# Likelihood Ratios and Transformation of Probability Associated with Two-Parameter Wiener Processes

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## 1. Introduction

Let $X_t$, $0 \leq t \leq 1$, be a standard Wiener process defined on a probability space $(\Omega, \{\mathfrak{F}_t\}, \mathscr{P}_0)$. Let $\mathscr{P}$ be a probability measure on $(\Omega, \mathfrak{F}_1)$ equivalent to $\mathscr{P}_0$. $E$ and $E_0$ will denote expectation relative to $\mathscr{P}$ and $\mathscr{P}_0$ respectively. Let $\mathfrak{F}_{xt}$ denote $\sigma(X_s, 0 \leq s \leq t)$. The following set of results is by now well known [see e.g., 3]:

(a) If $\dfrac{d\mathscr{P}}{d\mathscr{P}_0} = \exp\left\{\int_0^1 \phi_s\, dX_s - \tfrac{1}{2}\int_0^1 \phi_s^2\, ds\right\}$ where $\phi$ is an $\{\mathfrak{F}_t\}$ adapted process,

then $W_t = X_t - \int_0^t \phi_s\, ds$ is a standard Wiener process with respect to $\{\Omega, \{\mathfrak{F}_t\}, \mathscr{P}\}$.

(b) Under some additional conditions such as $\int_0^1 E\phi_s^2\, ds < \infty$, the likelihood ratio is expressible as

$$L_t = E_0\left(\frac{d\mathscr{P}}{d\mathscr{P}_0}\,\bigg|\,\mathfrak{F}_{xt}\right) = \exp\left\{\int_0^t \hat{\phi}_s\, dX_s - \tfrac{1}{2}\int_0^t \hat{\phi}_s^2\, ds\right\}$$

where $\hat{\phi}_t = E(\phi_t | \mathfrak{F}_{xt})$.

(c) Even without the hypotheses of (a) and (b), the likelihood ratio (viz., the projection of $\dfrac{d\mathscr{P}}{d\mathscr{P}_0}$ on the $\sigma$-field generated by $X_s$, $0 \leq s \leq t$) is of the form

$$L_t = \exp\left\{\int_0^t v_s\, dX_s - \tfrac{1}{2}\int_0^t v_s^2\, ds\right\}$$

where $v$ is an $\{\mathfrak{F}_{xt}\}$ adapted process and $V_t = X_t - \int_0^t v_s\, ds$ is a standard Wiener process with respect to $(\Omega, \{\mathfrak{F}_{xt}\}, \mathscr{P})$.

The purpose of this paper is to consider these and related problems for Wiener process with a two-dimensional parameter. An attempt in this direction was begun in [4] but the effort was only partly successful. It revealed the far more complex structure of the stochastic calculus in the two-parameter case, and a full elucidation of the form of the Radon-Nikodym derivative and likelihood ratio had to await the development of the calculus as presented in [6] and in Section 2 of this paper.

Let $R_+^2$ denote the positive quadrant of the plane. For two points $a=(a_1,a_2)$ and $b=(b_1,b_2)$ we denote

$a \prec b$    if $a_1 \leqq b_1$   and   $a_2 \leqq b_2$,

$a \ll b$    if $a_1 < b_1$   and   $a_2 < b_2$,

$a \wedge b$    if $a_1 \leqq b_1$   and   $a_2 \geqq b_2$,

$a \not\wedge b$    if $a_1 < b_1$   and   $a_2 > b_2$.

Furthermore, we shall adopt the notations

$a \otimes b = (a_1, b_2)$,

$a \wedge b = (\min(a_1, b_1), \min(a_2, b_2))$,

$a \vee b = (\max(a_1, b_1), \max(a_2, b_2))$.

Observe that if $a \wedge b$ then $a \otimes b = a \wedge b$ and $b \otimes a = a \vee b$. Note also that $a \otimes b \otimes c = a \otimes c$. Finally, for a fixed point $z_0$ in $R_+^2$, $R_{z_0}$ will denote the rectangle $\{z : z \prec z_0, z \in R_+^2\}$.

Let $(\Omega, \mathfrak{F}, \mathscr{P})$ be a probability space and let $\{\mathfrak{F}_z, z \in R_{z_0}\}$ be a family of $\sigma$-subfields such that

$F_1$)  $z' \succ z \Rightarrow \mathfrak{F}_{z'} \supset \mathfrak{F}_z$,

$F_2$)  $\mathfrak{F}_0$ contains all null sets of $\mathfrak{F}$ where 0 denotes the origin,

$F_3$)  $\mathfrak{F}_z = \bigcap\limits_{z' \succ z} \mathfrak{F}_{z'}$ for every $z$,

$F_4$)  $\mathfrak{F}_{z \otimes z_0}$ and $\mathfrak{F}_{z_0 \otimes z}$ are independent given $\mathfrak{F}_z$.

For each $z$, $\mathfrak{F}_z^1$ will denote $\mathfrak{F}_{z \otimes z_0}$ and $\mathfrak{F}_z^2$ will denote $\mathfrak{F}_{z_0 \otimes z}$.

Let $\{X_z, z \in R_{z_0}\}$ be a stochastic process defined on $(\Omega, \mathfrak{F}, \mathscr{P})$ and adapted to $\{\mathfrak{F}_z\}$ (i.e., for each $z$, $X_z$ is $\mathfrak{F}_z$-measurable). For $b \gg a$ let $(a, b]$ denote the rectangle $\{z : a \ll z \prec b\}$ and $X(a, b]$ the increment $X_b - X_{a \otimes b} - X_{b \otimes a} + X_a$.

*Definition.* $\{X_z, \mathfrak{F}_z, z \in R_{z_0}\}$ is said to be:

  $M_1$)  a *martingale* if $E\{X_{z'} | \mathfrak{F}_z\} = X_z$ almost surely,

  $M_2$)  a *weak martingale* if $E\{X(z, z'] | \mathfrak{F}_z\} = 0$,

  $M_3$)  a *strong martingale* if $E\{X(z, z'] | \mathfrak{F}_z^1 \vee \mathfrak{F}_z^2\} = 0$,

  $M_4$)  an *adapted i-martingale* if $E\{X(z, z'] | \mathfrak{F}_z^i\} = 0$, $i = 1, 2$,

  $M_5$)  a *Wiener process* if $\{X_z, \mathfrak{F}_z, z \in R_{z_0}\}$ is a strong martingale, and $X$ is a Gaussian process with $EX_z = 0$ and

$$EX(A) X(B) = \text{Area}(A \cap B) \quad \text{for all rectangles } A \text{ and } B.$$

We note that if $X$ satisfies condition $M_4$ it is said to be an $i$-martingale whether or not it is $\{\mathfrak{F}_z\}$ adapted. In (1)–(5) the conditions are to hold for all $z$ and all $z' > z$. With these definitions, we can easily verify that a process is a martingale if and only if it is both an adapted 1-martingale and an adapted 2-martingale. A strong martingale is also a martingale, and an adapted one or two martingale is also a weak martingale. We owe most of these definitions to [1].

Let $(\Omega, \{\mathfrak{F}_z\})$ be a measurable space on which two probability measures $\mathscr{P}$ and $\mathscr{P}_0$ are defined. Let $\{X_z, \mathfrak{F}_z, z \in R_{z_0}\}$ be a Wiener process under $\mathscr{P}_0$ and let $\mathfrak{F}_{xz}$ denote the $\sigma$-field generated by $\{X_\zeta, \zeta < z\}$. We shall attempt to answer the following questions:

(a) Suppose that $\mathscr{P}$ and $\mathscr{P}_0$ are equivalent and

$$\frac{d\mathscr{P}}{d\mathscr{P}_0} = \exp\{ \int_{R_{z_0}} \phi_\zeta \, dX_\zeta - \tfrac{1}{2} \int_{R_{z_0}} \phi_\zeta^2 \, d\zeta \},$$

how does $X$ behave under $\mathscr{P}$?

(b) With whatever additional assumptions which might be necessary, is it possible to obtain an explicit expression for the likelihood ratio

$$L_z = E_0 \left\{ \frac{d\mathscr{P}}{d\mathscr{P}_0} \middle| \mathfrak{F}_{xz} \right\}?$$

(c) If we do not assume that $\mathscr{P}$ and $\mathscr{P}_0$ are equivalent, but only that their restrictions on $\mathfrak{F}_{xz}$ are equivalent, can the general form of the Radon-Nikodym derivative on $\mathfrak{F}_{xz}$ be found?

We believe that these questions are answered with reasonable completeness by the results of this paper. We are satisfied that the form of these results is quite general, even if the conditions under which they are proved may not be the best possible. The order of our presentation will be as follows: The stochastic calculus required for the paper will be summarized in Section 2. In Section 3 we shall obtain a series of formulas which provide an answer to (c), and in Section 4 a generalization to the exponential formula for Wiener processes. In Section 5 we shall give an interpretation for these formulas in terms of some conditional moments of the process $X$ under the $\mathscr{P}$-measure. Finally, in Section 6 an application of these results to the following hypothesis testing problem which arises in signal detection will be considered:

$H_0$: The observation $\{\xi_z, z \in R_{z_0}\}$ is a white Gaussian noise.

$H$: The observation is of the form $\xi_z = \theta_z + \eta_z$ where $\eta$ is a white Gaussian noise and $\theta$ is a random signal.

It will be shown that in this case the likelihood ratio is expressible in terms of

$$\hat\theta_z = E(\theta_z | \mathfrak{F}_{xz}) \quad \text{and} \quad \rho(z, z') = \operatorname{cov}(\theta_z \, \theta_{z'} | \mathfrak{F}_{xz \vee z'}).$$

## 2. Stochastic Calculus for 2-Parameter Wiener Processes

As in Section 1, define a Wiener process $\{W_z, \mathfrak{F}_z, z \in R_{z_0}\}$ as a strong martingale such that $W$ is also a Gaussian process with $EW_z = 0$ and

$$EW_z W_{z'} = \text{Area}(R_{z \vee z'}). \tag{2.1}$$

Provided that a separable version is chosen, a Wiener process is sample continuous, and for rectangles $A$ and $B$

$$EW(A) W(B) = \text{Area}(A \cap B). \tag{2.2}$$

Let $\{\phi_z, z \in R_{z_0}\}$ be a process satisfying the following conditions:

(a) $\phi$ is a bimeasurable function of $(\omega, z)$ and $\qquad\qquad\qquad\qquad$ (2.3)

(b) $\int\limits_{R_{z_0}} E\phi_z^2 \, dz < \infty$ or

(b') $\mathscr{P}(\{\omega : \sup\limits_z |\phi(\omega, z)| < \infty\}) = 1$

and for each $z$

either $(c_0)$ $\phi_z$ is $\mathfrak{F}_z$-measurable
or $\quad(c_i)$ $\phi_z$ is $\mathfrak{F}_z^i$-measurable, $i = 1, 2$.

We shall denote by $\mathscr{H}_i (i = 0, 1, 2)$ the space of functions $\phi$ satisfying conditions (a), (b), and $(c_i)$, and by $\mathscr{H}_i'$ if $(c_i)$ is replaced by $(c_i')$.

For $\phi \in \mathscr{H}_i$ the integral $\int\limits_{R_{z_0}} \phi_z \, dW_z$ is well-defined, and if we set

$$(\phi \circ W)_z = \int\limits_{R_z} \phi_\zeta \, dW_\zeta$$
$$= \int\limits_{R_{z_0}} I(s \prec z) \phi_\zeta \, dW_\zeta \tag{2.4}$$

then the process $\phi \circ W$ is a strong martingale for $\phi \in \mathscr{H}_0$ and an adapted $i$-martingale $(i = 1, 2)$ for $\phi \in \mathscr{H}_i$. Furthermore, if we define

$$M_z = (\phi \circ W)_z (\psi \circ W)_z - \int\limits_{R_z} \phi_\zeta \psi_\zeta \, d\zeta. \tag{2.5}$$
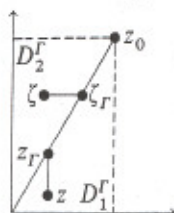
Then $M$ is a martingale if $\phi, \psi \in \mathscr{H}_0$, an adapted $i$-martingale $(i = 1, 2)$, if $\phi, \psi \in \mathscr{H}_i$ [1].

If $\phi \in \mathscr{H}_i'$ then there exists a sequence $\{\phi_n\}$ in $\mathscr{H}_i$ and that $\phi_n \to \phi$ almost surely and $\phi_n \circ W$ converges uniformly with probability 1. Hence, for $\phi \in \mathscr{H}_i'$ $\phi \circ W$ can be defined as the uniform limit of a sequence of continuous strong martingales (resp. $i$-martingales). Convergence being uniform, $\phi \circ W$ is sample continuous. We shall call $(\phi \circ W)$ under these conditions a local martingale (or local $i$-martingale).

The integral $\phi \circ W$ can be generalized still further. Let $\Gamma$ be an increasing path connecting the origin to $z_0$. For each $z \in R_{z_0}$ let $z_\Gamma$ denote the smallest point on $\Gamma$ greater than $z$ (with respect to the ordering $\succ$). The path $\Gamma$ divides $R_{z_0}$ into two parts, say $D_i^\Gamma$, $i = 1, 2$, where $D_1^\Gamma$ is the area below $\Gamma$ and $D_2^\Gamma$ is the area to the left of $\Gamma$, i.e.,

$$D_1^\Gamma = \{\zeta \in R_{z_0} : \zeta \otimes \zeta_\Gamma = \zeta_\Gamma\},$$

$$D_2^\Gamma = \{\zeta \in R_{z_0} : \zeta_\Gamma \otimes \zeta = \zeta_\Gamma\},$$



We shall say a process $X$ is $\Gamma$-*adapted* if:

for each $z \in R_{z_0}$, $X_z$ is $\mathscr{F}_{z\Gamma}$-measurable and a $\Gamma$-*martingale* if $\qquad$ (2.3 $c_\Gamma$)

$(M_6)$   $X$ is $\Gamma$-adapted and

$$E\{X(z, z'] | \mathscr{F}_{z\Gamma}\} = 0 \quad \text{whenever } z' \succ z.$$

Let $\mathscr{H}_\Gamma$ denote the space of functions $\phi$ which satisfy conditions (2.3a), (2.3b) and (2.3 $c_\Gamma$). For $\phi \in \mathscr{H}_\Gamma$ define

$$\phi_{iz}^\Gamma = \phi_z \quad \text{if } z \in D_i^\Gamma, \quad i = 1, 2; \quad = 0 \text{ otherwise.} \qquad (2.6)$$

Then $\phi_i^\Gamma \in \mathscr{H}_i$ and $\phi_z = \phi_{1z}^\Gamma + \phi_{2z}^\Gamma$ for almost all $z$.

**Proposition 2.1.** *Let $\Gamma$ be an increasing path connecting the origin $0$ to the final point $z_0$. Let $\phi \in \mathscr{H}_\Gamma$ and define*

$$(\phi \circ W)_z^\Gamma = (\phi_1^\Gamma \circ W)_z + (\phi_2^\Gamma \circ W)_z, \quad z \in R_{z_0}. \qquad (2.7)$$

*Then,*

(a) $(\phi \circ W)^\Gamma$ *is a $\Gamma$-martingale.*

(b) $(\phi \circ W)^\Gamma$ *is a martingale (one-parameter) on the path $\Gamma$.*

(c) *If $\Gamma$ and $\Gamma'$ are two increasing paths connecting $0$ and $z_0$ and $\phi$ is both $\Gamma$ and $\Gamma'$ adapted then*

$$(\phi \circ W)_{z_0}^\Gamma = (\phi \circ W)_{z_0}^{\Gamma'}.$$

*Proof.* (a) Let $X_z = (\phi \circ W)_z^\Gamma$ and $X_{iz} = (\phi_i^\Gamma \circ W)_z$. Suppose that $z \in D_1$. Then $\mathscr{F}_{z\Gamma} \subseteq \mathscr{F}_z^1$ so that

$$E\{X_1(z, z'] | \mathscr{F}_{z\Gamma}\} = 0.$$

On the other hand $X_2(z, z'] = X_2(z_\Gamma, z']$. Hence,

$$E\{X_2(z, z'] | \mathscr{F}_{z\Gamma}\} = 0.$$

Therefore, $E\{X(z, z'] | \mathscr{F}_{z\Gamma}\} = 0$ and $X$ is a $\Gamma$-martingale. For $z \in D_2^\Gamma$ the same argument with 1 and 2 reversed suffices.

(b) Let $z' \succ z$ and let $z, z' \in \Gamma$. We can write

$$X_{z'} - X_z = X(z, z'] + X(0 \otimes z, z \otimes z'] + X(z \otimes 0, z' \otimes z].$$

Observe that

$$(0 \otimes z)_r = (z \otimes 0)_r = z$$

so that the $\Gamma$-martingale property of $X$ implies

$$E\{(X_{z'} - X_z) | \mathscr{F}_z\} = 0, \qquad z, z' \in \Gamma.$$

(c) Observe that $\phi_i^{\Gamma}$ and $\phi_i^{\Gamma'}$ differ only on the sets $(D_1^{\Gamma} \cap D_2^{\Gamma'})$ and $(D_2^{\Gamma} \cap D_1^{\Gamma'})$, and that for every $\zeta$ in these sets $\zeta_{\Gamma} \wedge \zeta_{\Gamma'} = \zeta$. Since $\phi$ is adapted to both paths, for every $\zeta$ in these sets $\phi_\zeta$ is measurable with respect to $\mathscr{F}_\zeta = \mathscr{F}_{\zeta_\Gamma} \cap \mathscr{F}_{\zeta_{\Gamma'}}$. Hence, for $i \neq j$

$$\int_{D_i^{\Gamma} \cap D_j^{\Gamma'}} \phi_{i\zeta}^{\Gamma} \, dW_\zeta = \int_{D_i^{\Gamma} \cap D_j^{\Gamma'}} \phi_\zeta \, dW_\zeta = \int_{D_i^{\Gamma} \cap D_j^{\Gamma'}} \phi_{j\zeta}^{\Gamma'} \, dW_\zeta$$

and

$$\begin{aligned}
(\phi \circ W)_{z_0}^{\Gamma} - (\phi \circ W)_{z_0}^{\Gamma'} &= \int_{D_1^{\Gamma} \cap D_2^{\Gamma'}} (\phi_{1\zeta}^{\Gamma} - \phi_{2\zeta}^{\Gamma'}) \, dW_\zeta \\
&\quad + \int_{D_2^{\Gamma} \cap D_1^{\Gamma'}} (\phi_{2\zeta}^{\Gamma} - \phi_{1\zeta}^{\Gamma'}) \, dW_\zeta \\
&= 0.
\end{aligned}$$

Part (c) of Proposition 2.1 implies that if $\phi$ is adapted to more than one $\Gamma$ the stochastic integral $(\phi \circ W)^{\Gamma}$ is independent of $\Gamma$ so that the superscript can be dropped. Therefore, for $\phi \in \mathscr{H}_\Gamma$ we can unambiguously define

$$\phi \circ W = \phi_1^{\Gamma} \circ W + \phi_2^{\Gamma} \circ W. \tag{2.8}$$

So defined, $\phi \circ W$ is a $\Gamma$-martingale and a one-parameter martingale on $\Gamma$ for which a sample-continuous version can be chosen. Furthermore, it follows from (2.5) that

$$(\phi \circ W)_z (\phi \circ W)_z - \int_{R_z} \phi_\zeta \psi_\zeta \, d\zeta \tag{2.9}$$

is also a $\Gamma$-martingale and a one-parameter martingale on $\Gamma$.

Next, we shall define multiple integrals of the form

$$\psi \circ \mu \tilde{\mu} = \int \psi_{\zeta,\zeta'} \, d\mu(\zeta) \, d\tilde{\mu}(\zeta') \tag{2.10}$$

where $\mu$ and $\tilde{\mu}$ can each be $W$ or the Lebesgue measure. Denote by $\mathscr{H}$ the space of functions $\psi_{\zeta,\zeta'}$, $(\zeta, \zeta') \in R_{z_0}^2$, which satisfy

(a) $\psi$ is a measurable process and for each $(\zeta, \zeta') \psi_{\zeta,\zeta'}$ is $\mathscr{F}_{\zeta \vee \zeta'}$-measurable.

(b) $\displaystyle \int_{R_{z_0} \times R_{z_0}} I(\zeta \wedge \zeta') E \psi_{\zeta,\zeta'}^2 \, d\zeta \, d\zeta' < \infty.$ \tag{2.11}

For $\psi \in \mathscr{H}$ the integral $\psi \circ \mu \tilde{\mu}$ is defined as follows [6]:

(1) $\psi \in \mathscr{H}$ is said to be a *simple function* if there exist rectangles $A$ and $B$ such that $(\zeta, \zeta') \in A \times B \Rightarrow \zeta \wedge \zeta'$, and that $\psi$ is a constant $\psi_0$ on $A \times B$ and is zero

elsewhere. For a simple function $\psi$ we define

$$\psi \circ \mu \tilde{\mu} = \psi_0 \, \mu(A) \, \tilde{\mu}(B).$$

(2) Denote by $\mathscr{H}$ the space of functions $\psi$ which are sums of simple functions. For $\psi \in \mathscr{H}$ $\psi \circ \mu \tilde{\mu}$ is defined by linearity.

(3) If $\psi \in \mathscr{H}$ and $\psi_{\zeta, \zeta'} = 0$ unless $\zeta \wedge \zeta'$, then $\psi \circ \mu \tilde{\mu}$ is defined as the quadratic limit of an approximating sequence in $\mathscr{H}$.

(4) Finally, for a general $\psi \in \mathscr{H}$, we set

$$\hat{\psi}_{\zeta, \zeta'} = I(\zeta, \zeta') \psi_{\zeta, \zeta'}$$

and define $\psi \circ \mu \tilde{\mu} = \hat{\psi} \circ \mu \tilde{\mu}$.

**Proposition 2.2.** *Let $\psi \in \mathscr{H}$ and define*

$$X_z = \int\limits_{R_z \times R_z} \psi_{\zeta, \zeta'} \, dW_\zeta \, dW_{\zeta'},$$

$$Y_{1z} = \int\limits_{R_z \times R_z} \psi_{\zeta, \zeta'} \, d\zeta \, dW_{\zeta'},$$

$$Y_{2z} = \int\limits_{R_z \times R_z} \psi_{\zeta, \zeta'} \, dW_\zeta \, d\zeta'.$$

*Then, $X$, $Y_1$, $Y_2$ are respectively a martingale, an adapted 1-martingale, and an adapted 2-martingale for which almost surely sample continuous versions can be chosen. Furthermore, let*

$$f_1(z, \zeta') = \int\limits_{R_z} I(\zeta \wedge \zeta') \psi_{\zeta, \zeta'} \, dW_\zeta,$$

$$f_2(z, \zeta) = \int\limits_{R_z} I(\zeta \wedge \zeta') \psi_{\zeta, \zeta'} \, dW_{\zeta'},$$

$$g_1(z, \zeta') = \int\limits_{R_z} I(\zeta \wedge \zeta') \psi_{\zeta, \zeta'} \, d\zeta,$$

$$g_2(z, \zeta) = \int\limits_{R_z} I(\zeta \wedge \zeta') \psi_{\zeta, \zeta'} \, d\zeta'.$$

*Then,*

$$X_z = \int\limits_{R_z} f_1(z, \zeta') \, dW_{\zeta'}$$

$$= \int\limits_{R_z} f_2(z, \zeta) \, dW_\zeta,$$

$$Y_{1z} = \int\limits_{R_z} g_1(z, \zeta') \, dW_{\zeta'}$$

$$= \int\limits_{R_z} f_2(z, \zeta) \, d\zeta,$$

$$Y_{2z} = \int\limits_{R_z} g_2(z, \zeta) \, dW_\zeta$$

$$= \int\limits_{R_z} f_1(z, \zeta') \, d\zeta'.$$

*Proof.* For $\psi \in \mathcal{H}$ let $\{\psi_n\}$ be a sequence in $\bar{\mathcal{H}}$ such that

$$\|\psi_n - \psi\|^2 = \int_{R_a} E(\psi_{n,\zeta,\zeta'} - \psi_{\zeta,\zeta'})^2 \, d\zeta \, d\zeta' \xrightarrow[n \to \infty]{} 0$$

and define $f_{in}$ and $g_{in}$ by using $\psi_n$ in (3.8). Then

$$\int_{R_z} E[f_{in}(z,\zeta) - f_i(z,\zeta)]^2 \, d\zeta \leq \|\psi_n - \psi\|^2 \xrightarrow[n \to \infty]{} 0$$

and

$$\int_{R_z} E[g_{in}(z,\zeta) - g_i(z,\zeta)]^2 \, d\zeta \leq \text{Area}(R_z) \|\psi_n - \psi\|^2 \xrightarrow[n \to \infty]{} 0.$$

Hence, if we denote $X_{nz} = \int_{R_z \times R_z} \psi_{n\zeta,\zeta'} \, dW_\zeta \, dW_{\zeta'}$, then

$$E[X_z - \int_{R_z} f_1(z,\zeta') \, dW_{\zeta'}]^2 \leq 2E(X_z - X_{nz})^2 + 2\|\psi_n - \psi\|^2 \xrightarrow[n \to \infty]{} 0.$$

Similarly,

$$E[Y_{1z} - \int_{R_z} f_2(z,\zeta) \, d\zeta]^2 \leq 2E(Y_{1z} - Y_{1nz})^2 + 2\,\text{Area}(R_z)\|\psi_n - \psi\|^2 \xrightarrow[n \to \infty]{} 0.$$

These two cases are prototypical of all the others.

The martingale-properties can be proved using approximations, but they also follow directly from the iterated integrals by using Proposition 2.1. Continuity is proved by showing that a subsequence of $\{\psi_n\}$ can be so chosen that the resulting approximations of $X$ and $Y_i$ converge uniformly almost surely.

*Remark.* Proposition 2.2 might be viewed as stochastic Fubini's theorems.

If $u \in \mathcal{H}_\Gamma$ and $v$ is $\Gamma$-adapted, then

$$X_z = X_0 + \int_{R_z} u_\zeta \, dW_\zeta + \int_{R_z} v_\zeta \, d\zeta \tag{2.12}$$

is a sample-continuous semimartingale on $\Gamma$. As such the differentiation formula for one-parameter continuous semimartingale applies. If we parameterize $\Gamma$ by $\{z(t); 0 \leq t \leq 1\}$ then for a twice continuously differentiable function $F$

$$F(X_{z(t)}) - F(X_0) = \int_0^t F'(X_{z(s)}) \, dX_{z(s)} + \frac{1}{2} \int_0^t f''(X_{z(s)}) \, d\langle X, X \rangle_{z(s)}. \tag{2.13}$$

From (2.9) we know that the quadratic variation $\langle X, X \rangle$ on $\Gamma$ is given by

$$\langle X, X \rangle_{z(t)} = \int_{R_{z(t)}} u_\zeta^2 \, d\zeta$$

and (2.13) can now be expressed free of the parametrization as

$$F(X_z) - F(X_0) = \int_{R_z} F'(X_{\zeta_\Gamma}) \, dX_\zeta + \frac{1}{2} \int_{R_z} F''(X_{\zeta_\Gamma}) u_\zeta^2 \, d\zeta \quad z \in \Gamma. \tag{2.14}$$

Generalization to a collection of process $X_{kz}$ of the form (2.12) follows in an obvious way.

Let $\Gamma$ be an increasing path and consider a multiple integral

$$\psi \circ \mu \tilde{\mu} = \int_{R_{z_0} \times R_{z_0}} \psi_{\zeta,\zeta'} \, d\mu(\zeta) \, d\tilde{\mu}(\zeta') \tag{2.15}$$

where $d\mu(\zeta)$, $d\tilde{\mu}(\zeta) = d\zeta$ or $dW(\zeta)$ and $\psi \in \mathcal{H}$. We can rewrite it as

$$
\begin{aligned}
\psi \circ \mu \tilde{\mu} = & \int_{\zeta' \otimes \zeta \in D_1^{\Gamma}} \psi_{\zeta,\zeta'} \, I(\zeta \wedge \zeta') \, d\mu(\zeta) \, d\tilde{\mu}(\zeta') \\
& + \int_{\zeta' \otimes \zeta \in D_2^{\Gamma}} \psi_{\zeta,\zeta'} \, I(\zeta \wedge \zeta') \, d\mu(\zeta) \, d\tilde{\mu}(\zeta') \\
= & \int_{R_z} \int_{\zeta' \otimes \zeta \in D_1^{\Gamma}} \psi_{\zeta,\zeta'} \, I(\zeta \wedge \zeta') \, d\mu(\zeta) \, d\tilde{\mu}(\zeta) \\
& + \int_{R_z} \int_{\zeta' \otimes \zeta \in D_2^{\Gamma}} \psi_{\zeta,\zeta'} \, I(\zeta \wedge \zeta') \, d\tilde{\mu}(\zeta') \, d\mu(\zeta).
\end{aligned} \tag{2.16}
$$

It follows that for every increasing path $\Gamma \psi \circ WW$ can be expressed as

$$\psi \circ WW = u_\Gamma \circ W \tag{2.17}$$

while $\psi \circ W\mu$ and $\psi \circ \mu W$ $(d\mu(\zeta) = d\zeta)$ are of the form

$$
\begin{aligned}
\psi \circ W\mu \\
\psi \circ \mu W
\end{aligned} = u_\Gamma \circ W + v_\Gamma \circ \mu. \tag{2.18}
$$

Therefore, a process defined by

$$
\begin{aligned}
X_z = X_0 + & \int_{R_z} \phi_\zeta \, dW_\zeta + \int_{R_z} \theta_\zeta \, d\zeta + \int_{R_z \times R_z} \psi_{\zeta,\zeta'} \, dW_\zeta \, dW_{\zeta'} \\
& + \int_{R_z \times R_z} f_{\zeta,\zeta'} \, d\zeta \, dW_{\zeta'} + \int_{R_z \times R_z} g_{\zeta,\zeta'} \, dW_\zeta \, d\zeta'
\end{aligned} \tag{2.19}
$$

where $\phi \in \mathcal{H}_0$ and $\psi$, $f$, $g \in \mathcal{H}$, can be reexpressed for any increasing $\Gamma$ as

$$X_z = X_0 + \int_{R_z} u(\Gamma, \zeta) \, dW_\zeta + \int_{R_z} v(\Gamma, \zeta) \, d\zeta \tag{2.20}$$

where $u \in \mathcal{H}_\Gamma$. As such, $X$ is clearly a sample-continuous semimartingale on $\Gamma$.

Therefore, a collection of processes defined by

$$
\begin{aligned}
X_{kz} = X_{k0} + & \int_{R_z} \phi_{k\zeta} \, dW_\zeta + \int_{R_z} \theta_{k\zeta} \, d\zeta \\
& + \int_{R_z \times R_z} \psi_{k,\zeta,\zeta'} \, dW_\zeta \, dW_{\zeta'} + \int_{R_z \times R_z} f_{k,\zeta,\zeta'} \, d\zeta \, dW_{\zeta'} \\
& + \int_{R_z \times R_z} g_{k,\zeta,\zeta'} \, dW_\zeta \, d\zeta'
\end{aligned} \tag{2.21}
$$

can be reexpressed for any increasing path $\Gamma$ in the form

$$X_{kz} = X_{k0} + \int_{R_z} u_k(\Gamma,\zeta)\,dW_\zeta + \int_{R_z} v_k(\Gamma,\zeta)\,d\zeta, \quad k=1,2,\ldots,m \tag{2.22}$$

where $u_k(\Gamma, z)$ and $v_k(\Gamma, z)$ are $\mathfrak{F}_{zr}$-measurable for each $z \in R_{z_0}$.

If $F: R^n \to R$ is a function with continuous mixed partials up to the second order then the differentiation formula on $\Gamma$ becomes

$$F(X_z) = F(X_0) + \int_{R_z} F_k(X_{\zeta r})[u_k(\Gamma,\zeta)\,dW_\zeta + v_k(\Gamma,\zeta)\,d\zeta]$$

$$+ \tfrac{1}{2}\int_{R_z} F_{k\,l}(X_{\zeta r})\,u_k(\Gamma,\zeta)\,u_l(\Gamma,\zeta)\,d\zeta \tag{2.23}$$

where summation over repeated indices is implied.

The differentiation formula takes on a special form if $\Gamma$ is a vertical or horizontal path. Let $\Gamma_h$ be made up of a horizontal line across the whole width of $R_{z_0}$ together with two vertical segments which connects it to $0$ and $z_0$. For simplicity we shall call $\Gamma_h$ a horizontal path. Observe that for any $z \in \Gamma_h$ $R_z \subset D_1^{\Gamma_h}$ and $\zeta \in R_z \Rightarrow \zeta_{\Gamma_h} = \zeta \otimes z$. The functions $u_k$ and $v_k$ in (2.23) can now be written explicitly as

$$u_k(\Gamma_h,\zeta) = \phi_{k\zeta} + \int_{R_z} I(\zeta' \wedge \zeta)[\psi_{k,\zeta',\zeta}\,dW_{\zeta'} + f_{k,\zeta',\zeta}\,d\zeta']$$

$$v_k(\Gamma_h,\zeta) = \theta_{k\zeta} + \int_{R_z} I(\zeta' \wedge \zeta)\,g_{k,\zeta',\zeta}\,dW_{\zeta'}. \tag{2.24-1}$$

It is now convenient to adopt the notations

$$u_k(z,\zeta) = u_k(\Gamma_h,\zeta),$$

$$v_k(z,\zeta) = v_k(\Gamma_h,\zeta).$$

Observe that because of the term $I(\zeta' \wedge \zeta)$ in the integral

$$u_k(z,\zeta) = u_k(\zeta \otimes z,\zeta)$$

$$v_k(z,\zeta) = v_k(\zeta \otimes z,\zeta). \tag{2.25-1}$$

The differentiation formula (2.23) now takes on the form

$$F(X_z) = F(X_0) + \int_{R_z} F_k(X_{\zeta \otimes z})[u_k(z,\zeta)\,dW_\zeta + v_k(z,\zeta)\,d\zeta]$$

$$+ \tfrac{1}{2}\int_{R_z} F_{k\,l}(X_{\zeta \otimes z})\,u_k(z,\zeta)\,u_l(z,\zeta)\,d\zeta. \tag{2.26-1}$$

Similarly, we can define a vertical path $\Gamma_v$ and find

$$\bar{u}_k(z,\zeta) = u_k(\Gamma_v,\zeta) = \phi_{k\zeta} + \int_{R_z} I(\zeta \wedge \zeta')[\psi_{k,\zeta,\zeta'}\,dW_{\zeta'} + g_{k,\zeta,\zeta'}\,d\zeta'],$$

$$\bar{v}_k(z,\zeta) = v_k(\Gamma_v,\zeta) = \theta_{k\zeta} + \int_{R_z} I(\zeta \wedge \zeta')f_{k,\zeta,\zeta'}\,d\zeta'.$$

Now, $u_k$ and $v_k$ satisfy the conditions

$$\tilde{u}_k(z, \zeta) = \tilde{u}_k(z \otimes \zeta, \zeta)$$
$$\tilde{v}_k(z, \zeta) = \tilde{v}_k(z \otimes \zeta, \zeta)$$

(2.25-2)

and the differentiation formula has the form

$$F(X_z) = F(X_0) + \int_{R_z} F_k(X_{z \otimes \zeta})[\tilde{u}_k(z, \zeta) \, dW_\zeta + \tilde{v}_k(z, \zeta) \, d\zeta]$$
$$+ \tfrac{1}{2} \int_{R_z} F_{k\,l}(X_{z \otimes \zeta}) \, \tilde{u}_k(z, \zeta) \, \tilde{u}_l(z, \zeta) \, d\zeta.$$

(2.26-2)

## 3. Likelihood Ratio Formulas on Increasing Paths

Let $(\Omega, \mathfrak{F})$ be a measurable space and $\{X_z, z \in R_{z_0}\}$ a family of measurable functions. Let $\mathfrak{F}_{xz} = \sigma(X_\zeta, \zeta \in R_z)$ and assume $\mathfrak{F}_{xz_0} = \mathfrak{F}$. Let $\mathscr{P}$ and $\mathscr{P}_0$ be two equivalent probability measures on $(\Omega, \mathfrak{F})$ such that under $\mathscr{P}_0$, $X$ is a Wiener process. Denote the likelihood ratio by

$$L_z = E_0 \left\{ \frac{d\mathscr{P}}{d\mathscr{P}_0} \middle| \mathfrak{F}_{xz} \right\}.$$

(3.1)

Then $L$ is a positive $(\{\mathfrak{F}_{xz}\}, \mathscr{P}_0)$ martingale. In addition, we shall assume

$$E_0 L_z^2 < \infty, \quad \forall z \in R_{z_0}$$

(3.2)

so that we can invoke the representation theorem of [5] and write $L$ in the form

$$L_z = 1 + \int_{R_z} \alpha_\zeta \, dX_\zeta + \int_{R_z \times R_z} \beta_{\zeta, \zeta'} \, dX_\zeta \, dX_{\zeta'}.$$

(3.3)

Whence it follows that $L$ can be chosen to be almost surely sample-continuous. The square-integrability condition of $L$ is made necessary by the fact that unlike the one-parameter case the stochastic-integral representation for Wiener-martingales has been proved only for square-integrable martingales and not for martingales in general. Because of this, it is not yet clear whether all Radon-Nikodym derivatives on a Wiener space are sample continuous. However, we believe that the square-integrability condition (3.2) can be weakened and that the form that we shall derive is valid for all continuous likelihood ratios.

Equation (3.3) can be put in the form

$$L_z = 1 + \int_{R_z} L_{\zeta' \otimes z} u(z, \zeta') \, dX_{\zeta'}$$

(3.4-1)

with

$$u(z, \zeta') = \frac{1}{L_{\zeta' \otimes z}} \left[ \alpha_{\zeta'} + \int_{R_z} I(\zeta \wedge \zeta') \beta_{\zeta, \zeta'} \, dX_\zeta \right].$$

(3.5-1)

Alternatively, (2.4-1) and (2.5-1) can be recast into the form

$$L_z = 1 + \int_{R_z} L_{z \otimes \zeta}\, \tilde{u}(z, \zeta)\, dX_\zeta, \tag{3.4-2}$$

$$\tilde{u}(z, \zeta) = \frac{1}{L_{z \otimes \zeta}} \left[ \alpha_\zeta + \int_{R_z} I(\zeta \wedge \zeta')\, \beta_{\zeta, \zeta'}\, dX_{\zeta'} \right]. \tag{3.5-2}$$

We recognize (3.4-1) as a representation of $L$ as a 1-martingale, and (3.4-2) a representation as a 2-martingale. Since $L_z > 0$ almost surely, we can now apply the differentiation formula (2.26) to $\ln L_z$ and get

$$
\begin{aligned}
\ln L_z &= \int_{R_z} u(z, \zeta')\, dX_{\zeta'} - \tfrac{1}{2} \int_{R_z} u^2(z, \zeta')\, d\zeta' \\
&= \int_{R_z} \tilde{u}(z, \zeta)\, dX_\zeta - \tfrac{1}{2} \int_{R_z} \tilde{u}^2(z, \zeta)\, d\zeta.
\end{aligned}
$$

It follows that we have

$$L_z = \exp\left\{ \int_{R_z} u(z, \zeta')\, dX_{\zeta'} - \tfrac{1}{2} \int_{R_z} u^2(z, \zeta')\, d\zeta' \right\} \tag{3.6-1}$$

$$= \exp\left\{ \int_{R_z} \tilde{u}(z, \zeta)\, dX_\zeta - \tfrac{1}{2} \int_{R_z} \tilde{u}^2(z, \zeta)\, d\zeta \right\}. \tag{3.6-2}$$

Equation (3.7) is reminiscent of the exponential formula in one dimension, and indeed it is precisely that. We note from (3.5-1) that

$$u(z, \zeta') = u(\zeta' \otimes z, \zeta')$$

so that the exponent in (3.6-1) is a semimartingale on horizontal lines. Thus, (3.6-1) can be considered a representation of $L$ as a positive martingale on horizontal paths, and (3.6-2) as a representation on a vertical path. Thus, the similarity of (3.6) to the exponential formula for one-parameter Wiener processes comes as no surprise. Indeed, the representation (3.6) can be generalized to any increasing path.

Let $\Gamma$ be an increasing path connecting the origin and $z_0$. For any point $z \in R_{z_0}$, $z_\Gamma$ will denote the smallest point on $\Gamma$ greater or equal to $z$. We say $\{\phi_z, z \in R_{z_0}\}$ is $\mathfrak{F}_\Gamma$-adapted if for each $z$ $\phi_z$ is $\mathfrak{F}_{z_\Gamma}$-measurable. In Section 2, stochastic integrals for $\mathfrak{F}_\Gamma$-adapted integrands have been defined. Using this definition, we can rewrite (3.3) for $z \in \Gamma$ as

$$L_z = 1 + \int_{R_z} L_{\zeta_\Gamma}\, u_\Gamma(\zeta)\, dX_\zeta \tag{3.7}$$

where (c.f. (2.20))

$$
\begin{aligned}
u_\Gamma(\zeta) = (L_{\zeta_\Gamma})^{-1} \Big[ \alpha_\zeta &+ \int_{\zeta \otimes \zeta' \in D_1^\Gamma} \beta_{\zeta', \zeta}\, I(\zeta' \wedge \zeta)\, dX_{\zeta'} \\
&+ \int_{\zeta' \otimes \zeta \in D_2^\Gamma} \beta_{\zeta, \zeta'}\, I(\zeta \wedge \zeta')\, dX_{\zeta'} \Big].
\end{aligned}
\tag{3.8}
$$

Observe that only one of the two integrals in the definition of $u_\Gamma$ is non-zero. For $\zeta \in D_1^\Gamma$, $\zeta' \otimes \zeta$ cannot be in $D_2^\Gamma$, and for $\zeta \in D_2^\Gamma$, $\zeta \otimes \zeta'$ cannot be in $D_1^\Gamma$. So defined $u_\Gamma(\zeta)$ is $\mathfrak{F}_{\zeta_\Gamma}$-measurable, and an application of the one-dimensional differentiation rule to the path $\Gamma$ yields

$$L_z = \exp\{ \int_{R_z} u_\Gamma(\zeta) \, dX_\zeta - \tfrac{1}{2} \int_{R_z} u_\Gamma^2(\zeta) \, d\zeta \} \tag{3.9}$$

for all $z \in \Gamma$.

**Theorem 3.1.** *Let $(\Omega, \mathfrak{F}, \mathscr{P}_0)$ be a probability space and $\{X_z, z \in R_{z_0}\}$ a Wiener process. Let $\mathfrak{F}_{xz}$ denote the $\sigma$-field generated by $\{X_\zeta, \zeta \prec z\}$ and assume $\mathfrak{F} = \mathfrak{F}_{xz_0}$.*

*(a) Suppose $\mathscr{P}$ is a probability measure equivalent to $\mathscr{P}_0$ such that the likelihood ratio*

$$L_z = E_0 \left\{ \frac{d\mathscr{P}}{d\mathscr{P}_0} \middle| \mathfrak{F}_{xz} \right\}$$

*is $\mathscr{P}_0$-square-integrable (i.e., $E_0 L_z^2 < \infty$, $\forall z \prec z_0$). Then for any increasing path $\Gamma$ there exists an $\mathfrak{F}_\Gamma$-adapted process $u_\Gamma$ so that for all $z \in \Gamma$*

$$L_z = 1 + \int_{R_z} u_\Gamma(\zeta) L_{\zeta_\Gamma} \, dX_\zeta \tag{3.7}$$

*and*

$$L_z = \exp\{ \int_{R_z} u_\Gamma(\zeta) \, dX_\zeta - \tfrac{1}{2} \int_{R_z} u_\Gamma^2(\zeta) \, d\zeta \}. \tag{3.9}$$

*(b) Conversely, let $\Gamma$ be an increasing path and $u_\Gamma$ an $\mathfrak{F}_\Gamma$-adapted process satisfying*

$$\int_{R_{z_0}} u_\Gamma^2(\alpha) \, d\zeta < \infty \text{ almost surely } \mathscr{P}_0.$$

*Define for $z \in \Gamma$*

$$L_z = \exp\{ \int_{R_z} u_\Gamma(\zeta) \, dX_\zeta - \tfrac{1}{2} \int_{R_z} u_\Gamma^2(\zeta) \, d\zeta \}.$$

*Suppose that $E_0 L_{z_0} = 1$. Then $\dfrac{d\mathscr{P}}{d\mathscr{P}_0} = L_{z_0}$ defines a probability measure $\mathscr{P}$ and $E_0(L_{z_0} | \mathfrak{F}_{xz}) = L_z$.*

*Proof.* (a) Since $L_z$ is a $\mathscr{P}_0$-square-integrable $\mathfrak{F}_{xz}$-martingale, we can write as in (3.3)

$$L_z = 1 + \int_{R_z} \alpha_\zeta \, dX_\zeta + \int_{R_z \times R_z} \beta_{\zeta, \zeta'} \, dX_\zeta \, dX_{\zeta'}.$$

Define $u_\Gamma$ by (3.8). Then (3.7) follows. An application of the one-parameter differentiation formula to $\ln L_z$ on $\Gamma$ yields (3.9).

(b) Conversely, if $\int_{R_z} u_\Gamma^2(\zeta)\,d\zeta < \infty$ almost surely $(\mathscr{P}_0)$ then $M_z = \int_{R_z} u_\Gamma(\zeta)\,dX_\zeta$ is well-defined as a local martingale on $\Gamma$ with

$$\langle M, M \rangle_z = \int_{R_z} u_\Gamma^2(\zeta)\,d\zeta.$$

Hence, $L_z = e^{M_z - \frac{1}{2}\langle M, M \rangle_z}$ defines a probability measure if $EL_{z_0} = 1$. Since

$$L_z = 1 + \int_{R_z} L_{\zeta_\Gamma}\, u_\Gamma(\zeta)\,dX_\zeta$$

it follows that $E_0(L_{z_0} | \mathfrak{F}_{xz}) = L_z$, almost surely.

Let $Y_z = \int_{R_z} \alpha(\zeta)(dX_\zeta - u_\Gamma(\zeta)\,d\zeta)$, where $\alpha$ is a bounded deterministic function. Then, under $\mathscr{P}_0$, $Y_z$ can be considered a semimartingale on $\Gamma$, and the one-parameter differentiation rule (2.14) yields

$$L_z Y_z = \int_{R_z} L_{\zeta_\Gamma}[\alpha(\zeta) + Y_{\zeta_\Gamma}\, u_\Gamma(\zeta)]\,dX_\zeta$$

so that $L_z Y_z$ is a $\mathscr{P}_0$-martingale on $\Gamma$. Therefore $Y_z$ is a $\mathscr{P}$-martingale on $\Gamma$. This gives us the interpretation

$$E[(dX_\zeta - u_\Gamma(\zeta)\,d\zeta) | \mathfrak{F}_{xz_\Gamma}] = 0$$

or

$$u_\Gamma(\zeta)\,d\zeta = E[dX_\zeta | \mathfrak{F}_{xz_\Gamma}]. \tag{3.10}$$

Specializing to horizontal and vertical paths yields an interpretation for the functions $u$ and $\tilde{u}$ in ( .6) as follows.

$$u(z, \zeta')\,d\zeta' = E[dX_{\zeta'} | \mathfrak{F}_{x,\zeta' \otimes z}], \tag{3.11-1}$$

$$\tilde{u}(z, \zeta)\,d\zeta = E[dX_\zeta | \mathfrak{F}_{x,z \otimes \zeta}]. \tag{3.11-2}$$

A more precise statement of (3.10) or (3.11) can be made as follows: For a fixed $\Gamma$ define a $\Gamma$-*martingale* $Y$ by the property

$$E\{Y(z, z') | \mathfrak{F}_{zr}\} = 0 \quad \text{for all } z' \gg z.$$

This generalizes the concept of $i$-martingale (adapted or non-adapted). Now, a precise statement of (3.10) or (3.11) is given by

**Theorem 3.2.** *Let* $u_\Gamma$, $X$, $\mathscr{P}$ *and* $\mathscr{P}_0$, *be as in Theorem* 3.1. *Then*

$$Y_z = X_z - \int_{R_z} u_\Gamma(\zeta)\,d\zeta$$

*is a* $\Gamma$-*martingale with respect to* $\mathscr{P}$.

*Proof.* Fix two points $z \prec z'$, and for $\{\alpha : \alpha \succ z$ and $\alpha \in \Gamma\}$ define

$$M_\alpha = \int_{R_\alpha} I(z \prec \zeta \prec z')[dX_\zeta - u_\Gamma(\zeta)\,d\zeta].$$

Since $M_a$ is a $\mathscr{P}$-martingale on the portion of $\Gamma$ from $z_r$ to $z_0$ we have

$$E(M_{z_0}|\mathfrak{F}_{z_r})=M_{z_r}.$$

Since $M_{z_0}=Y(z,z']$ and $M_{z_r}=0$, the desired result follows.

Before proceeding to the derivation of a two-dimensional exponential formula for $L_z$, consider the special case where $u_\Gamma(\zeta)=\phi_\zeta$ is independent of path. In that case the formula (3.9) becomes

$$L_z=\exp\{\int_{R_z}\phi_\zeta\,dX_\zeta-\tfrac{1}{2}\int_{R_z}\phi_\zeta^2\,d\zeta\}$$

which being path independent is already a full-fledged two-dimensional exponential formula. Needless to say, the condition that $u_\Gamma$ be independent of path is a severe one and the circumstances under which this obtains will become apparent in the next section.


## 4. A Two-Dimensional Exponential Formula

The exponential formulas for the likelihood ratio given by (3.6) and (3.9) are two dimensional in form, but clearly one-dimensional in spirit. Our next objective is to derive a formula which is inherently two-dimensional. The starting point is (3.5-1) and (3.4-2). Observe that (3.4-2) is in the form of (2.20) for a vertical path so that by considering (3.4-2) on a vertical path through $\zeta'$, we get

$$L_{\zeta'\otimes z}=L_{\zeta'}+\int_{R_z}I(\zeta\wedge\zeta')\,L_{\zeta'\otimes\zeta}\,\tilde{u}(\zeta'\otimes\zeta,\zeta)\,dX_\zeta. \tag{4.1}$$

If we denote

$$Y_{z,\zeta'}=\alpha_{\zeta'}+\int_{R_z}I(\zeta\wedge\zeta')\,\beta_{\zeta,\zeta'}\,dX_\zeta \tag{4.2}$$

then (3.5-1) acquires the form

$$u(z,\zeta')=\left(\frac{1}{L_{\zeta'\otimes z}}\right)Y_{z,\zeta'}. \tag{4.3}$$

For a fixed $\zeta'$, $L_{\zeta'\otimes z}$ and $Y_{z,\zeta'}$ are 2-martingales, and we can apply (2.26-2) to get

$$u(z,\zeta')=\left(\frac{\alpha_{\zeta'}}{L_{\zeta'}}\right)+\int_{R_z}I(\zeta\wedge\zeta')\frac{1}{L_{\zeta'\otimes\zeta}}\beta_{\zeta,\zeta'}\,dX_\zeta$$

$$-\int_{R_z}I(\zeta\wedge\zeta')\frac{Y_{z\otimes\zeta,\zeta'}}{L_{\zeta'\otimes\zeta}^2}L_{\zeta'\otimes\zeta}\,\tilde{u}(\zeta'\otimes\zeta,\zeta)\,dX_\zeta$$

$$-\int_{R_z}I(\zeta\wedge\zeta')\frac{1}{L_{\zeta'\otimes\zeta}}\tilde{u}(\zeta'\otimes\zeta,\zeta')\,\beta_{\zeta,\zeta'}\,d\zeta$$

$$+\int_{R_z}I(\zeta\wedge\zeta')\left[\frac{Y_{z\otimes\zeta,\zeta'}}{L_{\zeta'\otimes\zeta}^3}\right]L_{\zeta'\otimes\zeta}^2\,\tilde{u}^2(\zeta'\otimes\zeta,\zeta)\,d\zeta.$$

Observe that because of the term $I(\zeta \wedge \zeta')$ in the integrals, we have

$$u(z, \zeta') = u(\zeta' \otimes z, \zeta')$$

so that

$$\frac{Y_{z \otimes \zeta, \zeta'}}{L_{\zeta' \otimes \zeta}} = u(z \otimes \zeta, \zeta') = u(\zeta' \otimes \zeta, \zeta').$$

Then it follows that we can write for $z > \zeta'$

$$u(z, \zeta') = \theta_{\zeta'} + \int_{R_z} I(\zeta \wedge \zeta') \rho(\zeta, \zeta') [dX_\zeta - \tilde{u}(\zeta' \otimes \zeta, \zeta) \, d\zeta] \qquad (4.4\text{-}1)$$

where

$$\theta_{\zeta'} = u(\zeta', \zeta') = \left( \frac{\alpha_{\zeta'}}{L_{\zeta'}} \right) \qquad (4.5)$$

and

$$\rho(\zeta, \zeta') = \frac{\beta(\zeta, \zeta')}{L_{\zeta' \otimes \zeta}} - u(\zeta' \otimes \zeta, \zeta') \, \tilde{u}(\zeta' \otimes \zeta, \zeta). \qquad (4.6)$$

By symmetry we can also write

$$\tilde{u}(z, \zeta) = \theta_\zeta + \int_{R_z} I(\zeta \wedge \zeta') \rho(\zeta, \zeta') \, dX_{\zeta'}$$
$$- \int_{R_z} I(\zeta \wedge \zeta') \rho(\zeta, \zeta') \, u(\zeta' \otimes \zeta, \zeta') \, d\zeta'. \qquad (4.4\text{-}2)$$

Equation (4.4-1) yields

$$u^2(z, \zeta') = \theta_{\zeta'}^2 + 2 \int_{R_z} I(\zeta \wedge \zeta') u(\zeta' \otimes \zeta, \zeta') \rho(\zeta, \zeta') [dX_\zeta - \tilde{u}(\zeta' \otimes \zeta, \zeta) \, d\zeta]$$
$$+ \int_{R_z} I(\zeta \wedge \zeta') \rho^2(\zeta, \zeta') \, d\zeta. \qquad (4.7)$$

Putting (4.4-1) and (4.7) into (3.6-1) yields

$$L_z = \exp \Big\{ \int_{R_z} \theta_{\zeta'} \, dX_{\zeta'} - \tfrac{1}{2} \int_{R_z} \theta_{\zeta'}^2 \, d\zeta' - \tfrac{1}{2} \int_{R_z \times R_z} \rho^2(\zeta, \zeta') \, d\zeta \, d\zeta'$$
$$+ \int_{R_z \times R_z} \rho(\zeta, \zeta') [dX_\zeta - \tilde{u}(\zeta' \otimes \zeta, \zeta) \, d\zeta] [dX_{\zeta'} - u(\zeta' \otimes \zeta, \zeta') \, d\zeta'] \Big\} \qquad (4.8)$$

which is the two-dimensional exponential formula that we have sought.

Given $\rho$ and $\theta$, (4.4) can be viewed as a pair of linear integral equations with unknowns $u$ and $\tilde{u}$. Indeed, if we set $u(a \otimes b, a) = h(a, b)$ and $\tilde{u}(a \otimes b, b) = \tilde{h}(a, b)$, (4.4) can be rewritten in the form

$$h(a, b) = h_0(a, b) + \int_{R_{a \otimes b} \times R_{a \otimes b}} G_a(\zeta, \zeta') \, h(\zeta', \zeta) \, d\zeta \, d\zeta',$$

$$\tilde{h}(a, b) = \tilde{h}_0(a, b) + \int_{R_{a \otimes b} \times R_{a \otimes b}} \tilde{G}_b(\zeta, \zeta') \, \tilde{h}(\zeta', \zeta) \, d\zeta \, d\zeta'.$$

If $\rho$ is bounded then so are $G$ and $\tilde{G}$, in which case Picard iteration converges, and the existence and uniqueness of $h$ and $\tilde{h}$ are not in question. Therefore, if $\rho$ is assumed to be bounded then (4.8) can be viewed as an expression of $L_z$ in terms of $\theta$ and $\rho$.

Summarizing, we have the following:

**Theorem 4.1.** *Let $\{L_z, z \in R_{z_0}\}$ be an almost surely positive square-integrable martingale defined on $(\Omega, \mathfrak{F}, \mathscr{P}_0)$ where $\mathfrak{F}$ is generated by a Wiener process $\{X_z, z \in R_{z_0}\}$. Let $L_0 = 1$. Then, there exist functions $\theta$, $\rho$, $u$ and $\tilde{u}$ satisfying (4.4) such that $L_z$ can be expressed by (4.8). Further, $L$ satisfies*

$$L_z = 1 + \int_{R_z} L_\zeta \, \theta_\zeta \, dX_\zeta$$

$$+ \int_{R_z \times R_z} L_{\zeta' \otimes \zeta} [\rho(\zeta, \zeta') + u(\zeta' \otimes \zeta, \zeta') \, \tilde{u}(\zeta' \otimes \zeta, \zeta)] \, dX_\zeta \, dX_{\zeta'}. \tag{4.9}$$

Conversely, let $\theta_z$ be an $\mathfrak{F}_{xz}$-measurable function defined for $z \in R_{z_0}$ and $\rho(z, z')$ be an $\mathfrak{F}_{xz \vee z'}$-measurable function defined for all $z$, $z' \in R_{z_0}$ such that $z \wedge z'$. Suppose that (4.4) has unique solutions for $u$ and $\tilde{u}$, and when $\theta$, $\rho$, $u$ and $\tilde{u}$ are substituted into (4.8), it yields an $L_z$ satisfying $E_0 L_{z_0} = 1$. Then, $L_z$ is a positive martingale which is the unique solution to (4.9).

**Corollary.** *Let $\{X_z, z \in R_{z_0}\}$ be a Wiener process defined on $(\Omega, \mathfrak{F}, \mathscr{P}_0)$ and denote $\mathfrak{F}_{xz} = \sigma(X_\zeta, \zeta \prec z)$. Let $\mathscr{P}$ be a probability measure on $(\Omega, \mathfrak{F})$ such that the restrictions of $\mathscr{P}$ and $\mathscr{P}_0$ to $\mathfrak{F}_{xz_0}$ are equivalent. Suppose that the likelihood ratio*

$$L_z = E_0 \left( \frac{d\mathscr{P}^x}{d\mathscr{P}_0^x} \bigg| \mathfrak{F}_{xz} \right)$$

*is $\mathscr{P}_0$ square-integrable. Then, it satisfies (4.8) and (4.9).*

*Proof.* The fact that $L_z$ satisfies (4.8) has already been proved by the steps leading to (4.8). To obtain (4.9), we return to (3.3) and use (4.5) and (4.6) to identify $\alpha$ and $\beta$. Finally, to go from (4.8) to (4.9), we rewrite (4.8) using (4.4) to get back to (2.6-1), viz.,

$$L_z = \exp\left\{ \int_{R_z} u(z, \zeta') \, dX_{\zeta'} - \tfrac{1}{2} \int_{R_z} u^2(z, \zeta') \, d\zeta' \right\}.$$

If $E_0 L_{z_0} = 1$, this implies (3.4-1), i.e.,

$$L_z = 1 + \int_{R_z} L_{\zeta' \otimes z} \, u(z, \zeta') \, dX_{\zeta'}.$$

Now, we can use (4.1) and (4.4-1), whence (4.9) follows.

We observe that if $\rho \equiv 0$ then (4.8) degenerates into the form given at the end of Section 3. In that case $u(z, \zeta) = \tilde{u}(z, \zeta) = \theta_\zeta$, and $u_\Gamma$ is indeed independent of the path. This situation arises when and only when $L_z$ satisfies the equation

$$L_z = L + \int_{R_z} \theta_\zeta L_\zeta \, dX_\zeta + \int_{R_z \times R_z} \theta_\zeta \, \theta_{\zeta'} \, L_{\zeta \vee \zeta'} \, dX_\zeta \, dX_{\zeta'}.$$

## 5. Interpretation of the Functions $\theta$ and $\rho$

The interpretation of $\theta$ comes immediately from those of $u$ and $\tilde{u}$ and the relationship $\theta(\zeta)=u(\zeta,\zeta)=\tilde{u}(\zeta,\zeta)$. We have from (3.11)

$$\theta(\zeta)\,d\zeta=E(dX_\zeta|\mathfrak{F}_{x\zeta}). \tag{5.1}$$

The interpretation of $\rho$ is more obscure. A hint as to what it should be comes from comparing (4.9) with Equation (4.12) of [4]. (In the latter equation the factor $\frac{1}{2}$ is due to a slightly different definition of the stochastic integral of the second type.) These equations are similar, and the comparison suggests that while $u$ and $\tilde{u}$ are conditional expectations of $dX$ given $\sigma$-fields of various kinds, $\rho$ should be the covariance of such conditional expectations. Specifically, we should have

$$u(\zeta'\otimes\zeta,\zeta')\,d\zeta'=E(dX_{\zeta'}|\mathfrak{F}_{x,\zeta'\otimes\zeta}), \tag{5.2-1}$$

$$\tilde{u}(\zeta'\otimes\zeta,\zeta)\,d\zeta=E(dX_\zeta|\mathfrak{F}_{x,\zeta'\otimes\zeta}), \tag{5.2-2}$$

$$\rho(\zeta,\zeta')\,d\zeta\,d\zeta'=E[(dX_\zeta-\tilde{u}(\zeta'\otimes\zeta,\zeta))(dX_{\zeta'}-u(\zeta'\otimes\zeta,\zeta'))|\mathfrak{F}_{x,\zeta'\otimes\zeta}] \tag{5.3}$$

for all $\zeta$, $\zeta'$ in $R_{z_0}$ such that $\zeta\wedge\zeta'$. We note that because $\zeta\wedge\zeta'$, $\zeta'\otimes\zeta$ can be replaced by $\zeta\vee\zeta'$ as is done in [4].

To verify (5.3) precisely, we must show that if

$$Y_z=\int_{R_z\times R_z} f(\zeta,\zeta')\{[dX_\zeta-\tilde{u}(\zeta'\otimes\zeta,\zeta)\,d\zeta]$$

$$\cdot[dX_{\zeta'}-u(\zeta'\otimes\zeta,\zeta')\,d\zeta']-\rho(\zeta,\zeta')\,d\zeta\,d\zeta'\} \tag{5.4}$$

where $f$ is any bounded deterministic function, then $Y$ is a weak martingale with respect to $(\{\mathfrak{F}_{xz}\},\mathscr{P})$, or equivalently, $Y_z L_z$ is a weak martingale with respect to $(\{\mathfrak{F}_{xz}\},\mathscr{P}_0)$. To do this we follow the procedure of Section 2, by first writing $Y_z$ and $L_z$ in the form of (2.22) for $I_h$ and then representing the integrands as stochastic integrals of the form (2.24-1).

Define

$$v(z,\zeta')=\int_{R_z} I(\zeta\wedge\zeta')f(\zeta,\zeta')[dX_\zeta-\tilde{u}(\zeta'\otimes\zeta,\zeta)\,d\zeta], \tag{5.5-1}$$

$$w(z,\zeta')=\int_{R_z} I(\zeta\wedge\zeta')f(\zeta,\zeta')$$

$$\cdot[u(\zeta'\otimes\zeta,\zeta')(dX_\zeta-\tilde{u}(\zeta'\otimes\zeta,\zeta)\,d\zeta)-\rho(\zeta,\zeta')\,d\zeta]. \tag{5.6-1}$$

Then,

$$Y_z=\int_{R_z} [v(z,\zeta')\,dX_{\zeta'}+w(z,\zeta')\,d\zeta']. \tag{5.7-1}$$

Similarly, we can also write

$$Y_z=\int_{R_z} [\tilde{v}(z,\zeta)\,dX_\zeta+\tilde{w}(z,\zeta)\,d\zeta] \tag{5.7-2}$$

with

$$\tilde{v}(z,\zeta) = \int_{R_z} I(\zeta \wedge \zeta') f(\zeta,\zeta') [dX_{\zeta'} - u(\zeta' \otimes \zeta, \zeta') \, d\zeta'], \tag{5.5-2}$$

$$\tilde{w}(z,\zeta) = \int_{R_z} I(\zeta \wedge \zeta') f(\zeta,\zeta')$$

$$\cdot \{\tilde{u}(\zeta' \otimes \zeta, \zeta) [dX_{\zeta'} - u(\zeta' \otimes \zeta, \zeta') \, d\zeta'] - \rho(\zeta,\zeta') \, d\zeta'\}. \tag{5.6-2}$$

Using (3.4.1) and applying the differentiation rule for 1-semimartingale, we get

$$L_z Y_z = \int_{R_z} L_{\zeta' \otimes z} [v(z,\zeta') + u(z,\zeta') Y_{\zeta' \otimes z}] \, dX_{\zeta'}$$

$$+ \int_{R_z} L_{\zeta' \otimes z} [w(z,\zeta') + u(z,\zeta') v(z,\zeta')] \, d\zeta'. \tag{5.8}$$

From (5.5-1), (5.6-1) and (4.4-1), we get

$$w(z,\zeta') + u(z,\zeta') v(z,\zeta')$$

$$= \int_{R_z} I(\zeta \wedge \zeta') [v(\zeta' \otimes \zeta, \zeta') \rho(\zeta,\zeta') + 2f(\zeta,\zeta') u(\zeta' \otimes \zeta, \zeta')]$$

$$\cdot [dX_\zeta - \tilde{u}(\zeta' \otimes \zeta, \zeta) \, d\zeta].$$

It follows from (4.1) that

$$L_{\zeta' \otimes z} [w(z,\zeta') + u(z,\zeta') v(z,\zeta')]$$

$$= \int_{R_z} I(\zeta \wedge \zeta') L_{\zeta' \otimes \zeta} [\tilde{u}(uv+w) + (v\rho + 2fu)] \, dX_\zeta$$

where the arguments of the functions in the integrand are $(\zeta,\zeta')$ for $f$ and $\rho$, $(\zeta' \otimes \zeta, \zeta')$ for $u$, $v$ and $w$, and $(\zeta' \otimes \zeta, \zeta)$ for $\tilde{u}$. Thus, (5.8) can now be written as

$$L_z Y_z = \int_{R_z} L_{\zeta' \otimes z} [v(z,\zeta') + u(z,\zeta') Y_{\zeta' \otimes z}] \, dX_{\zeta'} + \int_{R_z} G(z,\zeta) \, dX_\zeta.$$

Symmetry dictates that $G(z,\zeta)$ must be such that

$$L_z Y_z = \int_{R_z} L_{\zeta' \otimes z} [v(z,\zeta') + u(z,\zeta') Y_{\zeta' \otimes z}] \, dX_{\zeta'}$$

$$+ \int_{R_z} L_{z \otimes \zeta} [\tilde{v}(z,\zeta) + \tilde{u}(z,\zeta) Y_{z \otimes \zeta}] \, dX_\zeta \tag{5.9}$$

which is clearly a weak martingale with respect to $\mathscr{P}_0$.

# 6. Random Signal in Additive White Gaussian Noise

The following situation often arises in signal processing problems. The observation is represented by a process $\xi_z$

$$\xi_z = \theta_z + \eta_z$$

where $\theta$ is a random process representing the signal and $\eta$ is a white Gaussian noise. To deal with such a model, we can integrate both sides of the equation and get

$$X_z = \int_{R_z} \theta_\zeta \, d\zeta + W_z, \quad z \in R_{z_0} \tag{6.1}$$

where $X$ represents the observed process and $W$ is a Wiener process. Let $(\Omega, \mathfrak{F}, \mathscr{P})$ be the probability space in which the processes $X$, $\theta$ and $W$ are defined. For problems in signal detection and filtering it is useful to introduce a probability measure $\mathscr{P}_0$ on $(\Omega, \mathfrak{F})$ with respect to which $X$ itself is a Wiener process.

**Lemma.** *Let $(\Omega, \mathfrak{F}, \mathscr{P})$ be a probability space and let $\{\mathfrak{F}_z, z \in R_{z_0}\}$ be a family of $\sigma$-fields such that $\theta_z$ is $\mathfrak{F}_z$-measurable for each $z$ and $\{W_z, z \in R_{z_0}\}$ is a standard Wiener process with respect to $\{\mathfrak{F}_z\}$. Define*

$$V_z = \exp\left\{ -\int_{R_z} \theta_\zeta \, dW_\zeta - \tfrac{1}{2} \int_{R_z} \theta_\zeta^2 \, d\zeta \right\} \tag{6.2}$$

*and assume that $|\theta_\zeta(\omega)| \leqq c$ for almost all $(\zeta, \omega)$. Then for $\alpha \geqq 1$ we have*

$$1 \leqq EV_z^\alpha \leqq \exp\left[ \left( \frac{\alpha^2 - \alpha}{2} \right) c^2 \operatorname{Area}(R_z) \right]. \tag{6.3}$$

*Proof.* Using the differentiation rule (B.2-1), we can write

$$V_z^\alpha = 1 - \int_{R_z} V_{\zeta \otimes z}^\alpha \, \theta_\zeta \, dW_\zeta + \tfrac{1}{2}(\alpha^2 - \alpha) \int_{R_z} \theta_\zeta^2 \, V_{\zeta \otimes z}^\alpha \, d\zeta.$$

Now set

$$I_n(z) = 1 \quad \text{if } \sup_{\zeta \in R_z}(V_{\zeta \otimes z}) \leqq n$$

$$= 0 \quad \text{otherwise}$$

and define

$$V_{nz} = I_n(z) \, V_z.$$

If we denote

$$U_{nz} = \exp\left\{ -\int_{R_z} \theta_\zeta \, I_n(\zeta \otimes z) \, dW_\zeta - \tfrac{1}{2} \int_{R_z} \theta_\zeta^2 \, I_n(\zeta \otimes z) \, d\zeta \right\}$$

then because $I_n(z) = 1$ implies $I_n(\zeta \otimes z) = 1$ for all $\zeta$ in $R_z$, we have

$$V_{nz}^\alpha = I_n(z) \, V_z^\alpha = I_n(z) \, U_{nz}^\alpha$$

$$\leq U_{nz}^\alpha$$

$$= 1 - \int_{R_z} U_{n,\zeta\otimes z}^\alpha \, \theta_\zeta \, I_n(\zeta\otimes z) \, dW_\zeta$$

$$\qquad + \tfrac{1}{2}(\alpha^2 - \alpha) \int_{R_z} U_{n,\zeta\otimes z}^\alpha \, \theta_\zeta^2 \, I_n(\zeta\otimes z) \, d\zeta$$

$$= 1 - \int_{R_z} V_{n,\zeta\otimes z}^\alpha \, \theta_\zeta \, dW_\zeta + \tfrac{1}{2}(\alpha^2 - \alpha) \int_{R_z} \theta_\zeta^2 \, V_{n,\zeta\otimes z}^\alpha \, d\zeta$$

and

$$E\dot{V}_{nz}^\alpha \leq 1 + \tfrac{1}{2}(\alpha^2 - \alpha) \, c^2 \int_{R_z} EV_{n,\zeta\otimes z}^\alpha \, d\zeta$$

or

$$EV_{n,st}^\alpha \leq 1 + \frac{c^2}{2}(\alpha^2 - \alpha) t \int_0^s EV_{n,\sigma t}^\alpha \, d\sigma$$

and

$$EV_{n,st}^\alpha \leq \exp\frac{c^2}{2}(\alpha^2 - \alpha) \, t \, s$$

and the right hand side of (5.3) follows from Fatou's lemma.

Since $\int_{R_z} E[V_{\zeta\otimes z}^\alpha \, \theta_\zeta]^2 \, d\zeta < \infty$, the stochastic integral $\int_{R_z} V_{\zeta\otimes z}^\alpha \, \theta_\zeta \, dW_\zeta$ has zero mean so that

$$EV_z^\alpha = 1 + \tfrac{1}{2}(\alpha^2 - \alpha) \int_{R_z} E(\theta_\zeta^2 \, V_{\zeta\otimes z}^\alpha) \, d\zeta \geq 1.$$

**Theorem 6.1.** *Under the conditions of the above lemma, define a measure* $\mathscr{P}_0$ *by*

$$\frac{d\mathscr{P}_0}{d\mathscr{P}} = V_{z_0}$$

*where* $V_z$ *is given by (6.2). Define* $X_z$ *by (6.1). Then,*
   (a) $\mathscr{P}_0$ *is a probability measure,*
   (b) $X_z$ *is a Wiener process under* $\mathscr{P}_0$,
   (c) $\mathscr{P}_0 \sim \mathscr{P}$ *and*

$$\frac{d\mathscr{P}}{d\mathscr{P}_0} = \exp\{\int_{R_{z_0}} \theta_\zeta \, dX_\zeta - \tfrac{1}{2} \int_{R_{z_0}} \theta_\zeta^2 \, d\zeta\}. \tag{6.4}$$

*Proof.* (a) From (6.3) we have $EV_{z_0} = 1$. Since $V_{z_0}$ is clearly positive, $\mathscr{P}_0$ is a probability measure.
   (b) To prove $X_z$ is a $\mathscr{P}_0$-Wiener process it is enough to show that

$$E_0 \exp\{i \int_{R_{z_0}} u(\zeta) \, dX_\zeta\} = \exp\{-\tfrac{1}{2} \int_{R_{z_0}} u^2(\zeta) \, d\zeta\}$$

for all bounded deterministic $u$. Now,

$$E_0 \exp\{i \int_{R_{z_0}} u(\zeta)\, dX_\zeta\} = E[V_{z_0} \exp\{i \int_{R_{z_0}} u(\zeta)\, dX_\zeta\}]$$

$$= \exp\{-\tfrac{1}{2} \int_{R_{z_0}} u^2(\zeta)\, d\zeta\}\, E[\exp\{-\int_{R_{z_0}} [\theta_\zeta - iu(\zeta)]\, dW_\zeta - \tfrac{1}{2} \int_{R_{z_0}} [\theta_\zeta - iu(\zeta)]^2\, d\zeta\}].$$

Since $u$ is bounded (by $u_0$ say)

$$|\exp\{-\int_{R_{z_0}} [\theta_\zeta - iu(\zeta)]\, dW_\zeta - \tfrac{1}{2} \int_{R_{z_0}} [\theta_\zeta - iu(\zeta)]^2\, d\zeta\}|$$

$$= [V_{z_0}| \exp\{\tfrac{1}{2} \int_{R_{z_0}} u^2(\zeta)\, d\zeta\} \leq V_{z_0} \exp\{\tfrac{1}{2} u_0^2\, \text{Area}(R_{z_0})\}.$$

Hence,

$$\exp\{-\int_{R_z} [\theta_\zeta - iu(\zeta)]\, dW_\zeta - \tfrac{1}{2} \int_{R_z} [\theta_\zeta - iu(\zeta)]^2\, d\zeta\}$$

is a square-integrable $\mathscr{P}$-martingale and

$$E_0 \exp\{i \int_{R_{z_0}} u(\zeta)\, dX_\zeta\} = \exp\{-\tfrac{1}{2} \int_{R_{z_0}} u^2(\zeta)\, d\zeta\}$$

as was to be proved.

(c) Since $X$ is a $\mathscr{P}_0$-Wiener process and $\theta$ is bounded

$$\frac{1}{V_z} = \exp\{\int_{R_z} \theta_\zeta\, dX_\zeta - \tfrac{1}{2} \int_{R_z} \theta_\zeta^2\, d\zeta\}$$

must satisfy (5.3) with $E_0$ replacing $E$ and $\dfrac{1}{V_z}$ replacing $V_z$. Thus,

$$E_0 \left(\frac{1}{V_{z_0}}\right) = 1$$

and part (c) is proved.

Now, let $\mathfrak{F}_{xz}$ denote the $\sigma$-subfield generated by $\{X_\zeta, \zeta \in R_z\}$ and denote

$$L_z = E_0 \left(\frac{d\mathscr{P}}{d\mathscr{P}_0}\bigg| \mathfrak{F}_{xz}\right). \tag{6.5}$$

Let

$$\Lambda_z = \frac{1}{V_z} = \exp\{\int_{R_z} \theta_\zeta\, dX_\zeta - \tfrac{1}{2} \int_{R_z} \theta_\zeta^2\, d\zeta\} \tag{6.6}$$

which is of the form (3.8) with $\rho \equiv 0$. Hence, (3.9) and (3.4) yield

$$\Lambda_z = 1 + \int_{R_z} \Lambda_\zeta \theta_\zeta\, dX_\zeta + \int_{R_z \times R_z} \Lambda_{\zeta' \otimes \zeta} \theta_\zeta \theta_{\zeta'}\, dX_\zeta\, dX_{\zeta'}. \tag{6.7}$$

which was also derived in [4].

Now, denote

$$\hat{\theta}(\zeta|z) = E(\theta_\zeta | \mathfrak{F}_{xz}) \tag{6.8}$$

and

$$R(\zeta, \zeta'|z) = E[(\theta_\zeta - \hat{\theta}(\zeta|z))(\theta_{\zeta'} - \hat{\theta}(\zeta'|z))].$$

Then our principal results on the likelihood formulas can be summarized as follows:

**Theorem 6.2.** *Under the conditions of Theorem 6.1, the likelihood ratio $L_z$ defined by (6.5) has the alternative representation*

$$
L_z = 1 + \int_{R_z} L_\zeta \, \hat{\theta}(\zeta|\zeta) \, dX_\zeta
$$
$$
+ \int_{R_z \times R_z} L_{\zeta' \otimes \zeta} [\hat{\theta}(\zeta|\zeta' \otimes \zeta) \, \hat{\theta}(\zeta'|\zeta' \otimes \zeta) + R(\zeta, \zeta'|\zeta' \otimes \zeta)] \, dX_\zeta \, dX_{\zeta'}. \tag{6.10}
$$

*and*

$$
L_z = \exp\Big\{ \int_{R_z} \hat{\theta}(\zeta|\zeta) \, dX_\zeta - \tfrac{1}{2} \int_{R_z} \hat{\theta}^2(\zeta|\zeta) \, d\zeta - \tfrac{1}{2} \int_{R_z \times R_z} R^2(\zeta, \zeta'|\zeta' \otimes \zeta) \, d\zeta \, d\zeta'
$$
$$
+ \int_{R_z \times R_z} R(\zeta, \zeta'|\zeta' \otimes \zeta) [dX_\zeta - \hat{\theta}(\zeta|\zeta' \otimes \zeta) \, d\zeta] [dX_{\zeta'} - \hat{\theta}(\zeta'|\zeta' \otimes \zeta) \, d\zeta'] \Big\}. \tag{6.11}
$$

*Furthermore, the conditional moment $\hat{\theta}$ satisfies the equations*

$$
\hat{\theta}(\zeta'|\zeta' \otimes z) = \hat{\theta}(\zeta'|\zeta') + \int_{R_z} I(\zeta \wedge \zeta') R(\zeta, \zeta'|\zeta' \otimes \zeta) [dX_\zeta - \hat{\theta}(\zeta|\zeta' \otimes \zeta) \, d\zeta]
$$
$$
\hat{\theta}(\zeta|z \otimes \zeta) = \hat{\theta}(\zeta|\zeta) + \int_{R_z} I(\zeta \wedge \zeta') R(\zeta, \zeta'|\zeta' \otimes \zeta) [dX_{\zeta'} - \hat{\theta}(\zeta'|\zeta' \otimes \zeta) \, d\zeta']. \tag{6.12}
$$

*Proof.* First, we note that $L_z = E_0[\Lambda_z | \mathscr{F}_{xz}]$. Hence, from (6.7) we have

$$
L_{z0} = 1 + E_0\Big[ \int_{R_{z0}} \theta_\zeta \, \Lambda_\zeta \, dX_\zeta \Big| \mathscr{F}_{xz0} \Big]
$$
$$
+ E_0\Big[ \int_{R_{z0} \times R_{z0}} \theta_\zeta \, \theta_\zeta \, \Lambda_{\zeta' \otimes \zeta} \, dX_\zeta \, dX_{\zeta'} \Big| \mathscr{F}_{xz0} \Big].
$$

Let $S_n$ be a sequence of rectangular partitions of $R_{z0}$, i.e.,

$$
S_n = \{\zeta_{ij}^{(n)}\}, \qquad \zeta_{ij}^{(n)} = (a_i^{(n)}, b_j^{(n)})
$$

with

$$
\max_i (\max(a_{i+1}^{(n)} - a_i^{(n)}), \quad \max_j (b_{n+1}^{(n)} - b_j^{(n)})) \xrightarrow[n \to \infty]{} 0.
$$

For $\zeta = \zeta_{ij}^{(n)}$ denote $\Delta_\zeta = (\zeta_{i,j}^{(n)}, \zeta_{i+1, j+1}^{(n)}]$. Then,

$$E_0\left[\int_{R_{z_0}}\theta_\zeta\, \Lambda_\zeta\, dX_\zeta|\mathscr{F}_{xz_0}\right]=E_0\left[\lim_{n\to\infty}\text{q.m.}\sum_{\zeta\in S_n}\theta_\zeta\, \Lambda_\zeta\, X(\Delta_\zeta)|\mathscr{F}_{xz_0}\right]$$

$$=\lim_{n\to\infty}\text{q.m.}\ E_0\left[\sum_{\zeta\in S_n}\theta_\zeta\, \Lambda_\zeta\, X(\Delta_\zeta)|\mathscr{F}_{xz_0}\right]$$

$$=\lim_{n\to\infty}\text{q.m.}\sum_{\zeta\in S_n}E_0(\theta_\zeta\, \Lambda_\zeta|\mathscr{F}_{xz_0})\, X(\Delta_\zeta).$$

Write $\mathscr{F}_{xz_0}=\mathscr{F}_{x\zeta}\vee\mathscr{F}_{x\zeta}^+$. Because under $\mathscr{P}_0$, $X$ is a Wiener process, $\mathscr{F}_{x\zeta}^+$ and $\mathscr{F}_\zeta$ are $\mathscr{P}_0$ independent. Hence,

$$E_0(\theta_\zeta\, \Lambda_\zeta|\mathscr{F}_{xz_0})=E_0(\theta_\zeta\, \Lambda_\zeta|\mathscr{F}_{x\zeta})$$

$$=\hat{\theta}(\zeta|\zeta)\, L_\zeta.$$

Now, rather than defining $\hat{\theta}(\zeta|\zeta)$ as a Radon-Nikodym derivative of

$$\mu(A)=\int_A\theta_\zeta(\omega)\,\mathscr{P}(d\omega),\qquad A\in\mathscr{F}_{x\zeta}$$

with respect to $\mathscr{P}$ for a fixed $\zeta$, define $\hat{\theta}(\zeta|\zeta)$ as a Radon-Nikodym derivative of

$$\tilde{\mu}(A)=\int_A\theta(z,\omega)\,\mathscr{P}(d\omega)\, dz,\qquad A\in G_x$$

with respect to $d\mathscr{P}\, dz$ measure, where $G_x$ is the $\sigma$-field of $(z,\omega)$ sets generated by $\{\mathscr{F}_{xz}\}$ progressively measurable processes. Then, $\hat{\theta}$ is by definition progressively measurable with respect to $\{\mathscr{F}_{xz}\}$ and

$$E_0\left[\int_{R_{z_0}}\theta_\zeta\, \Lambda_\zeta\, dX_\zeta|\mathscr{F}_{xz_0}\right]=\lim_{n\to\infty}\text{q.m.}\sum_{\zeta\in S_n}\hat{\theta}(\zeta|\zeta)\, L_\zeta\, X(\Delta\zeta)$$

$$=\int_{R_{z_0}}\hat{\theta}(\zeta|\zeta)\, L_\zeta\, dX_\zeta.$$

Similarly,

$$E_0\left[\int_{R_{z_0}\times R_{z_0}}\theta_\zeta\theta_{\zeta'}\, \Lambda_{\zeta'\otimes\zeta}\, dX_\zeta\, dX_{\zeta'}|\mathscr{F}_{xz_0}\right]$$

$$=E_0\left[\lim_{n\to\infty}\text{q.m.}\sum_{\substack{\zeta\wedge\zeta'\\ \zeta,\zeta'\in S_n}}\theta_\zeta\theta_{\zeta'}\, \Lambda_{\zeta'\otimes\zeta}\, X(\Delta\zeta)\, X(\Delta\zeta')|\mathscr{F}_{xz_0}\right]$$

$$=\lim_{n\to\infty}\text{q.m.}\sum_{\substack{\zeta\wedge\zeta'\\ \zeta,\zeta'\in S_n}}E_0[\theta_\zeta\theta_{\zeta'}\, \Lambda_{\zeta'\otimes\zeta}|\mathscr{F}_{xz_0}]\, X(\Delta\zeta)\, X(\Delta\zeta')$$

$$=\lim_{n\to\infty}\text{q.m.}\sum E_0[\theta_\zeta\theta_{\zeta'}\, \Lambda_{\zeta'\otimes\zeta}|\mathscr{F}_{x\zeta'\otimes\zeta}\, X(\Delta\zeta)\, X(\Delta\zeta')$$

$$=\lim_{n\to\infty}\text{q.m.}\sum L_{\zeta'\otimes\zeta}\, E[\theta_\zeta\theta_{\zeta'}|\mathscr{F}_{x\zeta'\otimes\zeta}]\, X(\Delta\zeta)\, X(\Delta\zeta')$$

$$=\int_{R_{z_0}\times R_{z_0}}L_{\zeta'\otimes\zeta}[\hat{\theta}(\zeta'|\zeta'\otimes\zeta)\,\hat{\theta}(\zeta'|\zeta'\otimes\zeta)+R(\zeta,\zeta'|\zeta'\otimes\zeta)]\, dX_\zeta\, dX_{\zeta'}.$$

Hence, (6.10) is proved, and (6.11) follows from (4.8) if we compare (6.10) with (4.9).

We note that the functions $\theta$, $u$ and $\tilde{u}$ in (4.9) are now identified as follows:

$$\theta(\zeta) = \hat{\theta}(\zeta|\zeta),$$

$$u(\zeta' \otimes \zeta, \zeta') = \hat{\theta}(\zeta'|\zeta' \otimes \zeta),$$

$$\tilde{u}(\zeta' \otimes \zeta, \zeta) = \theta(\zeta|\zeta' \otimes \zeta).$$

Therefore, (4.4) takes on the form of (6.12).

Finally, if $\theta$ and $W$ are jointly Gaussian under $\mathscr{P}$, then $R(\zeta, \zeta'|\zeta' \vee \zeta)$ is a deterministic function. By using (6.12-2) in (6.12-1), we get for $z'_2 > z_2$

$$\hat{\theta}(z|z_1, z'_2) = \hat{\theta}(z|z)$$

$$+ \int_0^{z_1} \int_{z_2}^{z_2} R(\zeta, z|z \times \zeta)[dX_\zeta - \hat{\theta}(\zeta|\zeta)\, d\zeta]$$

$$- \int_{\zeta \in R_z} \int_{\zeta' \in (z_2, z'_2)} \left[ \int_0^{\zeta_1} R(\zeta', z|z \times \zeta') \right.$$

$$\left. \cdot R(\zeta', \zeta|\zeta \times \zeta')\, d\zeta'_1 \right] \hat{\theta}(\zeta|\zeta_1, \zeta_2)\, d\zeta\, d\zeta_2$$

which is a linear equation in $\{\hat{\theta}(z|z_1, z'_2),\ z \in R_{z_0},\ z'_2 > z_2\}$ with a deterministic kernel. Hence, given $\{\hat{\theta}(z|z),\ z \in R_{z_0}\}$, $\{\hat{\theta}(z|z_1, z'_2),\ z'_2 > z_2,\ z \in R_{z_0}\}$ is uniquely determined, and by symmetry so is $\{\hat{\theta}(z|z'_1, z_2),\ z'_1 > z_1,\ z \in R_{z_0}\}$. It follows that $\{L_z, z \in R_{z_0}\}$ is completely determined by $\hat{\theta}(z|z),\ z \in R_z$. This implies, for example, that a detector for testing between the hypotheses:

H: $X_z = \int_{R_z} \phi_\zeta\, d\zeta + W_z$ and $W$ is a Wiener process,

$H_0$: $X_z$ is a Wiener process

can be implemented by a filtering operation which yields $\hat{\theta}(z|z),\ z \in R_{z_0}$. Equation (6.12) represents a constraint on the various conditional moments. The existence of such a constraint is surprising and could hardly have been predicted a priori. As such, (6.12) has considerable interest in its own right.

We observe that a natural concomitant of the likelihood ratio formulas is the behavior of martingales under such transformation of measures. Theorems of the Girsanov type [2], representation theorems for martingales and weak martingales are all to be expected. Much of this body of results is already in hand and will be reported in a subsequent paper.

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