \alpha(\omega)\ I_A(t) \quad , A \subset \hat{T}^m$$

$$\theta(t,\omega) = \beta(\omega)\ I_B(t) \quad , B \subset \hat{T}^p$$

are said to be <u>comparable</u> if each pair (A_i, B_j) is either equal or disjoint modulo sets of zero Lebesgue measure, and <u>similar</u> if $m = p$ and $(B_1, B_2, \ldots, B_m)$ is a permutation of $(A_1, A_2, \ldots, A_m)$. Call ϕ <u>simple</u> if $\phi = \sum_{k=1}^{K} \phi_k$ and each ϕ_k is atomic.

For an atomic function ϕ define

$$(2.5) \qquad \phi \circ W^m = \alpha \prod_{i=1}^{m} W(A_i)$$

So define, $\phi \circ W^m$ has the following property:

<u>Lemma 2.1</u>. Let ϕ and θ be comparable atomic functions in $L_a^2(\hat{T}^m x\Omega)$ and $L_a^2(T^p x\Omega)$ of the form (2.4). Then

$$(2.6) \qquad E(\phi \circ W^m)\ (\theta \circ W^p) = 0$$

unless ϕ and θ are similar. In the latter case,

$$(2.7) \qquad E(\phi \circ W^m)\ (\theta \circ W^m) = \int_{\hat{T}^m} E\tilde{\theta}_t \tilde{\theta}_t\ dt \overset{\text{def.}}{=} \langle \tilde{\phi}, \tilde{\theta} \rangle$$

where $\tilde{\phi}$ denotes the symmetrization of ϕ, i.e.,

$$(2.8) \qquad \tilde{\phi}_t = \frac{1}{m!} \sum_{\Pi} \phi_{\Pi(t)} \quad , \Pi(t) = \text{permutation of } t$$

Proof: First, assume ϕ and θ to be similar. Then,

$$(\phi \circ W^m)\ (\theta \circ W^m) = \alpha\beta \prod_{i=1}^{m} W^2(A_i)$$

and $\alpha\beta$ is measurable with respect to $F(S_{A_1 A_2 \ldots A_m})$. Therefore, condition c_1 implies that

$$E[(\phi \circ W^m)(\theta \circ W^m) | F(S_{A_1 A_2 .. A_m})]$$

$$= \alpha\beta \prod_{i=1}^{m} E\, W^2(A_i)$$

$$= \alpha\beta \prod_{i=1}^{m} \mu(A_i)$$

and (2.7) follows.

Next, suppose that ϕ and θ are comparable but not similar. With no loss of generality assume $m \geq p$. Consider two possibilities:

(a) There exists a B_j (say B_1) such that

$$\mu(B_1 \cap [\bigcup_{i=1}^{m} A_i \cup S_{A_1 A_2 .. A_m}]) = 0$$

(b) For every $j \leq p$

$$\mu(B_j \cap [\bigcup_{i=1}^{m} A_i \cup S_{A_1 A_2 .. A_m}]) \neq 0$$

For case (a), let

$$D = \bigcup_{i=1}^{m} A_i \bigcup_{j=2}^{p} B_j \cup S_{A_1 A_2 .. A_m} \cup S_{B_1 B_2 \ldots B_p}$$

Then, with probability 1

$$E[(\phi \circ W^m)(\theta \circ W^p) | F(D)] = \alpha\beta \prod_{i=1}^{m} W(A_i) \prod_{j=2}^{p} W(B_j)[EW(B_1)] = 0$$

and (2.6) is verified.

For case (b) we shall prove that $S_{A_1A_2..A_m} \supset S_{B_1B_2..B_p}$. Since ϕ and θ are comparable but not similar and $m \geq p$, there must exist an A_i(say A_1) such that $\mu(A_1 \cap B_j) = 0$ for every j. Hence, $W(A_1)$ is independent of $\alpha\beta \prod_{i=2}^{m} W(A_i) \prod_{j=1}^{p} W(B_j)$ and (2.6) is again proved.

To prove $S_{A_1A_2..A_m} \supset S_{B_1B_2..B_p}$ for case (b), let $D \in C$ be any set such that

$$D \cap A_i \neq \phi \text{ for every } i$$

then, $D \supset S_{A_1A_2..A_m}$ by definition. The defining condition for case (b) implies that for each j

either $B_j \cap \bigcup_i A_i \neq \phi$

which implies $B_j = A_i$ for some i

which in turn implies $D \cap B_j \neq \phi$

or $B_j \cap S_{A_1A_2..A_m} = \phi$

which implies $D \cap B_j \neq \phi$

Therefore,

$$D \cap A_i \neq \phi \text{ for every } i \Rightarrow D \cap B_j \neq \phi \text{ for every } j$$

and $S_{A_1A_2..A_m} \supset S_{B_1B_2...B_p}$ ∎

Lemma 2.2. For atomic functions ϕ and θ that are not necessarily comparable, we can write

$$(2.9) \qquad \phi = \sum_{k=1}^{K} \phi_k$$

$$\phi = \sum_{\lambda=1}^{L} \theta_\lambda$$

where ϕ_k, θ_λ are atomic and the set $\{\phi_k, \theta_\lambda\}$ is pairwise comparable. For any atomic ϕ and θ in L_a^2 the isometry

$$(2.10) \qquad E(\phi \circ W^m)(\theta \circ W^p) = \delta_{mp} \langle \tilde{\phi}, \tilde{\theta} \rangle$$

holds.

Proof: ϕ and θ, being atomic, are of the form

$$\phi = \alpha \, I_{A_1 x A_2 x \ldots x A_m}$$

$$\theta = \beta \, I_{B_1 x B_2 x \ldots x B_p}$$

where $A_1, A_2, \ldots A_m, B_1, \ldots, B_p$ are rectangles in T. Since a union of rectangles is always a union of disjoint rectangles, there exist disjoint rectangles $D_1, D_2, \ldots, D_q$ such that each A_i or B_j is the union of some of the D_ν's. Hence (2.9) follows, with

$$\phi_k = \alpha \, I_{D_{k1} x D_{k2} x \ldots x D_{km}}$$

$$\theta_\lambda = \beta \, I_{D_{\lambda 1} x D_{\lambda 2} x \ldots x D_{\lambda p}}$$

where $D_{ki} \subset A_i$ and $D_{\lambda j} \subset B_j$ for every i and j. It follows that α is $F(S_{D_{k1} D_{k2} \ldots D_{km}})$-measurable and β is $F(S_{D_{\lambda 1} D_{\lambda 2} \ldots D_{\lambda m}})$-measurable for each k and λ. From lemma 2.1 we have

$$E(\phi_k \circ W^m)(\phi_\lambda \circ W^p) = \delta_{mp} \langle \tilde{\phi}_k, \tilde{\theta}_\lambda \rangle$$

and (2.10) follows from the bilinearity of $\langle \quad \rangle$. ¤

<u>Lemma 2.3</u>. Under conditions c_2 and c_3 the subset of simple functions is dense in $L_a^2(\hat{T}^m x \Omega)$.

A proof of this result is given in the appendix A.

Theorem 2.1. There is a unique linear map denoted by $\phi \circ W^m$ of $\phi \in L_a^2(\hat{T}^m x\Omega)$ into the space of square-integrable random variables such that

(a) For an atomic function $\phi = \alpha I_A$

$$\phi \circ W^m = \alpha \prod_i W(A_i)$$

(b) Symmetry:

$$\phi \circ W^m = \tilde{\phi} \circ W^m$$

(c) Isometry:

$$E(\phi \circ W^m)(\theta \circ W^p) = \langle \tilde{\phi}, \tilde{\theta} \rangle \delta_{mp}$$

Proof: First, any simple function ϕ is by definition of the form $\phi = \sum_{k=1}^{K} \phi_k$, where ϕ_k are atomic. Bilinearity of $\langle \quad \rangle$ then implies the isometry (2.10) for simple functions ϕ and θ. Let ϕ be any function from $L_a^2(\hat{T}^m x\Omega)$. Lemma 3.2 implies that there exists a sequence $\{\phi^{(n)}\}$ of simple functions such that

$$\phi^{(n)} \xrightarrow[n\to\infty]{L_a^2} \phi$$

Hence, $\{\phi^{(n)}\}$ is Cauchy. The isometry (2.10) then implies that $\{\phi^{(n)} \circ W^m\}$ is mean-square convergent as a sequence of random variables, and we take the limit to be $\phi \circ W^m$. Verification of the properties follows from the isometric property in a striaghtforward way. ∎

Remark: Observe that the isometry property of the multiple stochastic integral implies uniqueness up to equivalence of integrand. That is, if $\phi \circ W^m = \theta \circ W^m$ then

$$\|\phi-\theta\|^2 = \int_{\hat{T}^m} E(\tilde{\phi}_t-\tilde{\theta}_t)^2 \, dt = 0$$

Theorem 2.2. (Projection) For any $B \in \mathcal{R}^n(T)$

(2.12) $$E(\phi\circ W^m|F(B)) = E(\phi|F(B))\, I_{B^m} \circ W^m$$

Proof: It is enough to prove this for an atomic ϕ. Let $\phi = \alpha\, I_{A_1 x A_2 x \ldots x A_m}$. Then

$$E(\phi\circ W^m|F(B)) = E(\alpha \prod_{i=1}^{m} W(A_i)|F(B))$$

$$= E\{\alpha\, E[\prod_{i=1}^{m} W(A_i)|F(B\cup S_{A_1\ldots A_m})]|F(B)\}$$

$$= E\{\alpha \prod_{i=1}^{m} W(A_i\cap B)|F(B)\}$$

$$= E(\alpha|F(B)) \prod_{i=1}^{m} W(A_i\cap B)$$

$$= E(\phi|F(B))\, I_{B^m} \circ W^m$$ ◻

Corollary. If $B \in C$ then

$$E(\phi\circ W^m|F(B)) = \phi\, I_{B^m} \circ W^m$$

Proof: If $B \in C$ then $t_i \in B$ for each i implies $B \supset S_{t_1 t_2 \ldots t_m}$. Hence, $t \in B^m \Rightarrow \phi_t$ is $F(B)$-measurable and $E(\phi|F(B))\, I_{B^m} = \phi\, I_{B^m}$ a.s. ◻

Let $\{(\phi\circ W^m)_B, B \in C\}$ be the set-parameterized process defined by

$$(\phi\circ W^m)_B = \phi I_{B^m} \circ W^m$$

Then the corollary to Theorem 2.2 implies that $\{(\phi\circ W^m)_B, B \in C\}$ is a martingale. We shall call $(\phi\circ W^m)_B$ the <u>indefinite integral</u> of $\phi \circ W^m$.

3. <u>Relationship with Multiple Wiener Integrals and Representation of Wiener Functionals</u>

Let $\tilde{T}^m$ denote the set of m-tuples of distinct points in T. Let $\theta(t)$, $t \in \tilde{T}^m$ satisfy

$$\int_{\tilde{T}^m} \theta^2(t)\, dt < \infty$$

Let $\theta \square W^m$ denote a multiple Wiener integral of order m.

<u>Theorem 3.1</u>. For a given C satisfying condition $c_1 - c_3$, a multiple Wiener integral can be represented as

$$(3.1) \qquad \theta \square W^m = \sum_{k=1}^{m} \binom{m}{k} \theta_k \circ W^k$$

where

$$(3.2) \qquad \theta_k(t_1,t_2,\cdots,t_k,\omega) = (\tilde{\theta}(t_1,t_2,\cdots,t_k,\cdot)\, I_{S^{m-k}_{t_1 t_2 \cdot\cdot t_k}}(\cdot) \square W^{m-k})(\omega)$$

and $\theta_k \circ W^k$ is a multiple stochastic integral defined relative to C.

Proof: Let $\Pi\theta$ denote the transformation of θ by a premutation of its arguments. Suppose for some permutation Π

$$\Pi\theta = I_{A_1 x A_2 x \ldots x A_m}$$

where $A_1,\ldots\ldots,A_k$ are C-independent rectangles and $A_{k+1},\ldots,A_m$ are distinct rectangles contained in $S_{A_1A_2..A_k}$. Then, symmetry implies that

$$\theta \,\Box\, W^m = \Pi\theta \,\Box\, W^m = [\prod_{i=k+1}^{m} W(A_i)] \prod_{i=1}^{k} W(A_i)$$

$$= h_k \circ W^k$$

when

$$h_k(t_1,\ldots,t_k,\omega) = I_{A_1x\ldots xA_k}(t_1,t_2,\ldots,t_k)[I_{A_{k+1}x..xA_m} \,\Box\, W^{m-k}](\omega)$$

$$= \Pi\theta(t_1,t_2,\ldots,t_k,\cdot) \,\Box\, W^{m-k}$$

The isometry of multiple stochastic integrals implies that both k and the two sets $\{A_1,A_2,\ldots,A_k\}$ and $\{A_{k+1},A_{k+2},\ldots A_m\}$ are unique. The integer k is unique because otherwise we would have

$$E(\theta\Box W^m)^2 = E(h_k\circ W^k)\ (h_{k'}\circ W^{k'}) = 0$$

The collection $\{A_1,A_2,\ldots,A_k\}$ is unique because otherwise we would have

$$\theta \,\Box\, W^m = h_k \circ W^k = g_k \circ W^k$$

and $h_k g_k \equiv 0$. It follows that

$$\sum_{\text{all } \Pi} [(\Pi\theta)(t_1,t_2,\ldots,t_k,\cdot)\ I_{S^{m-k}_{t_1t_2..t_k}}(\cdot) \,\Box\, W^{m-k}] \circ W^k$$

$$= k!(m-k)!\ \theta \,\Box\, W^m$$

$$= m!\ \theta_k \circ W^k$$

where

$$\theta_k(t_1,t_2,\ldots,t_k,\omega) = [\tilde{\theta}(t_1,t_2,\ldots,t_k,\cdot)\, I_{S^{m-k}_{t_1 t_2 \ldots t_k}}(\cdot) \,\square\, W^{m-k}](\omega)$$

Hence,

$$\theta \,\square\, W^m = \binom{m}{k} \quad \theta_k \circ W^k$$

In appendix B, it is proved that linear combinations of such θ's are dense in $L^2(T^m)$. Thus, the theorem is proved. ¤

<u>Corollary 1</u>. Let $F_W(A)$ denote the σ-field generated by $\{W(B),\ B \subset A\}$. Then, every square-integrable $F_W(T)$-measurable random variable Z has a representation of the form

$$(3.3) \qquad Z = EZ + \sum_{m=1}^{\infty} Z_m \circ W^m$$

where $Z_m \circ W^m$ are stochastic integrals defined with respect to the same C that satisfies conditions $c_1 - c_3$.

Proof: This corollary follows immediately from the main theorem and the well known result [5] that Z has a representation in a series of multiple Wiener integrals.

<u>Corollary 2</u>. For $f \in L^2(T)$, define

$$(3.4) \qquad \hat{f}^m(t_1,\ldots,t_m) = \prod_{i=1}^{m} f(t_i)$$

and set

$$(3.5) \qquad W_m(f,A) = (\hat{f}^m \,\square\, W^m)_A$$

Then, for $A \in C$

$$(3.6) \qquad W_m(f,A) = \sum_{k=1}^{m} \binom{m}{k} [\hat{f}^k(\cdot)\, W_{m-k}(f,S_{\cdot}) \circ W^k]_A$$

Proof: Observe that f^k is symmetric and

$$\hat{f}^m(t_1,t_2,\ldots,t_m) = \hat{f}^k(t_1,t_2,\ldots,t_k)\, \hat{f}^{m-k}(t_{k+1},\ldots,t_m)$$

Hence, (3.1) yields (3.6) for $A = T$, and the rest follows from the projection property (Theorem 2.2).

Corollary 3. For $f \in L^2(T)$ define

$$(3.7) \qquad L(f,A) = \exp\{(f \square W)_A - \frac{1}{2} \int_A f^2(t)\, dt\}$$

Then, for $A \in C$

$$(3.8) \qquad L(f,A) = 1 + \sum_{m=1}^{\infty} \frac{1}{m!} [\hat{f}^m(\cdot) L(f,S_{\cdot}) \circ W^m]_A$$

Proof: For multiple Wiener integrals (C = {all closed sets }}(3.8) reduces to

$$(3.9) \qquad L(f,A) = 1 + \sum_{m=1}^{\infty} \frac{1}{m!} W_m(f,A)$$

which is well known [5]. For the general case, we use (3.6) in (3.9) and write

$$L(f,A) = 1 + \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{k=1}^{m} \binom{m}{k} [\hat{f}^k W_{m-k}(f,S_{\cdot}) \circ W^k]_A$$

$$= 1 + \sum_{k=1}^{\infty} \frac{1}{k!} [\hat{f}^k \sum_{j=0}^{\infty} \frac{1}{j!} W_j(f,S.) \circ W^k]_A$$

$$= 1 + \sum_{k=1}^{\infty} \frac{1}{k!} [\hat{f}^k L(f,S.) \circ W^k]_A \qquad ¤$$

The expansion formula (3.8) for exponentials of the form (3.7) can be extended with the Wiener integral $f \,\square\, W$ in the exponent being replaced by a stochastic integral $f \circ W$. The result can be stated as follows:

Proposition 3.2. Equation (3.8) remains valid for $f \in L_a^2(T\times\Omega)$ such that f is bounded.

Proof: Define f to be a discrete simple function if f is a simple function

$$f(t,\omega) = \sum_{i=1}^{k} \alpha_i(\omega)\, I_{A_i}(t)$$

such that $P(\alpha_i \in J) = 1$ for some finite set J. Such a function may be written as $f(t,\omega) = g(t,\alpha(\omega))$ where $\alpha = (\alpha_1,\ldots,\alpha_k)$ and

$$g(t,c) = \sum_{i=1}^{k} c_i I_{A_i}(t) \text{ for } c \in J^k.$$

Then $g(\cdot,c) \in L^2(T)$ for each $c \in J^k$ so by Corollary 3 of Theorem 3.1,

$$(3.10) \qquad L(g(\cdot,c),A) = 1 + \sum_{m=1}^{\infty} \frac{1}{m!} [\hat{g}^m(\cdot,c) L(g(\cdot,c),S.) \circ W^m]_A.$$

This equality holds in $L^2(\Omega,F,P)$ for each $c \in J^k$ and hence it continues to hold in $L^2(\Omega,F,P)$ if c is replaced by the random vector $\alpha(\omega)$. By proposition C in appendix C, replacing c by $\alpha(\omega)$ in the stochastic integrals is equivalent to replacing c by $\alpha(\omega)$ in each of the integrands and then forming the stochastic integrals. (To apply propositon C to the mth term on the right of (3.10), let $B_i = S_{A_i}$). This verifies equation (3.8) if f is a discrete simple function.

Conclude that $E[L(f,A)] = 1$ if f is discrete and simple. Moreover, if $p \geq 1$ and $|f(\cdot,\cdot)| \leq \Gamma$ for some constant Γ , then

$$L(f,A)^p = L(pf,A)\ \exp(\tfrac{1}{2}(p^2-p)\int_A f(t)^2 dt)$$
$$\leq L(pf,A)\ \exp(\tfrac{1}{2}(p^2-p)\Gamma^2\mu(T)$$

so that

$$(3.11) \qquad E[L(f,A)^p] \leq \exp(\tfrac{1}{2}(p^2-p)\Gamma^2\mu(T))$$

Now choose any $f \in L_a^2(T\times\Omega)$ with $|f(\omega,t)| \leq \Gamma$. Then there is a sequence of discrete simple functions $f_j \to f$ in $L_a^2(T\times\Omega)$ such that $|f_j(\omega,t)| \leq \Gamma$ for each j. Hence $(f_j \circ W)_A \to (f\circ W)_A$ a.s. in $L^2(\Omega)$ so that taking a subsequence if necessary, we can assume that $(f_j\circ W)_A \to (f\circ W)_A$ with probability one. Thus $L(f_j,A) \to L(f,A)$ with probability one. By the estimate (3.11), the collection of random variables $\{L(f_j,A)^p : p \geq 1\}$ is uniformly integrable for each $p \geq 1$ so that $L(f_j,A) \to L(f,A)$ in $L^p(\Omega)$ for each $p > 1$. Moreover, $\hat{f}_j^m \to \hat{f}^m$ in $L_a^p(\hat{T}^m\times\Omega)$ for each $p \geq 1$ since these functions are uniformly bounded. Now (3.8) is true for f replaced by f_j, and it is then easily verified for f by taking the limit in $L^2(\Omega)$ term by term as $j \to +\infty$. ¤

4. A Likelihood Ratio Formula

Let $\{Z_t, t \in T\}$ be a bounded process defined on $(\Omega,\mathcal{F},P)$ and let $\{W(A), A \subset T\}$ be a Wiener process defined on the same space. Let $\mathcal{F}(A) = \sigma(\{W(B), B \subset A\}, \{Z_t, t \in A\})$. We assume that $A \cap A' = \phi \Rightarrow W(A')$ in $\mathcal{F}(A)$-independent. For any collection $\mathcal{C}$ the support

S_t contains t. Hence, Z_t is $F(S_t)$ measurable. For any C satisfying $c_1 - c_3$, the stochastic integral $Z \circ W$ is well-defined.

Now, let P' be a measure on (Ω, F) defined by:

$$(4.1) \qquad \frac{dP'}{dP} = \exp\{Z \circ W - \frac{1}{2} Z^2 \circ \mu\}$$

and set

$$(4.2) \qquad L(Z,A) = \exp\{(Z \circ W)_A - \frac{1}{2} (Z^2 \circ \mu)_A\}$$

For any C satisfying $c_1 - c_3$, proposition 3.2 yields

$$(4.3) \qquad L(Z,A) = 1 + \sum_{m=1}^{\infty} \frac{1}{m!} [\hat{Z}^m(\cdot) L(Z,S_{\cdot}) \circ W^m]_A$$

It follows that

$$(4.4) \qquad L(Z,A) = E(\frac{dP'}{dP} | F(A))$$

and P' is a probability measure.

Next, let $F_W(A) = \sigma(\{W(B), B \subset A\})$, and define the <u>likelihood ratio</u> by

$$(4.5) \qquad \Lambda(A) = E(\frac{dP'}{dP} | F_W(A))$$

We shall use (4.3) to derive an expression for $\Lambda(A)$.

<u>Proposition 4.1</u>. Let $t \in \hat{T}^m$ and define

$$(4.6) \qquad \tilde{Z}_m(t) = E'(Z(t_1)Z(t_2)..Z(t_m) | F_W(S_{t_1 t_2 \ldots t_m}))$$

Then the likelihood ratio is given by

$$(4.7) \qquad \Lambda(A) = 1 + \sum_{m=1}^{\infty} \frac{1}{m!} [\tilde{Z}_m(\cdot)\Lambda(S_\cdot) \circ W^m]$$

Proof: We begin by writing

$$\Lambda(A) = E[L(Z,A)|F_W(A)]$$

and using (4.3). Observe that with P-measure 1,

$$E\{[\hat{Z}^m(\cdot)L(Z,S_\cdot) \circ W^m]_A|F_W(A)\} = E[\hat{Z}^m(\cdot)L(Z,S_\cdot)|F_W(A)] \circ W^m$$

Now, for $t = (t_1,t_2,\ldots,t_m) \in A^m$,

$$E[\hat{Z}^m(t)L(Z,S_t)|F_W(A)]$$

$$= E[Z(t_1)Z(t_2)\ldots Z(t_m)L(Z,S_{t_1t_2..t_m})|F_W(A)]$$

$$= E[Z(t_1)Z(t_2)\ldots Z(t_m)L(Z,S_{t_1t_2..t_m})|F_W(S_{t_1t_2..t_m})]$$

$$= \Lambda(S_{t_1..t_m})E'[Z(t_1)\ldots Z(t_m)|F_W(S_{t_1\ldots t_m})]$$

$$= \tilde{Z}_m(t)\Lambda(S_t)$$

and (4.7) follows. ¤

Two special cases are of particular interest. First, let $a \in \mathbb{R}^n$ be a fixed unit vector (i.e., $\|a\| = 1$) and let H_α denote the half space $\{t \in \mathbb{R}^n : (t,a) \geq \alpha\}$. Then, the collection $C = \{H_\alpha \cap T\}$ is a one-parameter family of sets such that $\hat{T}^m$ is vacuous for $m > 1$. That is, two or more points are always C-dependent. In this case the likelihood ratio formula reduces to

$$\Lambda(A) = 1 + [\tilde{Z}_1(\cdot)\Lambda(S_\cdot) \circ W]_A \quad , \quad A \in C$$

and an application of (3.8) yields

$$(4.8)\qquad \Lambda(A) = L(\tilde{Z}_1, A) = \exp\{(\tilde{Z}_1 \circ W - \frac{1}{2}\tilde{Z}_1^2 \circ \mu)_A\}$$

where

$$\tilde{Z}_1(t) = E'(Z(t) | F_W(S_t))$$

$$= E'(Z(t) | F_W(H_{(t,a)} \cap T))$$

In this case we see that the likelihood ratio is expressible as an exponential of the conditional mean.

The second case of special interest results from taking C = {all closed sets in T}. For this case

$$S_{t_1 t_2 \ldots t_m} = \{t_1, t_2, \ldots, t_m\}$$

Hence, with P-measure 1

$$\Lambda(S_{t_1 t_2 \ldots t_m}) = 1$$

and

$$\tilde{Z}_m(t) = E' Z(t_1) \ldots Z(t_m)]$$

Furthermore, if we assume that Z and W are independent processes under P then Z is identically distributed under P'. Hence, for that case we can write

$$(4.9)\qquad \Lambda(A) = 1 + \sum_{m=1}^{\infty} \frac{1}{m!} (\rho_m \square W^m)_A$$

where ρ_m is the mth moment

$$(4.10)\qquad \rho_m(t_1, t_2, \ldots, t_m) = E[Z(t_1) \ldots Z(t_m)] \quad .$$

Equation (4.9) provides a martingale representation of the likelihood ratio for the "additive white Gaussian noise" model under very general conditions. In the one-dimensional case, it was recently obtained in [7].

Equation (4.7) is an integral equation in that Λ occurs on both sides. In special cases [2,6,9] the equation can be converted to yield an exponential formula for Λ in terms of conditional moments.

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Appendix A: Proof that Simple Functions are Dense

The purpose of this appendix is to prove the following proposition:

Proposition A. Conditions c_2 and c_3 imply that the space of simple functions is dense in $L_a^2(\hat{T}^m x\Omega)$ for each $m \geq 1$.

Proof: We begin by introducing some additional notation. For $\varepsilon > 0$ and $t = (t_1,\ldots,t_m) \in T^m$, define the ε-support of t by

$$S_t^\varepsilon = S_{B(\varepsilon,t_1)B(\varepsilon,t_2)\ldots B(\varepsilon,t_m)} \quad ,$$

where $B(\varepsilon,t_i)$ denotes a ball with radius ε and center t_i, and define $S_t^{(-)} = \bigcup_{\varepsilon>0} S_t^\varepsilon$. Define $L_\varepsilon^2(\hat{T}^m x\Omega)$ the same way as $L_a^2(\hat{T}^m x\Omega)$ but with condition h_2 replaced by the stronger condition: (h_2^ε) for each $t \in \hat{T}^m$, ϕ_t is $F(S_t^\varepsilon)$-measurable. Finally, let $C_\varepsilon(\hat{T}^m x\Omega)$ be the subspace of $L_\varepsilon^2(\hat{T}^m x\Omega)$ consisting of $\phi \in L_\varepsilon^2(\hat{T}^m x\Omega)$ such that $\phi(\cdot,\omega)$ is continuous on $\hat{T}^m$ with probability one.

Proposition A is a consequence of the following sequence of lemmas.

□

Lemma A.1. $\bigcup_{\varepsilon>0} L_\varepsilon^2(\hat{T}^m x\Omega)$ is dense in $L_a^2(\hat{T}^m x\Omega)$ under conditions c_2 and c_3.

Proof: Let $f \in L_a^2(\hat{T}^m x\Omega)$ be bounded by a constant $\Gamma > 0$. For any $\varepsilon > 0$, there is a Borel measurable mapping $u(\cdot,\varepsilon)$ of the open set $\hat{T}^m$ into a finite subset of $\hat{T}^m$ such that $|u(\underline{x},\varepsilon) - \underline{x}| < \varepsilon$ for all $\underline{x} \in \hat{T}^m$. Define

$$f^\varepsilon(S) = E[f(s)|F(S_{u(s,\varepsilon)}^{2\varepsilon})] \quad s \in \hat{T}^m$$

A version of $f^\varepsilon(s)$ can be chosen for each s so that f^ε is a jointly measurable function of (s,ω). Indeed, for each fixed $t \in \hat{T}^m$ there exist versions of

$$g^\varepsilon(s,t) = E[f^\varepsilon(s)|F(S_t^{2\varepsilon})]$$

which are jointly measurable functions of (s,ε), and then $g^\varepsilon(s,u(s,\varepsilon))$ is a jointly measurable version of $f^\varepsilon(x)$. Also, f^ε can be assumed to be bounded by Γ. For each $s \in \hat{T}^m$, $f^\varepsilon(s)$ is measurable with respect to

$$F(S_{u(x,\varepsilon)}^{2\varepsilon}) \subset F(S_s^\varepsilon)$$

so that $f^\varepsilon \in L_\varepsilon^2(\hat{T}^m x\Omega)$.

Since $S_s^{3\varepsilon} \subset S_{u(s,\varepsilon)}^{2\varepsilon} \subset S_s^\varepsilon$, $\lim_{\varepsilon\downarrow 0} S_{u(s,\varepsilon)}^{2\varepsilon} = \lim_{\varepsilon\downarrow 0} S_s^\varepsilon = S_s^{(-)}$ for each $\underline{s} \in \hat{T}^m$. By the continuity of σ-fields generated by the Wiener process, $\lim_{\varepsilon\downarrow 0} F(S_s^\varepsilon) = F(S_s^{(-)})$. Then, by L^2-martingale convergence, for each $s \in \hat{T}^m$,

$$E[(f(s)-f^\varepsilon(s))^2] = E\ (E[f(s)|F(R_s)] - E[f(s)|F(R_{u(s,\varepsilon)}^{2\varepsilon})])$$

$$\underset{\varepsilon\downarrow 0}{\rightarrow} E\ (E[f(s)|F(R_s)] - E[f(s)|F(R_s^{(-)})])$$

By condition c_3, $\mu(R_\varepsilon - R_s^{(-)}) = 0$ and so also $E[(f(s)-f^\varepsilon(s))^2] \rightarrow 0$, for a.e. $s \in \hat{T}^m$. Since $(f(s)-f^\varepsilon(s))^2 \leq 4\Gamma^2$,

$$\|f-f^\varepsilon\|^2 = \int_{\hat{T}^m} E[(f(s)-f^\varepsilon(s))^2\, ds \rightarrow 0$$

by the Lebesgue Dominated Convergence Theorem. Thus, any bounded function $f \in L_a^2(\hat{T}^m x\Omega)$ is the limit of functions in $\bigcup_{\varepsilon>0} L_\varepsilon^2(\hat{T}^m x\Omega)$. Since the bounded functions in $L_a^2(\hat{T}^m x\Omega)$ are dense in $L_a^2(\hat{T}^m x\Omega)$, the lemma is established. ¤

<u>Lemma A.2.</u> $\bigcup_{\varepsilon>0} C_\varepsilon(\hat{T}^m x\Omega)$ is dense in $\bigcup_{\varepsilon>0} L_\varepsilon^2(\hat{T}^m x\Omega)$.

Proof: Let $f \in L^2_{2\varepsilon}(\hat{T}^m x\Omega)$ be bounded by some constant $\Gamma > 0$. Choose $V \in C^\infty(\mathbb{R}^{mn})$ such that $V \geq 0$, $V(x) = 0$ if $|x| \geq 1$, and $\int_{\mathbb{R}^n} V(x)dx = 1$. For $\delta > 0$, define $V^\delta \in C^\infty(\mathbb{R}^{mn})$ by $V^\delta(x) = (\frac{1}{\delta})^{mn} V(\frac{x}{\delta})$ and define a function f^δ on $\hat{T}^m$ by the convolution: $f^\delta(\cdot,\omega) = V^\delta * f(\cdot,\omega)$ for each fixed ω. Here the function $f(\cdot,\omega)$, which is a priori defined on $\hat{T}^m \subset T^m \subset (\mathbb{R}^n)^m \doteq \mathbb{R}^{mn}$, is extended to a function on all of $\mathbb{R}^{mn}$ by the convention $f(\underline{s},\omega) = 0$ if $\underline{s} \notin \hat{T}^m$. Note that f^δ is bounded by Γ and sample continuous, and since $V(x) = 0$ for $|x| \geq \delta$, $f^\delta \in C_{2\varepsilon-\delta}(\hat{T}^m x\Omega)$.

Observe that

$$\|f-f^\delta\|^2 = E[\int_{\hat{T}^m} |f(s) - f^\delta(s)|^2 \, ds]$$

$$\leq E\int_{\mathbb{R}^{mn}} |f(s) - V^\delta * f(s)|^2 \, ds]$$

$$\leq \int_{\mathbb{R}^{mn}} V\,(x)\; E[\int_{\mathbb{R}^{mn}} |f(s) - f(s-x)\;^2 \, ds] \, dx \tag{A.1}$$

Now $\int_{\mathbb{R}^{mn}} |f(s) - f(s-x)|^2 \, dx \to 0$ as $s \to 0$ for all ω since translations are continuous in $L^2(\mathbb{R}^{mn})$. Hence, the expertation in (A.1) converges to zero as $x \to 0$ by Lebesgues Bounded Convergence Theorem and so also $\|f-f^\delta\| \to 0$ as $\delta \to 0$. ¤

<u>Lemma A.3</u>. If $f \in C\;(\hat{T}^m x\Omega)$ for some $\varepsilon > 0$, then there is a sequence f^δ of simple function which converge to f in $L^2(\hat{T}^m x\Omega)$.

Proof: It suffices to prove the lemma under the additional assumption that f is bounded uniformly in (t,ω). Recall that under Condition

c_3, $\hat{T}^m$ is naturally identified with an open subset of $\mathbb{R}^{mn}$. For $\delta > 0$, let I_δ denote sets of the form

$$(I_{11}x\ldots xI_{1n})x\ldots x(I_{m1}x\ldots xI_{mn})$$

where each I_{ij} is an interval of the form $(k\delta,(k+1)\delta]$, and let $\hat{I}_\delta$ consist of $A \in I_\delta$ such that $A \subset \hat{T}^m$. Let $u(\cdot,\delta)$ be a function from $\hat{T}^m$ to $\hat{T}^m$ such that $u(x,\delta) = u(x',\delta) \in J$ whenever x, $x' \in J$ for some $J \in I_\delta$. Define $f^\delta(\underline{s}) = f(u(\underline{s},\delta))$ if $\underline{s} \in J$ for some $J \in \hat{I}_\delta$, and define $f^\delta(\underline{s}) = 0$ otherwise. For $\delta < \varepsilon/\sqrt{n}$, each of the m rectangles in T of a set in $\hat{I}_\delta$ has diameter less than ε so that $f^\delta \in S_a(\hat{T}^m x\Omega)$ for $\delta < \varepsilon/\sqrt{n}$. Furthermore, f^δ is bounded by the same constant that f is, and $f^\delta(s,\omega) \to f(s,\omega)$ as $\delta \to 0$ for each $(s,\omega) \in \hat{T}^m x\Omega$ by the sample continuity of f. So $f^\delta \to f$ in $L^2(\hat{T}^m x\Omega)$ as $\delta \to 0$ by dominated convergence. ¤

Appendix B

Let $\mathcal{I}_m$ denote the collection of subsets of T^m of the form $A_1 x \ldots x A_m$ such that each $A_i \in \mathcal{R}^n(T)$ and for some permutation Π,

1) $A_{\Pi(1)}, \ldots, A_{\Pi(k)}$ are C-independent, and

2) $A_{\Pi(k+1)}, \ldots, A_{\Pi(m)} \subset S_{A_{\Pi(1)} A_{\Pi(2)} \cdots A_{\Pi(k)}}$.

The purpose of this appendix is to prove the following proposition:

Proposition B. The linear span of $\{1_A : A \in \mathcal{I}_m\}$ is dense in $L^2(T^m)$ for each $m \geq 1$.

Proof: Consider the following two conditions on C:

(b_1) There is a countable subcollection of $\mathcal{I}_m$ which covers T^m a.e.

(b_2) There is a countable subcollection $\mathcal{I}_m^d$ of disjoint sets in $\mathcal{I}_m$ which covers T^m a.e.

By a sequence of lemmas it is shown below that conditions c_2 and $c_3 \Rightarrow$ condition $b_1 \Rightarrow$ condition $b_2 \Rightarrow$ the conclusion of Proposition B.

¤

Lemma B.1.

$$\bigcup_{\ell=1}^{m} \bigcup_{\Pi \in P(m)} \Pi \circ \{(\underline{x}, \underline{y}) : \underline{x} \in \hat{E}^{\ell},\ \underline{y} \in (S_{\underline{x}})^{m-\ell}\} = T^m \qquad (*)$$

Proof: Let $\underline{q} = (q_1, \ldots, q_m) \in T^m$. Choose a permutation $\underline{p} = (p_1, \ldots p_m) = \Pi(q_1, \ldots, q_m)$ so that for some ℓ with $1 \leq \ell \leq m$,

$$S_{\underline{q}} = S_{p_1, \ldots, p} \neq S_{p_1, \ldots, \hat{p}_1, \ldots, p_\ell} \text{ for } 1 \leq i \leq \ell$$

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conditions c_2 and

Proposition B.

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(*)

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$1 \leq \ell \leq m$,

ℓ

To show that $\underline{q}$ is contained in the left si

show that $p_1,\ldots,p_\ell$ are C-independent. Now, if

C-independent, then $p_i \in S_{p_1,\ldots,\hat{p}_i,\ldots,p_\ell}$ for

$$\{A \in C\colon p_1,\ldots,p \in A\} = \{A \in C\colon p_1,\ldots,\hat{p}_i,\ldots$$

Intersecting all the sets contained in this col

that

$$S_{p_1,\ldots,p_\ell} = S_{p_1,\ldots,\hat{p}_i,\ldots,p_\ell}$$

which contradicts our choice of $p_1,\ldots,p_\ell$. Thu

C-independent so that $\underline{p}$, and hence $\underline{q}$, is contai

(*).

Lemma B.2. Conditions c_2 and c_3 imply Conditio

Proof: Let I_m^0 denote the subsets of T^m of the

for some $\Pi \in P(m)$ and some $\ell > 0$,

a) $A_{\Pi},\ldots,A_{\Pi_\ell}$ are C-independent, closed r

have rational coordinates in $T \subset \mathbb{R}^n$,

b) $A_{\Pi(\ell+1)} = \ldots = A_{\Pi(m)} = S_{A_{\Pi(1)}A_{\Pi(2)}\cdots}$

Then I_m^0 is a countable subset of I_m and

$$\bigcup_{A\in I_m^0} A \supset \bigcup_{\ell=1}^{m} \bigcup_{\Pi\in P(m)} \Pi\circ\{(\underline{x},\underline{y}):\underline{x}\in\hat{T}^\ell,\ \underline{y}\in(S^{(-)})^{m-\ell}\}$$

$$= \bigcup_{\ell=1}^{m} \bigcup_{\Pi\in P(m)} \Pi\circ\{(\underline{x},\underline{y}):\underline{x}\in\hat{T}^\ell,\ \underline{y}\in(S_{\underline{x}})^{m-\ell}\} \qquad (B.1)$$

$$- \bigcup_{\ell=1}^{m} \bigcup_{\Pi\in P(m)} \Pi\circ S_{m,\ell}$$

where

$$S_{m,\ell} = \{(\underline{x},\underline{y}):\underline{x}\in\hat{T}^\ell,\ \underline{y}\in(R_{\underline{x}})^{m-\ell} - (R_{\underline{x}}^{(-)})^{m-\ell}\}\ .$$

The first term on the right hand side of (B.1) is equal to T^m by Lemma B.1. Thus, to complete the proof it must be shown that $\mu^m(S_{m,\ell}) = 0$ for all $m \geq 1$ and $1 \leq \ell \leq m$.

By Condition c_2,

$$F_\varepsilon = \{(\underline{x},\underline{y}):\underline{x}\in\hat{T}^\ell,\ \underline{y}\in(S_{\underline{x}}^\varepsilon)^{m-\ell}\}$$

is a closed subset of $\hat{T}^\ell \times T^{m-\ell}$ which increases as ε decreases to zero. Since $S_{m,\ell} = F_0 - \bigcup_{\varepsilon>0} F_\varepsilon$, it follows that $S_{m,\ell}$ is a Borel subset of T^m. By Condition c_3, the section

$$\{\underline{y}:(\underline{x},\underline{y}) \in S_{m,\ell}\} \subset T^{m-\ell}$$

of $S_{m,\ell}$ at $\underline{x}$ has Lebesgue measure zero for a.e. $\underline{y} \in \hat{T}^m$. Hence, by Fubini's theorem, $\mu^m(S_{m,\ell}) = 0$ for $1 \leq \ell \leq m$. ◻

Lemma B.3. Condition b_1 implies condition b_2.

Proof: Let $F_1, F_2, \ldots$ be a countable subcollection of $\mathcal{I}_m$ which covers T^m a.e.. Then the disjoint sets $D_i = F_i - \bigcup_{j=1}^{i-1} F_j$ $i \geq 1$ cover T^m a.e.. We claim that for each $i \geq 1$ there is a finite collection of disjoint sets $D_{i1}, \ldots, D_{in_i}$ in $\mathcal{I}_m$ such that $D_i = \bigcup_{j=1}^{n_i} D_{ij}$. Condition b is then satisfied with $\mathcal{I}_m^d = \{D_{ij} : i \geq 1, 1 \leq j \leq n_i\}$. It remains to prove the claim.

By induction, it suffices to establish the cliam for $i = 2$. Now $F_1 = A_1 x \ldots x A_m$ for some Borel sets $A_1, \ldots, A_m \subset T$. Thus, $F_1^i = \bigcup_{j=1}^{r} K_j$ where $K_1, \ldots, K_r$ are disjoint and each K_j is the product of m Borel subsets of T. In fact, F_1^i is the union of all sets of the form $B_1 x \ldots x B_m$ such that $B_i = A_i$ or $B_i = A_i^c$ for each i and such that $B_i = A_i^c$ for at least one i, and these sets are disjoint. So $D_2 = \bigcup_{j=1}^{k} K_j \cap F_2$. The sets $K_j \cap F_2$. The sets $K_j \cap F_2$ are disjoint sets in $\mathcal{I}_m$ as required so the claim is established. ¤

<u>Lemma B.4</u>. Condition b_2 implies that the linear span of $\{1_A : A \in \mathcal{I}_m\}$ is dense in $L^2(T^m)$.

Proof: Let $F = F_1 x \ldots x F_m$ where each $F_i \in R^n(T)$. Then $A \cap F \in \mathcal{I}_m$ for any $A \in \mathcal{I}_m$ and by Condition b_2,

$$1_F = \sum_{A \in \mathcal{I}_m^d} 1_{A \cap F} \quad \text{a.e. in } T^m.$$

Since the linear span of functions of the form 1_F is dense in $L^2(T^m)$, the lemma is established. ¤

Appendix C

Proposition C. Assume Conditions c_1 - c_3. Let $B_1,\ldots,B_k$ be closed subsets of T and suppose that $\alpha_i(\omega)$ is an $F(B_i)$ measurable random variable with values in a finite set J for $1 \leq i \leq k$. Suppose for each $c \in J^k$ that $h(\cdot,\cdot,c) \in L_a^2(\hat{T}^m x\Omega)$ and that

$$h(t,\cdot,c) = h(t,\cdot,c') \text{ a.s.}$$

whenever $c_i = c_i'$ for all i such that $B_i \not\subset S_t$. Then $h(\cdot,\cdot,\alpha(\cdot)) \in L_a^2(\hat{T}^m x\Omega)$ and

$$h(\cdot,\cdot,\alpha(\cdot))\circ W^m = h(\cdot,\cdot,c)\circ W^m\big|_{c=\alpha(\cdot)} \quad \text{a.s.}$$

Proof: For each $\theta \in \{0,1\}^k$, define

$$\hat{T}^m_\theta = \{t \in \hat{T}^m : B_i \subset S_t \Leftrightarrow \theta_i = 1 \text{ for } 1 \leq i \leq k\}$$

By condition c_2, the set $\{t: B \subset S_t\}$ is open for each i so that $\hat{T}^m_\theta$ is Borel for each θ. Since $\bigcup_\theta \hat{T}^m_\theta = \hat{T}^m$ it suffices to prove the lemma when

$$h(t,\cdot,c) = h(t,\cdot,c) I_{\hat{T}^m_\theta}(t)$$

for all t,c. Now, for definiteness, suppose that $\theta_i = 1$ for $1 \leq i \leq \ell$ and $\theta_i = 0$ for $\ell \leq i \leq k$. Let $\Pi: \mathbb{R}^k \to \mathbb{R}^\ell$ denote projection onto the first ℓ coordinates. Then for all $c \in J^k$,

$$h(t,\omega,c) = \tilde{h}(t,\omega,\Pi(c))$$

where $\tilde{h}(t,\omega,c) = h(t,\omega,(\Pi(c),j_0,\ldots,j_0))$ for some fixed $j_0 \in J$.

Thus,

$$h(\cdot,\cdot,\alpha(\cdot))\circ W^m = \tilde{h}(\cdot,\cdot,\Pi(\alpha(\cdot)))\circ W^m$$

$$= \sum_{b\in J^\ell} [\tilde{h}(\cdot,\cdot,b) I_{(\Pi(\alpha(\cdot))=b}] \circ W^m$$

$$= \sum_{b\in J^\ell} I_{(\Pi(\alpha(\cdot))=b)} (\tilde{h}(\cdot,\cdot,b)\circ W^m)$$

$$= (\tilde{h}(\cdot,\cdot,b)\circ W^m)|_{b=\Pi(\alpha(\cdot))}$$

$$= (h(\cdot,\cdot,c)\circ W^m)|_{c=\alpha(\cdot)}$$

The second equality is easily proven by approximating $\tilde{h}(\cdot,\cdot,b)$ in $L_a^2(\hat{T}^m x\Omega)$ for each b by simple functions which vanish off the open set $\{t\in\hat{T}^m : B_i \subset S_t \text{ for } 1 \le i \le \ell\}$. ¤