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THE SAMPLE FUNCTION CONTINUITY OF STOCHASTIC INTEGRALS IN THE PLANE[1]

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Sample continuity is proved for extended stochastic integrals involving a two-parameter Wiener process.

1. Introduction. Let R_+^2 be the positive quadrant of the plane. The partial orderings $\prec$ and $\lll$ denote: $(a, b) \prec (c, d)$ if $a \leqq c$ and $b \leqq d$, $(a, b) \lll (c, d)$ if $a < c$ and $b < d$. For $z \in R_+^2$, R_z denotes the rectangle $\{\zeta : (0, 0) \prec \zeta \prec z\}$. Let $(\Omega, \mathscr{F}, \mathscr{P})$ be a complete probability space and $\mathscr{F}_\zeta$, $\zeta \in R_+^2$ be sub-σ-fields of $\mathscr{F}$. We assume that

$$\mathscr{F}_z \subset \mathscr{F}_{z'} \quad \text{whenever} \quad z \prec z' ,$$

$$\mathscr{F}_z \quad \text{is right continuous} \quad (\mathscr{F}_z = \textstyle\bigcap_{z \lll z'} \mathscr{F}_{z'}) ,$$

$$\mathscr{F}_{(z_1,\infty)} \quad \text{and} \quad \mathscr{F}_{(\infty,z_2)} \quad \text{are conditionally independent given} \quad \mathscr{F}_z .$$

Let $(W_z, \mathscr{F}_z)$ be a separable two-parameter Wiener process on $(\Omega, \mathscr{F}, \mathscr{P})$. Stochastic integrals of the type $\int \phi\, dW$, $\int\int \psi\, dW\, dW$, $\int\int \psi\, dW\, d\zeta$ have been defined under the conditions $E \int \phi^2\, d\zeta < \infty$, $E \int\int \psi^2\, d\zeta\, d\zeta' < \infty$ and the existence of sample continuous versions of these stochastic integrals was demonstrated ([6], [7], cf. also [2]).

In an attempt to extend the definitions of the stochastic integrals by replacing $E(\cdot) < \infty$ with $(\cdot) < \infty$ a.s., it turned out that the standard one-parameter stopping argument does not go over to the multiparameter case. An extension was presented in [8], but it did not yield the sample function continuity of the extended stochastic integral. The purpose of this note is to prove the sample function continuity of the extended stochastic integral.

The stochastic integrals under $E(\cdot) < \infty$ were also defined for the case where W_z is replaced by certain general continuous (or right-continuous) square integrable two-parameter martingales ([2], [7]), and so was the extension of [8]. This note deals only with integration with respect to the Wiener process. A remark concerning the possibility of extending the approach to the case of integration with respect to general martingales is deferred to the end of the note.

2. Preliminaries. Let ϕ_ζ, $\zeta \in R_+^2$ be $\mathscr{F}_\zeta$ adapted, measurable and satisfy

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$\int_{R_+^2} \phi_\zeta^2 \, d\zeta < \infty$ a.s. Let $y^n(\omega)$ denote the rv

$$y^n(\omega) = 1 \quad \text{if} \quad \int_{R_+^2} \phi_\zeta^2 \, d\zeta \leqq n ,$$
$$= 0 \quad \text{otherwise.}$$

Let Y_ζ^n denote the two-parameter martingale

$$Y_\zeta^n = E(y^n \mid \mathscr{F}_\zeta) .$$

Since $\mathscr{F}_\zeta$ is right continuous, it follows ([5], the concluding remark to the proof of Theorem 21) that Y_ζ^n is right continuous in probability for every $\zeta \in R_+^2$; therefore, it follows by the same arguments as those of page 61 of [4] that there exists a separable and measurable version of $E(y^n \mid \mathscr{F}_\zeta)$; we will denote this version by Y_ζ^n. Let λ be a constant satisfying $0 < \lambda < 1$, and let

$$S_\zeta^n = 1 \quad \text{if} \quad Y_\zeta^n \geqq \lambda , \tag{1}$$
$$= 0 \quad \text{otherwise.}$$

LEMMA 1.

$$E \int_{R_+^2} S_\zeta^n \phi_\zeta^2 \, d\zeta \leqq \frac{n}{\lambda} \tag{2}$$

and

$$\lim_{n\to\infty} \mathrm{P}\{\inf_{\zeta \in R_+^2} S_\zeta^n = 1\} = 1 . \tag{3}$$

PROOF. Obviously,

$$E \int_{R_+^2} y^n \phi_\zeta^2 \, d\zeta \leqq n .$$

Let

$$\pi_\zeta^k = \phi_\zeta^2 \quad \text{if} \quad \phi_\zeta^2 \leqq k ,$$
$$= k \quad \text{otherwise.}$$

Therefore, by the monotone convergence and Fubini's theorems

$$E \int y^n \phi_\zeta^2 \, d\zeta = \lim_{k\to\infty} \int E(y^n \pi_\zeta^k) \, d\zeta \tag{4}$$
$$= \lim_{k\to\infty} E \int Y_\zeta^n \pi_\zeta^k \, d\zeta$$
$$= E \int Y_\zeta^n \phi_\zeta^2 \, d\zeta . \tag{5}$$

Therefore $E \int_{R_+^2} Y_\zeta^n \phi_\zeta^2 \, d\zeta \leqq n$, and (2) follows since $S_\zeta^n \leqq Y_\zeta^n/\lambda$. We turn now to the proof of (3). Note that $\mathrm{P}\{\inf S_\zeta^n = 1\} = \mathrm{P}\{\inf Y_\zeta^n \geqq \lambda\}$. Let $p^n = \mathrm{P}\{y^n = 1\}$, then $p^n \to 1$ as $n \to \infty$ and also $EY_\zeta^n = p^n$. By the maximal inequality for two-parameter martingales [1], the separability of Y_ζ^n and the fact that every countable dense set is a separating set

$$E\{\sup_{\zeta \in R_+^2} |Y_\zeta^n - p^n|^2\} \leqq c \sup_{\zeta \in R_+^2} E(Y_\zeta^n - p^n)^2$$
$$\leqq c(E(y^n)^2 - (p^n)^2)$$
$$= cp^n(1 - p^n) .$$

Therefore, for n large enough so that $p^n > \lambda$

$$\begin{aligned}
\mathrm{P}\{\inf Y_\zeta^n \geqq \lambda\} &= \mathrm{P}\{\inf (Y_\zeta^n - p^n) \geqq \lambda - p^n\} \\
&\geqq \mathrm{P}\{\sup |Y_\zeta^n - p^n| \leqq |p^n - \lambda|\} \\
&= 1 - \mathrm{P}\{\sup |Y_\zeta^n - p^n| > |p^n - \lambda|\} \\
&\geqq 1 - \frac{cp^n(1 - p^n)}{(p^n - \lambda)^2},
\end{aligned}$$

which proves (2) since $p^n \to 1$ as $n \to \infty$.

Write $(a, b) \curlywedge (c, d)$ if $a \leqq c$ and $b \geqq d$, and let $(a, b) \curlyvee (c, d)$ denote $(\max(a, c), \max(b, d))$. Let $\psi(\zeta, \zeta')$, $\zeta, \zeta' \in R_+^2$ be measurable, $\mathscr{F}_{\zeta \curlyvee \zeta'}$ adapted, such that $\psi(\zeta, \zeta') = 0$ if $\zeta \curlywedge \zeta'$ is not satisfied and

$$\int\int_{R_+^2 \times R_+^2} \psi^2(\zeta, \zeta')\, d\zeta\, d\zeta' < \infty \quad \text{a.s.}$$

Let y^n denote the rv

$$\begin{aligned}
y^n &= 1 \quad \text{if} \quad \int\int_{R_+^2 \times R_+^2} \psi^2(\zeta, \zeta')\, d\zeta\, d\zeta' \leqq n\,, \\
&= 0 \quad \text{otherwise,}
\end{aligned}$$

and let Y_ζ^n denote the separable and measurable version of the two-parameter martingale $E(y^n \mid \mathscr{F}_\zeta)$. S_ζ^n is as defined by (1).

LEMMA 2.

$$E \int\int_{R_+^2 \times R_+^2} S^n_{\zeta \vee \zeta'} \psi^2(\zeta, \zeta')\, d\zeta\, d\zeta' \leqq \frac{n}{\lambda} \tag{6}$$

and

$$\lim_{n\to\infty} \mathrm{P}\{\inf_{\zeta \in R_+^2} S_\zeta^n = 1\} = 1\,. \tag{7}$$

The proof is essentially the same as that of Lemma 1 and is therefore omitted.

REMARK. S_ζ^n as defined by (1) is not a stopping time (cf. [7]). Let

$$\begin{aligned}
T_z^n &= 1 \quad \text{if} \quad \inf_{\zeta <_z} S_\zeta^n = 1\,, \\
&= 0 \quad \text{if not,}
\end{aligned}$$

then T_z^n is a two-parameter stopping time and Lemmas 1 and 2 remain true with S_ζ^n replaced by T_ζ^n.

3. The sample function continuity of stochastic integrals. Let ϕ_ζ be as in the previous section, and $n'' > n' \geqq n$. Since $n' > n$ implies $\inf_{\zeta \in R_+^2}(S_\zeta^{n'} - S_\zeta^n) \geqq 0$ a.s., it follows that

$$\mathrm{P}\{\sup_{z \in R_+^2} |\int_{R_z} \phi_\zeta S_\zeta^{n'}\, dW_\zeta - \int_{R_z} \phi_\zeta S_\zeta^{n''}\, dW_\zeta| > \varepsilon\} \leqq \mathrm{P}\{\inf_{\zeta \in R_+^2} S_\zeta^n \neq 1\}$$

and therefore by (2) of Lemma 1, $\int_{R_z} \phi_\zeta S_\zeta^n\, dW_\zeta$ converges in probability uniformly to a continuous random function. Define this limit as the stochastic integral. In view of Proposition 4 of [8] the stochastic integral as defined here is for each $z \in R_+^2$ a.s. equal to the one defined in [8]. Similar arguments hold for $\int\int \psi\, dW\, dW'$ and $\int\int \psi\, dW\, d\zeta'$.

REMARK. In order to apply the argument of this note to prove the continuity (right-continuity) of $\int \phi\, dM$, $\int\int \phi\, dM\, dM$ when M is a continuous (right-continuous) martingale ([2]), we have to justify the passage from (4) to (5) with $d\zeta$ replaced by $d[M]_\zeta$. In the one-parameter case this follows directly by considering the dual predictable projection of $\int y^n \phi^2\, d[M]$ (Theorems V.31, V.17 and V.15 of [5]). Some partial results in this direction follow directly (for example, if $d[M]_\zeta = m_\zeta\, d\zeta$, and ϕ_ζ and m_ζ are $\mathscr{F}_{\zeta-}$-adapted), and it seems that in the general case this extension will become straightforward after the general theory of processes is extended to n-parameter processes.

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