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Representation and Transformation of Two-Parameter Martingales Under a Change of Measure

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1. Introduction

Let $(\Omega, \{\mathscr{F}_t, 0 \leqq t \leqq 1\}, P_0)$ be a probability space and let $\{X_t, 0 \leqq t \leqq 1\}$ be a P_0 local martingale. Doléans-Dade showed that the integral equation

$$L_t = 1 + \int_0^t L_{s-}\, dX_s$$

has a unique solution $\{L_t, 0 \leqq t \leqq 1\}$ which is a positive local martingale. If $E_0 L_1 = 1$ then L_t is a martingale and $\frac{dP}{dP_0} = L_1$ defines a new probability measure P with

$$L_t = E_0 \left(\frac{dP}{dP_0} \middle| \mathscr{F}_t \right).$$

Results concerning P_0 local martingales under P or P local martingales under P_0 have come to be known as Girsanov's theorem [4]. For example, if N_t is a continuous P_0 local martingale then $N_t - [N, X]_t$ is a P local martingale where $[N, X]_t$ is defined intrinsically as a quadratic variation process.

If $\{\mathscr{F}_t, 0 \leqq t \leqq 1\}$ is generated by a P_0 Wiener process W_t then there exists a P Wiener process $\tilde{W}_t$. Further, any P local martingale Z_t has the integral representation $Z_t = \int_0^t q_s\, d\tilde{W}_s$ where $\int_0^1 q_s^2\, ds < \infty$ a.s.

For processes with a two-dimensional parameter the martingale concept has several extensions [1, 9]. The purpose of this paper is to present some results of the Girsanov type for such processes defined on the probability space of a 2-parameter Wiener process. When possible, transformation results are stated and proved in an intrinsic (representation independent) form.

Preliminary material is presented in Sect. 2, while the main results are collected in Sect. 3. In the remaining two sections, the theorems regarding martingales and weak martingales, respectively, are proved.

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2. The Stochastic Calculus and Likelihood Ratios

The basic definitions of [1] will be used in this paper, and are summarized as follows. Let $R_+ = [0, \infty) \times [0, \infty)$ denote the positive quadrant of the plane. For two points $s = (s_1, s_2)$ and $s' = (s'_1, s'_2)$ in R_+, $s > s'$ will denote the condition $s_1 \geqq s'_1$ and $s_2 \geqq s'_2$, $s \curlywedge s'$ will denote the condition $s'_1 \geqq s_1$ and $s_2 \geqq s'_2$, $s \times s'$ will denote the point (s_1, s'_2), and $s \vee s'$ will denote the point $(\max(s_1, s'_1), \max(s_2, s'_2))$. $I(s \curlywedge s')$ will denote the indicator function of the set $\{s \curlywedge s'\}$. 0 will denote the origin in R_+ and R_z the rectangle $\{0 \prec s \prec z\}$. $R_z \otimes R_z$ is the set $\{(s, s'): s \in R_z, s' \in R_z, s \curlywedge s'\}$.

Let $\{W_z, z \in R_{z_0}\}$ be a standard Wiener process defined on $(\Omega, \{\mathscr{F}_z\}, P_0)$ where $\mathscr{F}_z = \{\sigma(W_s, s \prec z)$ completed with respect to $P_0\}$. A stochastic process $X = \{X_z, z \in R_{z_0}\}$ is said to be adapted if X_z is $\mathscr{F}_z$ measurable for each $z \in R_{z_0}$. In the following definitions, the process X is assumed to be adapted and for each z, X_z is integrable. X is a martingale if $z' > z$ implies that $E_0(X_{z'} | \mathscr{F}_z) = X_z$ a.s., X is an adapted 1-martingale (2-martingale) if $\{X_{s_1, s_2}, \mathscr{F}_{s_1 s_2}\}$ is a one parameter martingale in s_1 for each s_2 (in s_2 for each s_1). X is a weak martingale if $s' \succ s$ implies that $E[X_{s'} - X_{s' \times s} - X_{s \times s'} + X_s | \mathscr{F}_s] = 0$ a.s.

A proper 1-martingale (2-martingale) is a square integrable, sample continuous process $M_1(M_2)$ which is an adapted 1-martingale (2-martingale) and mean square differentiable in the 2-direction (1-direction).

A process Z is a local martingale if there is a sequence Z_n of square integrable martingales such that $Z_n(z, \omega) = Z(z, \omega)$ for $z \in R_{z_0}$ and $n \geqq N(\omega)$ where $N(\omega) < \infty$ a.s. A local i-martingale (proper local i-martingale, weak local martingale) is similarly defined as the limit of square integrable i-martingales (proper i-martingales, weak martingales) for $i = 1, 2$.

For $1 \leqq p \leqq +\infty$, define $\mathscr{L}_1^p$ to be the collection of adapted, measurable functions $q(s, \omega)$ on $(R_{z_0} \times \Omega, \mathscr{B}(R_{z_0}) \times \mathscr{F})$ (where $\mathscr{B}$ denotes Borel sets) such that $\int_{R_{z_0}} |q(s, \omega)|^p \, ds < +\infty$ a.s. if $p < +\infty$ or else $\sup_s |q(s, \omega)| < +\infty$ a.s. if $p = +\infty$. Define $\mathscr{L}_2^p$ to be the collection of measurable functions $r(s, s', \omega)$ on $(R_{z_0} \otimes R_{z_0} \times \Omega, \mathscr{B}(R_{z_0} \otimes R_{z_0}) \times \mathscr{F})$ such that $r(s, s')$ is $\mathscr{F}_{s \vee s'}$ measurable for each $s, s' \in R_{z_0}$ and, if $p < +\infty$, then $\int_{R_{z_0} \otimes R_{z_0}} |r(s, s')|^p \, ds \, ds' < +\infty$ a.s. or else, if $p = +\infty$, then $\sup_{s, s'} |r(s, s')| < +\infty$ a.s. Clearly $\mathscr{L}_i^p \supseteq \mathscr{L}_i^q$ if $p \leqq q$ for $i = 1, 2$.

By the Theorems of Wong and Zakai [see 2], all local martingales have the stochastic integral representation

$$Z_z = \int_{R_z} q(s) \, dW_s + \int_{R_z \otimes R_z} r(s, s') \, dW_s \, dW_{s'}, \tag{2.1}$$

and all proper local 1-martingales (2-martingales) are given by mixed area integrals

$$\int_{R_z \otimes R_z} \alpha(s, s') \, ds \, dW_{s'} \left(\int_{R_z \otimes R_z} \beta(s, s') \, dW_s \, ds' \right) \tag{2.2}$$

where $q \in \mathscr{L}_1^2$ and $r, \alpha, \beta \in \mathscr{L}_2^2$. For $2 \leqq p \leqq +\infty$, denote by S^p the class of random processes which can be expressed as the sum of processes of the form (2.1), (2.2) and an absolutely continuous process $B_z = \int_{R_z} b(s)\,ds$ where $q, b \in \mathscr{L}_2^p$, and $r, \alpha, \beta \in \mathscr{L}_2^p$. Clearly $\mathscr{S}^p \supseteq \mathscr{S}^q$ if $p \leqq q$. Define $\mathscr{S}^\omega = \bigcap_{p<+\infty} \mathscr{S}^p$. The processes in $\mathscr{S}^2$ are called semimartingales. Denoting Lebesgue measure by μ, a semi-martingale is conveniently written as

$$Z = W \circ r \circ W + q \circ W + \mu \circ \alpha \circ W + W \circ \beta \circ \mu + b \circ \mu. \tag{2.3}$$

Stochastic integrals can be defined with respect to semimartingales. If $\Psi \in \mathscr{L}_1^s$ and $Z \in \mathscr{S}^r$ for some $2 \leqq r, s, t \leqq +\infty$ such that $\frac{1}{r} + \frac{1}{s} = \frac{1}{t}$, then by a simple application of Hölder's inequality the stochastic integral

$$\begin{aligned}\Psi \circ Z = &\int_{R_z} q_s \Psi_s\, dW_s + \int_{R_z \otimes R_z} r_{s,s'} \Psi_{ss'}\, dW_s\, dW_{s'} \\ &+ \int_{R_z \otimes R_z} \alpha_{s,s'} \Psi_{ss'}\, ds\, dW_{s'} + \int_{R_z \otimes R_z} \beta_{s,s'} \Psi_{ss'}\, dW_s\, ds' + \int_{R_z} b_s \Psi_s\, ds\end{aligned}$$

is a semimartingale in $\mathscr{S}^t$.

A process is a weak martingale if and only if it the sum of a 1-martingale and a 2-martingale [7]. It follows that all semimartingales which are local weak martingales have the representation

$$Z = W \circ r \circ W + q \circ W + \mu \circ \alpha \circ W + W \circ \beta \circ \mu \tag{2.4}$$

A semimartingale is a one-parameter semimartingale in each direction with the semimartingale representations

$$Z_z = \int_{R_z} Z_{W1}(z, s')\, dW_{s'} + \int_{R_z} Z_{\mu 1}(z, s')\, ds' \tag{2.5}$$

$$Z_z = \int_{R_z} Z_{W2}(z, s)\, dW_s + \int_{R_z} Z_{\mu 2}(z, s)\, ds \tag{2.6}$$

where

$$\begin{aligned} Z_{W1}(z, s') &= q_{s'} + \int_{R_z} I(s \curlywedge s')\, r_{s,s'}\, dW_s + \int_{R_z} I(s \curlywedge s')\, \alpha_{s,s'}\, ds \\ Z_{\mu 1}(z, s') &= b_{s'} + \int_{R_z} I(s \curlywedge s')\, \beta_{s,s'}\, dW_s \\ Z_{W2}(z, s) &= q_s + \int_{R_z} I(s \curlywedge s')\, r_{s,s'}\, dW_{s'} + \int_{R_z} I(s \curlywedge s')\, \beta_{s,s'}\, ds' \\ Z_{\mu 2}(z, s) &= b_s + \int_{R_z} I(s \curlywedge s')\, \alpha_{s,s'}\, dW_{s'}. \end{aligned} \tag{2.7}$$

It is convenient to write (2.5) and (2.6) in the compact form

$$Z = Z_{W1} \circ W + Z_{\mu 1} \circ \mu, \tag{2.8}$$

$$Z = Z_{W2} \circ W + Z_{\mu 1} \circ \mu. \tag{2.9}$$

Note that if $Z = \mu \circ \alpha \circ W$ and $\bar{Z} = \mu \circ \bar{\alpha} \circ W$, then $Z \equiv \bar{Z}$ a.s. if and only if $Z_{W1}(z, s') = \bar{Z}_{W1}(z, s')$ for $(ds' \times dP$ measure) a.e. (s', ω) for each z. In this case $\bar{\alpha}$

will be called a version of α. Hence, α and β in (2.3) are uniquely determined by Z up to versions.

The composition $Y * X$ of two semimartingales X and Y is the process defined by

$$\begin{aligned}(Y*X)_z = &\int_{R_z \otimes R_z} Y_{W2}(s\vee s', s)\, X_{W1}(s\vee s', s')\, dW_s\, dW_{s'} \\ &+ \int_{R_z \otimes R_z} Y_{\mu 2}(s\vee s', s)\, X_{W1}(s\vee s', s')\, ds\, dW_{s'} \\ &+ \int_{R_z \otimes R_z} Y_{W2}(s\vee s', s)\, X_{\mu 1}(s\vee s', s')\, dW_s\, ds' \\ &+ \int_{R_z \otimes R_z} Y_{\mu 2}(s\vee s', s)\, X_{\mu 1}(s\vee s', s')\, ds\, ds' \end{aligned} \tag{2.10}$$

Formally, $Y*X$ satisfies $\partial_1 \partial_2 (Y*X) = \partial_2 Y \partial_1 X$. In abbreviated form, (2.10) can be expressed as

$$Y*X = W \circ Y_{W2} X_{W1} \circ W + \mu \circ Y_{\mu 2} X_{W1} \circ W + W \circ Y_{W2} X_{\mu 1} \circ \mu + \mu \circ Y_{\mu 2} X_{\mu 1} \circ \mu.$$

One may define quadratic variation processes for a semimartingale with representation (2.1) by

$$[Z, Z]_z = b^2 \circ \mu + \mu \circ r^2 \circ \mu \tag{2.11}$$

and, for $i=1$ or 2,

$$\langle Z, Z \rangle_{iz} = \int_{R_z} X_{Wi}^2(z, s)\, ds. \tag{2.12}$$

The definition (2.12) is consistent with the definition of quadratic variation for one-parameter semimartingales. Both $[Z, Z]$ and $\langle Z, Z\rangle_i$ are intrinsic to Z in the sense that they have representation free, quadratic variation interpretations [1, 2]. Define $[Z, \tilde{Z}]$ and $\langle Z, \tilde{Z}\rangle_i$ for semimartingales Z and $\tilde{Z}$ by bilinearity. If Z has representation (2.3) and

$$\tilde{Z} = W \circ \tilde{r} \circ W + \tilde{q} \circ W + \mu \circ \tilde{\alpha} \circ W + W \circ \tilde{\beta} \circ \mu + \tilde{b} \circ \mu \tag{2.13}$$

then

$$\langle Z, \tilde{Z}\rangle_1(z) = \int_{R_z} Z_{W1}(z, s')\, \tilde{Z}_{W1}(z, s')\, ds', \qquad z = (z_1, z_2) \tag{2.14}$$

and

$$\langle Z, \tilde{Z}\rangle_1 = [Z, \tilde{Z}] + (W \circ r + \mu \circ \alpha)\, \tilde{Z}_{W1} \circ \mu + (W \circ \tilde{r} + \mu \circ \tilde{\alpha})\, Z_{W1} \circ \mu. \tag{2.15}$$

To obtain (2.15) apply the 1-parameter differential formula [4] to the integrand in (2.14) as a function of z_2 for fixed z_1 and use (2.7). Similarly,

$$\langle Z, \tilde{Z}\rangle_2 = [Z, \tilde{Z}] + \mu \circ \tilde{Z}_{W2}(r \circ W + \beta \circ \mu) + \mu \circ Z_{W2}(\tilde{r} \circ W + \tilde{\beta} \circ \mu). \tag{2.16}$$

The differentiation formula of [8, 9] for semimartingales has been put into a representation free form [10]. Let $F: R \to R$ be a function with continuous derivatives through the fourth order, and let Z be given by (2.1). Let $F_k(x) = \frac{\partial^k}{\partial x^k} F(x)$. Then

$$\begin{aligned} F(Z) = & F(Z_0) + F_1(Z) \circ Z + F_2(Z) \circ (Z * Z) \\ & + \tfrac{1}{2} F_2(Z) \circ (\langle Z, Z \rangle_1 + \langle Z, Z \rangle_2 - [Z, Z]) \\ & + \tfrac{1}{2} F_3(Z) \circ (Z * \langle Z, Z \rangle_1 + \langle Z, Z \rangle_2 * Z + 2[Z, Z * Z]) \\ & + \tfrac{1}{4} F_4(Z) \circ (\langle Z, Z \rangle_2 * \langle Z, Z \rangle_1). \end{aligned} \tag{2.17}$$

If $Z = (Z_1, \ldots, Z_n)$ is a vector of n semimartingales and if $F: \mathbb{R}^n \to \mathbb{R}$ has continuous partial derivatives to fourth order, then (2.17) still yields the integral representation of $F(Z)$ if the terms are interpreted appropriately. For example, identify

$$F_1(Z) \circ Z = \sum_i \frac{\partial F}{\partial z_i} \circ Z_i$$

$$F_2(Z) \circ (Z * Z) = \sum_{i,j} \frac{\partial^2 F}{\partial z_i \partial z_j} \circ (Z_i * Z_j)$$

$$F_3(Z) \circ (Z * \langle Z, Z \rangle_1) = \sum_{i,j,k} \frac{\partial^3 F}{\partial z_i \partial z_j \partial z_k} \circ (Z_i * \langle Z_j, Z_k \rangle_1).$$

For $n=2$ and $F(z, \tilde{z}) = z\tilde{z}$, this yields

$$Z\tilde{Z} = Z_0 \tilde{Z}_0 + Z \circ \tilde{Z} + \tilde{Z} \circ Z + Z * \tilde{Z} + \tilde{Z} * Z + \langle Z, \tilde{Z} \rangle_1 + \langle Z, \tilde{Z} \rangle_2 - [Z, \tilde{Z}]. \tag{2.18}$$

This generalization of the differentiation formula given in [8, 9] may be proved, as in [8], by repeated application of the differential formula for one parameter martingales. (2.18) shows that $Z * \tilde{Z} + \tilde{Z} * Z$ is intrinsic to Z, $\tilde{Z}$ since all the other terms are. Thus the symmetrization of $*$ is an intrinsic operation.

The following proposition summarizes some properties of the operations defined on semimartingales. One consequence is that $\mathscr{S}^\omega$ is closed under all operations of the stochastic calculus defined so far.

Proposition 2.1 [3]. a) *Let $Z \in \mathscr{S}^p$ for $p \leqq 2$. Then $(s, s', \omega) \leadsto Z_{Wi}(s \vee s', s', \omega)$ and $(s, s', \omega) \leadsto Z_{\mu i}(s \vee s', s', \omega)$ are in $\mathscr{L}_2^p$ for $i = 1, 2$.* b) *If $Z \in \mathscr{S}^r$, $\tilde{Z} \in \mathscr{S}^s$ and $\Psi \in \mathscr{L}_1^s$, for some $2 \leqq r, s, t \leqq +\infty$ such that $\frac{1}{r} + \frac{1}{s} = \frac{1}{t}$, then $[Z, \tilde{Z}]$, $\langle Z, \tilde{Z} \rangle_1$, $\langle Z, \tilde{Z} \rangle_2$, $Z * \tilde{Z}$ and $\Psi \cdot Z$ are well-defined semimartingales in $\mathscr{S}^t$. If $s = +\infty$ (so $2 \leqq r = t < +\infty$) then $\Psi \circ Z$ is still a well defined semimartingale in $\mathscr{S}^t$.* c) *If $p \geqq 2$ and $Z = (Z, \ldots, Z_n)$ is a vector of n semimartingales, each in $\mathscr{S}^{4p}$, and if $F: \mathbb{R}^n \to \mathbb{R}$ has continuous partial derivatives to fourth order, then $F(Z) \in \mathscr{S}^p$ and all terms in (2.17) are semimartingales.*

Let P be a probability measure on $(\Omega, \{\mathscr{F}_z\})$ equivalent to P_0. Then if $E_0\left[\left(\frac{dP}{dP_0}\right)^2\right]$ is finite, the likelihood ratio

$$L_z = E_0 \left[\frac{dP}{dP_0} \middle| \mathscr{F}_z \right]$$

satisfies $L=e^X$ where [7]

$$X_z=\int_{R_z}\theta_s\,dW_s-\tfrac{1}{2}\int_{R_z}\theta_s^2\,ds-\tfrac{1}{2}\int_{R_z\otimes R_z}\rho_{s,s'}^2\,ds\,ds' + \int_{R_z\otimes R_z}\rho_{s,s'}[dW_s-\tilde{u}(s'\times s,s)\,ds][dW_{s'}-u(s'\times s,s')\,ds'] \tag{2.19}$$

for some functions ρ, θ, u and $\tilde{u}$. In an abbreviated form, (2.19) is expressed as

$$X=\theta\circ W-\tfrac{1}{2}\theta^2\circ\mu-\tfrac{1}{2}\mu\circ\rho^2\circ\mu+(W-\mu\tilde{u})\circ\rho\circ(W-u\mu).$$

If $\rho\in\mathscr{L}_2^\infty$ and $\theta\in\mathscr{L}_1^\infty$, then u and $\tilde{u}$ are uniquely determined by

$$u(z,s')=\theta_{s'}+\int_{R_z}I(s\curlywedge s')\,\rho_{s,s'}[dW_s-\tilde{u}(s'\times s,s)\,ds] \tag{2.20}$$

$$\tilde{u}(z,s)=\theta_s+\int_{R_z}I(s\curlywedge s')\,\rho_{s,s'}[dW_{s'}-u(s'\times s,s')\,ds'] \tag{2.21}$$

and X is a semimartingale. It then makes sense to write $L=e^X=\mathscr{E}(\rho,\theta)$. $L=e^X$ also has the representation [7]

$$e^X=e^X\circ(m+M_2*M_1) \tag{2.22}$$

where m is the local martingale part of X and $M_1(M_2)$ is the local 1-martingale (2-martingale) part of X. It follows from (2.19)–(2.21) that

$$X_{W1}(z,s')=(M_1)_{W1}(z,s')=u(z,s'), \tag{2.23}$$

$$X_{W2}(z,s)=(M_2)_{W2}(z,s)=\tilde{u}(z,s). \tag{2.24}$$

$L=e^X$ also has essentially one parameter representations, as expressed by

$$e^X_{W1}(z,s')=e^X_{s'\times z}u(z,s'), \tag{2.25}$$

$$e^X_{W2}(z,s)=e^X_{z\times s}\tilde{u}(z,s). \tag{2.26}$$

Proposition 2.2. *If $\theta\in\mathscr{L}_1^p$ and $\rho\in\mathscr{L}_2^p$ for some $p\geqq 4$, then there exists a unique solution $(u,\tilde{u})$ to equations (2.20) and (2.21) such that $u(s'\times s,s')$ and $\tilde{u}(s'\times s,s)$, as functions of (s,s',ω), are in $\mathscr{L}_2^p$. If in addition $p\geqq 8$, then $\mathscr{L}\in\mathscr{S}^{p/4}$ and if $p\geqq 10$, then $X\in\mathscr{S}^{p/5}$.*

Lemma 2.3. *Let T be an interval $[0,t_0]\subset\mathbb{R}$. Given $2\leqq p\leqq+\infty$, $g\in\mathscr{L}^p(T)$ and $K\in\mathscr{L}^p(T\times T)$, the integral equation*

$$f(s)=g(s)+\int_{t\leqq s}K(s,t)f(t)\,dt \qquad t\in T \tag{2.27}$$

has a unique solution $f\in\mathscr{L}^p(T)$. The solution satisfies

$$\|f\|_p\leqq\left[\sum_{n=0}^{\infty}\frac{(\|K\|_p\,\mu(T)^{\frac{p-2}{p}})^n}{n!}\right]. \tag{2.28}$$

Remark. The lemma and its proof easily carry over to more general partially ordered sets T. This generalization will be used without further mention.

Proof of Lemma 2.3. The proof will be accomplished by a version of Picard iteration. Define $\phi_0(s)=g(s)$ and $\phi_n(s)=\int_{t\leqq s} K(s,t)\,\phi_{n-1}(t)\,dt$ for $n\geqq 1$. For each fixed $s\in T$, define $a(s)=\|K(s,\cdot)\|_{p'}^{p}$, where p' is conjugate to p $\left(\text{i.e. } \frac{1}{p}+\frac{1}{p'}=1\right)$. The by Holder's inequality, for each s and $n\geqq 1$

$$|\phi_n(s)|^p \leqq a(s) \int_{t\leqq s} |\phi_{n-1}(t)|^p\,dt. \tag{2.29}$$

Also by Holder's inequality,

$$\|a\|_1=\int_T\Big(\int_T |K(s,t)|^{p'}\,dt\Big)^{p/p'}ds \leqq \|K\|_p^p\,\mu(T)^{p-2}. \tag{2.30}$$

Now (2.29) for $n=1$ implies that $|\phi_1(s)|^p \leqq a(s)\,\|g\|_p^p$ and then by induction (2.29) implies that

$$|\phi_n(s)|^p \leqq a(s) \int_0^s \int_0^{t_1} \dots \int_0^{t_{n-2}} a(t_1)\dots a(t_{n-1})\,dt_{n-1}\dots dt_1\,\|g\|_p^p.$$

Integrating each side of this inequality over T yields that

$$\|\phi_n\|_p^p \leqq \frac{\|a\|_1^n}{n!}\,\|g\|_p^p. \tag{2.31}$$

Hence, $\sum_{n=0}^{\infty}\|\phi_n\|_p$ is dominated by the right hand side of the inequality (2.28) so that the sum $f=\sum_{n=0}^{\infty}\phi_n$ converges to a unique element $f\in\mathscr{L}^p(T)$ which satisfies (2.27) and (2.28). The proof of uniqueness of the solution f follows similar reasoning. □

Proof of Proposition 2.2. Substituting Eq. (2.21) into Eq. (2.20) with $z=s'\times s$ yields that, for $(s,s')\in R_{z_0}\otimes R_{z_0}$,

$$\begin{aligned} U(s'\times s, s') = &\Big\{\theta_{s'} + \int_{R_{s'\times s}} I(t\wedge s')\,\rho_{t,s'}\big[dW_t - \big(\theta_t + \int_{R_{s'\times t}} I(t\wedge t')\,\rho_{t,t'}\,dW_{t'}\big)\,dt\big]\Big\}\\ &+ \int_{R_{s'\times s}} I(t\wedge s')\,\rho_{t,s'} \int_{R_{s'\times t}} I(t\wedge t')\,\rho_{t,t'}\,U(t'\times t, t')\,dt'\,dt. \end{aligned}$$

Let $T=R_{z_0}\otimes R_{z_0}\subset\mathbb{R}^4$ be equipped with Lebesgue measure and define a partial ordering on T by letting $(t,t')\leqq(s,s')$ if $t'\times t \prec s'\times s$. Part a of Proposition 2.1 insures that $(s,s')\rightsquigarrow \int_{R_{s'\times s}} I(t\wedge s')\,\rho_{t,s'}\,dW_t$ and $(t,s')\rightsquigarrow \int_{R_{s'\times t}} I(t\wedge t')\,\rho_{t,t'}\,dW_{t'}$ are each in $\mathscr{L}^p(T)$ for a.e. ω. Application of Holder's inequality then implies that, for a.e. ω, the term in curly braces in Eq. (2.31) is in $\mathscr{L}^{p/2}(T)$ for a.e. ω.

For each ω, define K by

$$K((s,s'),(t,t')) = I(t\wedge s')\,I(t\wedge t')\,\rho_{t,s'}\,\rho_{t,t'} \qquad \text{for } (s,s')\leqq(t,t')\in T.$$

Since $\rho\in L^p(T\times T)$ for a.e. ω, Holder's inequality implies that $K\in L^{p/2}(T\times T)$ for a.e. ω.

Now, for a.e. ω, Eq. (2.31) as an equation for $u(s' \times s, s'):(s, s') \in T)$ can be written as (2.27) where $K \in L^{p/2}(T \times T)$ and $g \in L^{p/2}(T)$. Hence, by Lemma 2.3, there is a unique solution $U \in L^{p/2}(T)$ for a.e. ω. Furthermore, the construction of the solution in the proof of Lemma 2.3 insures that $u(s' \times s, s', \omega) \in \mathscr{L}^{p/2}(T)$. A similar argument shows that $\tilde{u}(s' \times s, s, \omega) \in \mathscr{L}^{p/2}(T)$ as well. The remaining assertions of Proposition 2.2 follow from the representations (2.19) and (2.22) for X and L. $\square$

Technical Assumption. Throughout the remander of this paper assume that $P \sim P_0$ with $\frac{dP}{dP_0} = \mathscr{E}(\rho, \theta) = \exp(X) = L$. In addition, assume that one of the following is true:

A.1 $X \in \mathscr{S}^\infty$

A.2 θ, ρ, u and $\tilde{u}$ are a.s. bounded.

Finally, if A.1 is assumed, then the definition of semimartingale will be restricted to consist only of processes in $\mathscr{S}^\infty$.

Remarks. (1) By Proposition 2.2, condition A.1 is satisfied if $\theta \in \mathscr{L}_1^p$ and $\rho \in \mathscr{L}_2^p$ for all $p < +\infty$. Condition A.2 is satisfied if, for example, ρ, θ, u and $\tilde{u}$ are sample continuous. An open problem is to find reasonable conditions on ρ, θ to insure that A.2 is true. (2) The results of this paper which are true under assumption A.1 and the proviso that semimartingales are $\mathscr{S}^\infty$ processes are still true if the processes are only in $\mathscr{S}^p$ for some (rather than all) sufficiently large p, as determined by Proposition 2.1 and 2.2 in each case. (3) If ρ and θ are given and if $L = \mathscr{E}(\rho, \theta)$, then L is a positive local martingale. Included in the technical assumptions is the Girsonov condition $E[L_{z_0}] = 1$, which implies that L is actually a martingale. Conditions on ρ and θ which insure that $E[\mathscr{E}(\rho, \theta)] = 1$ appear to be severe when compared to the one parameter case, unless $\rho = 0$. (See [7], Sect. 5). (4) L is a strong martingale [9] if and only if $\rho \equiv 0$. Moreover, when probability P is used, $\mathscr{F}_z$ and $\mathscr{F}_{z'}$ are conditionally independent given $\mathscr{F}_z \cap \mathscr{F}_{z'}$ for all $z, z' \in R_{z_0}$ if and only if $\rho \equiv 0$. Much of the complexity of this paper disappears and more general results are easily established if L is restricted to be a strong martingale.

Lemma 2.4. *Let $Z = \{Z_z, z \in R_{z_0}\}$ be a semimartingale with the representation* (2.3). *Then the following identities hold and all terms are semimartingales.*

$$Z * e^X = e^X \circ (Z * M_1) \tag{2.32}$$

$$e^X * Z = e^X \circ (M_2 * Z) \tag{2.33}$$

$$\langle X, Z \rangle_1 = [X, Z] + (W - \mu\tilde{u}) \circ \rho Z_{W1} \circ \mu + W \circ ru \circ \mu + \mu \circ \alpha u \circ \mu \tag{2.34}$$

$$\langle X, Z \rangle_2 = [X, Z] + \mu \circ Z_{W2} \rho \circ (W - u\mu) + \mu \circ \tilde{u} r \circ W + \mu \circ \tilde{u} \alpha \circ \mu \tag{2.35}$$

$$\langle e^X, Z \rangle_1 = e^X \circ \{\langle X, Z \rangle_1 + M_2 * \langle X, Z \rangle_1 + [X * Z, X] + [X * X, Z]\} \tag{2.36}$$

$$\langle e^X, Z \rangle_2 = e^X \circ \{\langle X, Z \rangle_2 + \langle X, Z \rangle_2 * M_1 + [Z * X, X] + [X * X, Z]\} \tag{2.37}$$

$$\begin{aligned} e^X Z = Z \circ e^X + e^X \circ \{ & Z + M_2 * (Z + \langle Z, X \rangle_1) + (Z + \langle Z, X \rangle_2) * M_1 \\ & + \langle X, Z \rangle_1 + \langle X, Z \rangle_2 + [X, X * Z] + [X, Z * X] - [Z, X - X * X]\} \end{aligned} \tag{2.38}$$

$$e^{-X}=1+e^{-X}\circ\{-X*\langle X,X\rangle_1-\langle X,X\rangle_2*X+\langle X,X\rangle_1+\langle X,X\rangle_2 \\ -[X,X]-m+M_2*M_1\} \tag{2.39}$$

$$e^{-X}Z=Z\circ e^{-X}+e^{-X}\circ\{Z-\langle Z,X\rangle_1-\langle Z,X\rangle_2+[Z,X]-[X,X*Z] \\ -[X,Z*X]-(M_2-\langle X,X\rangle_2)*(Z-\langle X,Z\rangle_1) \\ -(Z-\langle X,Z\rangle_2)*(M_1-\langle X,X\rangle_1)\}. \tag{2.40}$$

Proof. Under the assumption A.1, each term in Eqs. (2.32)–(2.40) is a semimartingale by Proposition 2.1. If, instead, assumption A.2 is made, note that Z appears at most once in each of the terms in (2.32)–(2.40) so that each term has a semimartingale representation with integrands in $\mathscr{L}_1^2$ or $\mathscr{L}_2^2$, since each integrand is the product of a.s. bounded processes and is at most an a.s. square integrable process.

Now (2.23)–(2.26) imply (2.32) and (2.33), while (2.15), (2.16), (2.23) and (2.24) imply (2.34) and (2.35). Equation (2.36) is proved by applying the differentiation rule for one parameter semimartingales to the integrand in

$$\langle e^X, Z\rangle_1=e^X_{W1}Z_{W1}\circ\mu=\int_{R_z} e^X(s'\times z)\,u(z,s')\,Z_{W1}(z,s')\,ds'$$

as a function of z_2 for s' fixed. (2.37) follows similarly. (2.38) is obtained by applying the differentiation formula to $F(e^X, Z)=e^X Z$ and using (2.32), (2.33), (2.36) and (2.37). (2.39) is obtained by applying the differentiation formula to $F(X)=e^{-X}$ and (2.40) follows by applying the differentiation formula to $F(e^{-X},Z)=Ze^{-X}$. □

Local martingales, local i-martingales, and local weak martingales may be defined under the law P exactly as they were for law P_0. It follows that a process Z is a P local martingale (a P local i-martingale, $i=1$ or 2, a P local weak martingale) if and only if LZ is a P_0 local martingale (P_0 local i-martingale, P_0 local weak martingale).

The term semimartingale will always refer to the processes in $\mathscr{S}^2$ (or $\mathscr{S}^\omega$, if condition A.1 is assumed) which have stochastic integral representations with respect to the process $\{W_z\}$ on $(\Omega, \mathscr{F}, P_0)$. Equations (2.38) and (2.40) show that Z is a semimartingale if and only if $e^X Z$ is a semimartingale. For example, under assumption A.2, if Z is a P local martingale then Z is a semimartingale.

3. Compensation and Representation Theorems

The main results of this paper, which are summarized in this section, describe martingales, weak martingales and i-martingales under the change of measure $P_0\to P$ when subject to one of the technical assumptions described in Sect. 2. There are two types of results: one type concerns compensation (or transformation) of P_0 martingales, P_0 i-martingales and P_0 weak martingales to obtain P martingales, P i-martingales, and P weak martingales (and vice versa). The other type concerns the integral representation of P weak martingales and P martingales (i.e., the counterpart of (2.1) and (2.4) when P_0 is replaced by P).

Theorem 1 *(i-Martingale Compensation). Let Z and N be semimartingales, and let $i=1$ or 2.*

(a) *If Z is a P_0 local i-martingale, then (and only then) $Z-\langle X, Z\rangle_i$ is a P local i-martingale.*

(b) *If N is a P local i-martingale, then (and only then) $N+\langle X, N\rangle_i$ is a P_0 local i-martingale.*

If Z is a P_0 local i-martingale, then $\langle X, Z\rangle_i$ is the unique semimartingale with $(\langle X, Z\rangle_i)_{W_i}=0$ such that $Z-\langle X, Z\rangle_i$ is a P i-martingale.

Remark. Theorem 1 follows easily from the theorem on transformation of one parameter local martingales. It is also easily proved using the identities in Lemma 2.4.

It will be convenient to define some operators on the linear space of semimartingales. If Y is a semimartingale, let

$$T(Y)=\langle X, Y\rangle_1+\langle X, Y\rangle_2,$$

$$V(Y)=[X, X*Y+Y*X]-[Y, X-X*X],$$

let I denote the identity operator and define the linear operator Π by $\Pi=I+T+V$.

Theorem 2 (Martingale Compensation). (a) If a semimartingale Z is a P local martingale then $\Pi(Z)$ is the *unique* P_0 local martingale such that $Z-\Pi(Z)$ has no P_0 local martingale component (i.e., such that $[Z-\Pi(Z), Z-\Pi(Z)]=0$). (b) Conversely, if a semimartingale N is a P_0 local martingale, then there are unique proper P_0 local i-martingales n_i $i=1,2$ and a unique absolutely continuous semimartingale b such that

$$n_2+b=\langle N-n_1, X\rangle, \tag{3.1}$$

$$n_1+b=\langle N-n_2, X\rangle, \tag{3.2}$$

and $Z=N_1-n_1-n_2-b$ is the *unique* P local martingale such that $N-Z$ has no P_0 local martingale component. (c) Let N be as in (b). Then there exist unique semimartingales m_1 and m_2 such that

$$m_1=\langle N-m_2, X\rangle_2 \tag{3.3}$$

$$m_2=\langle N-m_1, X\rangle_1. \tag{3.4}$$

If b is the absolutely continuous semimartingale

$$b=[N, X-X*X]-[X, X*(N-m_1)+(N-m_2)*X] \tag{3.5}$$

then $n_i=m_i-b$, $i=1,2$, are proper P_0 local i-martingales satisfying (3.1) and (3.2). Hence Z in part (b) satisfies $Z=N-m_1-m_2+b$. (d) The linear operator Π is an invertible map of the space of semimartingales onto the space of semimartingales. For a semimartingale N, $\Pi^{-1}(N)$ has the representation

$$\Pi^{-1}(N)=(I-V)\sum_{n=0}^{\infty}(-1)^n T^n(N) \tag{3.6}$$

where the series in (3.6) converges pointwise in probability to a semimartingale. $N-\Pi^{-1}(N)$ and $Z-\Pi(Z)$ have no P_0-martingale components for any semimartingales N, Z. Π maps the space of P local martingales *onto* the space of P_0 local martingales.

Remark. Theorem 2 shows that martingales can be compensated under a change of measure in an intrinsic fashion. Unfortunately, the compensation of a P_0 local martingale to obtain a P local martingale requires the solution of Eqs. (3.1) and (3.2), or equivalently, the series representation (3.6). Given a P_0 local martingale N, when Z is described using Eqs. (3.1) and (3.2) as in (b), then the P local martingale property of Z is immediate from Theorem 1 (see proof). The equivalent system of Eqs. (3.3)–(3.5) or the series representation (3.6) for Z has the advantage of not requiring semimartingale decompositions using proper local i-martingales.

Theorem 3 *(Weak Martingale Compensation).* (a) *If a semimartingale N is a P_0 local weak martingale then*

$$N-\langle N, X\rangle_1-\langle N, X\rangle_2+[N, X+X*X] \tag{3.7}$$

and

$$\tilde{N}=N-\mu\circ\alpha u\circ\mu-\mu\circ\tilde{u}\beta\circ\mu-[N, X+X*X] \tag{3.8}$$

are P local weak martingales. The process in (3.1) *is determined from N and X by intrinsic operations. $\tilde{N}$ is the unique P local weak martingale such that $N-\tilde{N}$ is an absolutely continuous semimartingale.* (b) *If a semimartingale M is a P_0 local martingale, then $M-[M, X+X*X]$ is a representable P local weak martingale.*

Remark. It would be desirable to find an expression for $\tilde{N}$ in Theorem 3 which is intrinsically determined by N and X. The last part of the theorem shows that this is possible if N is a P_0 local martingale (rather than just a P_0 weak local martingale).

Theorem 4 *(Weak Martingale Representation). All semimartingales which are P local weak martingales may be represented as*

$$\begin{aligned} &q\circ(W-\theta\mu)+(W-\mu\tilde{u})\circ r\circ(W-u\mu)\\ &\quad-\mu\circ r\rho\circ\mu+\mu\circ a\circ(W-u\mu)+(W-\mu\tilde{u})\circ b\circ\mu. \end{aligned} \tag{3.9}$$

Remarks. (1) A consequence of Theorem 4 is that ρ, θ, u and $\tilde{u}$ have the interpretation first noted in [7] (E denotes expectation under law P):

$$\begin{aligned} \theta(s)\,ds&=E[dW_s\,|\,\mathscr{F}_s]\\ u(z,s')\,ds'&=E[dW_{s'}\,|\,\mathscr{F}_{s'\times z}]\\ \tilde{u}(z,s)\,ds&=E[dW_s\,|\,\mathscr{F}_{z\times s}]\\ \rho(s,s')\,ds\,ds'&=E[(dW_s-\tilde{u}(z,s)\,ds)\,(dW_{s'}-u(z,s')\,ds')\,|\,\mathscr{F}_{s\vee s'}]. \end{aligned}$$

(2) Since the σ-fields $\mathscr{F}_z$ are generated by a P_0 Wiener process, if $z'\curlywedge z$ then $\mathscr{F}_z$ and $\mathscr{F}_{z'}$ are conditionally independent given $\mathscr{F}_{z\times z'}$ (using probability measure P_0). However, when P_0 is replaced by P, the conditional independence is lost

unless ρ is identically zero. Indeed, this suggested by the interpretation given in Theorem 4 of ρ as a conditional correlation. As a result, the σ-fields $\{\mathscr{F}_z\}$ cannot be generated by a process which is Wiener on $(\Omega, \mathscr{F}_z, P)$.

A corollary to the representation (3.9) is that the class of representable P local weak martingales is stable under stochastic integration. Unless ρ is identically zero, the classes of P local martingales and representable P local i-martingales are not stable under the operation of stochastic integration. Hence, there does not exist a counterpart to the representation (3.9) for P local martingales.

If ρ is identically zero, then $\tilde{W}=W-\theta\circ\mu$ is a Wiener process on $(\Omega, \mathscr{F}_z, P_0)$ [7]. If $\tilde{W}$ generates the same σ-fields as W (i.e., "innovations equivalence" holds for W under P), then P local martingales may be expressed as $q\circ\tilde{W}+\tilde{W}\circ r\circ\tilde{W}$ by the Wong-Zakai representation theorem. However, innovations equivalence is not necessarily true.

Theorem 5 *(Martingale Representation when $\rho=0$). If ρ is identically zero, then any semimartingale which is a P local martingale may be represented as $q\circ\tilde{W}+\tilde{W}\circ r\circ\tilde{W}$ where $\tilde{W}=W-\theta\circ\mu$.*

4. Martingale Results

Theorems 2 and 5 will be proved in this section.

Proof of Theorem 2.a. Let Z be a semimartingale and a P local martingale. Using the definition of $\Pi(Z)$, Eq. (2.38) may be rewritten as

$$e^X Z=e^X\circ\Pi(Z)+e^X\circ\{M_2*(Z+(Z,X\rangle_1)+M_1*(Z+\langle X,Z\rangle_2)\}. \tag{4.1}$$

Now $e^X Z$ is a P_0-local martingale. In addition, by Theorem 3.1, $Z+\langle Z,X\rangle_i$ is a P_0 local i-martingale for $i=1,2$ so that

$$M_2*(Z+\langle Z,X\rangle_1) \quad \text{and} \quad (Z,+X,Z\rangle_2*M_1$$

are P_0 local martingales. So $e^X\circ(Z)$, and hence also $\Pi(Z)=e^{-X}\circ e^X\circ(\Pi(Z))$, is a P_0 local martingale since all the other terms in Eq. (4.1) are P_0 local martingales. The uniqueness assertion of Theorem 2a follows from the uniqueness of semimartingale representations.

Proof of Theorem 2b. Let N be a semimartingale and a P_0 local martingale and suppose that $Z=N-n_1-n_2-b$ is a semimartingale where n_i $i=1,2$ are proper P_0 local martingales and b is an absolutely continuous semimartingale. Then Z is a P local martingale if and only if Z is a P local i-martingale for $i=1,2$. Hence, by Theorem 1, Z is a P local martingale if and only if Eq. (3.1) and (3.2) are true. It remains only to prove the existence and uniqueness of solutions n_1, n_2 and b to (3.1) and (3.2). Such existence and uniqueness are implied by Theorem 2c. The uniqueness of the solution to (3.1) and (3.2) follows from the fact that if n_1, n_2 and b satisfy (3.1) and (3.2) then $m_1=n_1+b$ and $m_2=n_2+b$ satisfy (3.3)–(3.5) which have a unique solution by Theorem 2c. Thus, Theorem 2b will be proven once Theorem 2c is established.

Proof of Theorem 2c. If m_1 and m_2 are semimartingales satisfying (3.3) and (3.4), then $(m_1)_{W2}=(m_2)_{W1}=0$ so that m_1 and m_2 must have the semimartingale representations

$$m_1=\mu\circ f_1\circ W+b_1$$

$$m_2=W\circ f_2\circ\mu+b_2.$$

Let $N=W\circ r\circ W+q\circ W$ be the semimartingale representation for N. Using (2.34) and (2.35) and equating proper local i-martingale terms yield that (3.3) and (3.4) are equivalent to the four equations

$$\mu\circ f_1\circ W=\mu\circ\{\tilde{u}r+(N-m_2)_{W2}\,\rho\}\circ W \tag{4.2}$$

$$W\circ f_2\circ\mu=W\circ\{ru+\rho(N-m_1)_{W1}\}\circ\mu \tag{4.3}$$

$$b_1=[N,X]-\mu\circ\{(N-m_2)_{W2}\,\rho u+\tilde{u}f_2\}\circ\mu \tag{4.4}$$

$$b_2=[N,X]-\mu\circ\{\tilde{u}\rho(N-m_1)_{W1}+f_1u\}\circ\mu. \tag{4.5}$$

Now (4.2) and (4.3) are true if and only if there are versions of f_1 and f_2 so that, for $s\curlywedge s'$,

$$\begin{aligned} f_1(s,s')=&\{\tilde{u}(s,s')\,r(s,s')+\rho(s,s')\,(q(s)+\int_{R_{s'\times z}} I(s\curlywedge t')\,r(s,t')\,dW_{t'})\}\\ &-\rho(s,s')\int_{R_{s'\times z}} I(s\curlywedge t')\,f_2(s,t')\,dt' \end{aligned} \tag{4.6}$$

$$\begin{aligned} f_2(s,s')=&\{r(s,s')\,u(s,s')+\rho(s,s')\,(q(s')+\int_{R_{s'\times z}} I(t\curlywedge s')\,r(t,s')\,dW_t)\}\\ &-\rho(s,s')\int_{R_{s'\times z}} I(s'\curlywedge t)\,f_1(s',t)\,dt. \end{aligned} \tag{4.7}$$

Under either of the two possible technical assumptions of this paper, there is a unique solution (f_1,f_2) to the Eq. (4.6) and (4.7). If condition A.1 is assumed, so that $X, N, M_1, M_2\in\mathscr{S}^\omega$, then by Proposition 2.1a and Holder's inequality, the quantities in curly brackets in (4.6) and (4.7) are each in $\mathscr{L}_2^p$ for $2\leqq p<+\infty$. As in the proof of Proposition 2.2, a single iteration of equations (4.6) and (4.7) and an application of Lemma 2.3 then show that there is a unique solution (f_1,f_2) to Eq. (4.6) and (4.7) with $f_i\in\mathscr{L}_2^p$ for $2\leqq p<+\infty$.

Under the alternative assumption A.2, the quantities in curly brackets in (4.6) and (4.7) are each in $\mathscr{L}_2^2$ since each term is the product of a single term in $\mathscr{L}_2^2$ and a.s. bounded processes. A Picard iteration argument then implies the existence and uniqueness of a solution (f_1,f_2) to (4.6) and (4.7) such that $f_i\in\mathscr{L}_2^2$ for $i=1,2$.

In summary, under either of the assumptions A.1 or A.2 there is a unique solution (f_1,f_2) to (4.6) and (4.7), and $n_1=\mu\circ f_1\circ W$ and $n_2=W\circ f_2\circ\mu$ are the unique semimartingales satisfying (4.2) and (4.3). Finally, substitution of (4.7) and (4.6) into (4.4) and (4.5) respectively yields that $b_1\equiv b_2\equiv b$ a.s. where b is given by Eq. (3.5).

Proof of Theorem 2d. It is easy to check that $TV=V^2=0$ so that $\Pi=I+T+V=(I+T)(I+V)$ and $(I+V)$ is invertible with $(I+V)^{-1}=I-V$. The representation (3.6) can hence be established by proving that for any semimartingale Y, the series

$$\sum_{n=0}^{\infty}(-1)^n T^n(Y)$$

converges pointwise in probability to a semimartingale $R(Y)$, and proving that the resulting linear operator R satisfies $(I+T)R=R(I+T)=I$ (i.e. $R=(I+T)^{-1}$).

So suppose that Y is a semimartingale. By Lemma 2.4 and induction on n, $T^n Y$ is a semimartingale for all $n\geqq 0$. Let

$$T^n(Y)=q^{(n)}\circ W+W\circ r^{(n)}\circ W+\mu\circ f_1^{(n)}\circ W+W\circ f_2^{(n)}\circ\mu+b^{(n)}$$

be the semimartingale representation. By (2.34) and (2.35),

$$\begin{aligned}T(Y)=&\,W\circ(\rho\, Y_{W1}+r^{(0)}\tilde{u})\circ\mu+\mu\circ(Y_{W2}\,\rho+u r^{(0)})\circ W+2[X,Y]\\&+\mu\circ(f_2^{(0)}u-\tilde{u}\,\rho\, Y_{W1}+\tilde{u} f_1^{(0)}-Y_{W2}\,\rho u)\circ\mu.\end{aligned}$$

Hence $q^{(1)}=r^{(1)}=0$, and moreover, $q^{(n)}=r^{(n)}=0$ for all $n\geqq 1$. Also

$$f_1^{(1)}=Y_{W2}\,\rho+u\, r^{(0)},\qquad f_2^{(1)}=\rho\, Y_{W1}+r^{(0)}\tilde{u}.$$

Under assumption A.1 $f_i^{(i)}(1)\in\mathscr{L}_2^p$ for all $p<+\infty$, and under assumption A.2 $f_i^{(1)}(1)\in\mathscr{L}_2^{\infty}$, for $i=1,2$ by Propositions 2.1 and 2.2. Applying (2.34) and (2.35) again to compute $T^n(Y)=T(T^{n-1}(Y))$ and using the fact that $r^{(n)}=q^{(n)}=0$ for $n\geqq 1$ yield that, for $n\geqq 2$,

$$\begin{aligned}f_1^{(n)}(s,s')&=\rho(s,s')\int_{R_{s'}\times s} I(s\curlywedge r')\, f_2^{(n-1)}(s,r')\,dr'\\ f_2^{(n)}(s,s')&=\rho(s,s')\int_{R_{s'}\times s} I(s'\curlywedge r)\, f_1^{(n-1)}(s',r)\,dr\\ b_z^{(n)}&=\int_{R_z\otimes R_z} ds\,ds'\,\{(f_2^{(n-1)}(s,s')-f_1^{(n)}(s,s'))\,u(s'\times s')\\&\qquad+(f_1^{(n-1)}(s,s')-f_2^{(n)}(s,s'))\,\tilde{u}(s'\times s,s)\}.\end{aligned}\tag{4.8}$$

Under assumption A.1 the Picard iteration argument of Lemma 2.3 shows that the sum $\sum_{n=0}^{\infty} f_i^{(n)}$ converges a.s. in $L^p(R_z\otimes R_z)$ to a function f_i where $f_i\in\mathscr{L}_2^p$ for all $p<+\infty$, for $i=1,2$. Under the alternative assumption A.2, the fact that $f_i^{(1)}\in\mathscr{L}_2^2$ and $\rho\in\mathscr{L}_2^{\infty}$ implies by induction that $f_i^{(n)}\in\mathscr{L}_2^{\infty}$ for all $n\geqq 3$. Picard iteration then yields that $\sum_{n=0}^{\infty} f_i^{(n)}$ converges uniformly a.s. to a function $f_i\in\mathscr{L}_2^{\infty}$ for $i=1,2$.

Under either technical assumption, the stopping time provided in [6] and Lemma 7 of [5] then imply that the proper P_0 local i-martingale term of $\sum_{n=1}^{\infty} T^n(Y)$ converges pointwise in probability to the proper P_0 local i-martingale $\mu\circ f_1\circ W$ (if $i=1$) or $W\circ f_2\circ\mu$ (if $i=2$). The convergence of $\sum_n f_i^{(n)}$ and (4.5) imply the uniform convergence of $\sum_{n=0}^{\infty} b^{(n)}$ a.s. to an absolutely continuous process b. Thus the series (4.1) does indeed converge to a local semimartingale $R(Y)$.

Finally, note that

$$\begin{aligned} R(I+T)(Y) &= \left(\lim_{k\to\infty} \sum_{n=0}^{k} (-1)^n T^n\right)(I+T)(Y) \\ &= Y + \lim_{k\to\infty} T^{(k+1)}(Y) = Y \end{aligned}$$

where the limit is pointwise in probability. Similarly, $(I+T)R = I$. The proof of the representation (3.6) is complete.

The fact that $N - \Pi^{-1}(N)$ and $Z - \Pi(Z)$ have no P_0-martingale component for any semimartingales N and Z is immediate from the definition of Π and (3.6).

Now Π maps the space of P local martingales into the space of P_0 local martingales by part (a). To prove that this mapping is onto, let N be a P_0 local martingale, and let Z be the unique P local martingale such that $N - Z$ has no P_0 local martingale component. Z exists by part (b). Then N and $\Pi(Z)$ are each P_0 local martingales and $N - \Pi(Z) = N - Z + (Z - \Pi(Z))$ has no P_0 local martingale component. Hence $N = \Pi(Z)$.

Proof of Theorem 5. If ρ is identically zero then $X = \theta \circ W - \frac{1}{2}\theta^2 \circ \mu$ by (2.19). It follows that $T^n = 0$ for $n \geqq 3$ so the series defining R is finite. The result is (using (2.24) and (2.35),

$$\begin{aligned} &R(N)(I+T)^{-1} N \\ &\quad = N - \langle N, X\rangle_1 - \langle N, X\rangle_2 + \langle X, \langle N, X\rangle_1\rangle_2 + \langle X, \langle N, X\rangle_2\rangle_1 \\ &\quad = N - \langle N, X\rangle_1 - \langle N, X\rangle_2 + 2[N, X * X] \end{aligned}$$

and

$$\begin{aligned} VR(N) &= [X, R(N) * X + X * R(N)] - [R(N), X - X * X] \\ &= 0 - [N, X - X * X]. \end{aligned}$$

So for any semimartingale which is a P_0 local martingale $N = W \circ r \circ W + q \circ W$,

$$\begin{aligned} \Pi^{-1}(N) &= N - \langle N, X\rangle_1 - \langle N, X\rangle_2 + [N, X + X * X] \\ &= \tilde{W} \circ r \circ \tilde{W} + q \circ \tilde{W} \end{aligned} \tag{4.9}$$

where $\tilde{W} = W - \theta \circ \mu$. Since Π^{-1} maps onto the set of all semimartingale which are P local martingales, any such process has the representation (4.9). □

5. Weak Martingale Results

Theorems 3 and 4 will be proved in this section.

Proof of Theorem 3b. Suppose that M is a semimartingale and a P_0 local martingale. Then trivially

$$e^X(M - [M, X + X * X]) = (e^X M - [e^X, M]) - (e^X[M, X + X * X] - [e^X, M]) \tag{5.1}$$

Since M and e^X are both P_0 local martingales, $e^X M-[e^X, M]$ is a P_0 local weak martingale. (This may be proved by using integral representations, but is an "intrinsic" fact [1].) By (2.22),

$$[e^X, M]=e^X\circ[M, m+M_2*M_1]=e^X\circ[M, X+X*X]. \tag{5.2}$$

Using the differential formula (2.18) and substituting in (5.2) yield

$$\begin{aligned}e^X[M, X+X*X]-[e^X, M]=[M, X+X*X]\circ e^X+e^X*[M, X+X*X]\\ +[M, X+X*X]*e^X\end{aligned}$$

which is a P_0 local weak martingale. In view of (5.1), $e^X(M-[M, X+X*X])$ is a P_0 local weak martingale so that $M-[M, X+X*X]$ is a P local weak martingale.

Proof of Theorem 3a. Suppose that N is a semimartingale and a P_0 local weak martingale. Then N may be expressed as

$$N=M+\mu\circ\alpha\circ W+W\circ\beta\circ\mu=M+n_1+n_2 \tag{5.3}$$

where M is a P_0 local martingale, and $n_1=\mu\circ\alpha\circ W$ ($n_2=W\circ\beta\circ\mu$) is a proper P_0 local 1-martingale (proper P_0 local 2-martingale). By Theorem 1 and Theorem 3.6 the following semimartingales are all P_0 local weak martingales:

$$\begin{aligned}&n_1-\langle n_1, X\rangle_1=n_1-\langle n_1, X\rangle_1-\langle n_1, X\rangle_2+[n_1, X+X*X]\\ &n_2-\langle n_2, X\rangle_2=n_2-\langle n_2, X\rangle_1-\langle n_2, X\rangle_2+[n_2, X+X*X]\\ &\{M-\langle M, X\rangle_1\}+\{M-\langle M, X\rangle_2\}-\{M-[M, X+X*X]\}\\ &\quad=M-\langle M, X\rangle_1-\langle M, X\rangle_2+[M, X+X*X].\end{aligned}$$

The sum of these processes is

$$N-\langle N, X\rangle_1-\langle N, X\rangle_2+[N, X+X*X] \tag{5.4}$$

which is thus also a P_0 local weak martingale as advertised.

The proofs that $M-[M, X+X*X]$ and (5.4) are P local weak martingales have been intrinsic (essentially representation independent) and are thus likely to remain valid in a more general setting.

Since $[n_i, X+X*X]=0$ it follows that

$$\begin{aligned}&N-[N, X+X*X]-\mu\circ\alpha u\circ\mu-\mu\circ\tilde{u}\beta\circ\mu\\ &\quad=(M-[M, X+X*X])+(n_1-\mu\circ\alpha u\circ\mu)+(n_2-\mu\circ\tilde{u}\beta\circ\mu).\end{aligned} \tag{5.5}$$

The first term on the right is a P local weak martingale by Theorem 3b. It will be shown that the other two terms are also. Let $B_1=\mu\circ\alpha u\circ\mu$ and apply (2.33) to get

$$e^X n_1=n_1\circ e^X+e^X\circ\{n_1+M_2*(n_1+\langle X, n_1\rangle_1)+\langle X, n_1\rangle_1+[X*n_1, X]\}$$

and

$$e^X B_1 = B_1 \circ e^X + e^X \circ \{B_1 + M_2 * B_1 + B_1 * M_1\}.$$

Hence

$$e^X (n_1 - B_1) = K + e^X \circ \{\langle X, n_1\rangle_1 + [X * n_1, X] - B_1\}$$

where K is a P_0 local weak martingale. Applying (2, 3, 4) shows that

$$\langle X, n_1\rangle_1 + [X * n_1, X] - B_1 = W \circ \rho(n_1)_{W1} \circ \mu$$

is a P_0 local weak martingale, so that $e^X(n_1 - B_1)$ is one also. Therefore $n_1 - \mu \circ \alpha u \circ \mu$ (and similarly $n_2 - \mu \circ \tilde{u}\beta \circ \mu$) is a P local weak martingale. Therefore, each side of (5.5) is a P local weak martingale.

Finally, to prove the uniqueness assertion in Theorem 3a, it suffices to show that if B is an absolutely continuous semimartingale and also a P local weak martingale, then B is identically zero a.s. Since B is absolutely continuous, (2.38) yields

$$e^X B = B \circ e^X + e^X \circ \{B + M_2 * B + B * M_1\}$$

so that

$$B = e^{-X} \circ \{e^X B - B \circ e^X\} + M_2 * B + B * M_1$$

which shows that B is also a P_0 local weak martingale. Hence B is identically zero a.s. The proof of Theorem 3 is complete. □

Proof of Theorem 4. Given q, r, a and b as in (3.9), let $\alpha = a - \tilde{u}r$, $\beta = b - ru$ and $N = q \circ W + W \circ r \circ W + \mu \circ \alpha \circ W + W \circ \beta \circ \mu$. Then $\tilde{N}$ in (3.8) is equal to (3.9) which is thus a representable P local weak martingale.

Conversely, any representable P local weak martingale Z may be written $Z = N - B$ where N is a representable P_0 local weak martingale and B is a bounded variation process. Let $N = W \circ r \circ W + q \circ W + \mu \circ \alpha \circ W + W \circ \beta \circ \mu$ be the semimartingale representation of N. By the uniqueness assertion of Theorem 3.b, Z must equal $\tilde{N}$ of (3.8), which is equal to (3.9) with $a = \alpha + \tilde{u}r$ and $b = \beta + ru$. □

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