TEAMS, SIGNALING, AND INFORMATION THEORY*

Y. C. Ho
Pierce Hall G12j
Division of Applied Sciences
Harvard University
Cambridge, MA 02138

Marcia P. Kastner
College of Engineering
Boston University
110 Cummington Street
Boston, MA 02215

Eugene Wong
Department of Electrical Engineering and Computer Sciences
University of California
Berkeley, CA 94720

Abstract

The purpose of this paper is to unify results from three separate and at least superficially unrelated subject matters, namely, team decision theory, market signaling in economics, and the classical Shannon information theory.

1. Introduction

The study of the interaction between information and decision in many-person optimization problems called team theory was initiated by Marschak in the '50's. More recently, this has been extended and unified with work on decentralized or nonclassical stochastic control theory which emphasized the role of information structure in problems involving dynamics, or sequential order of actions. During the same period sporadic and not too successful attempts have been made to relate Shannon's information theory with feedback control system design. Again with the recent maturity of control theory as a subject in applied mathematics, the two disciplines begin to exhibit much closer connection than heretofore displayed, e.g. the Viterbi algorithm and the Kalman-Bucy filter, the recent work of Whittle and Rudge [9]. Lastly, one of the current interests in mathematical economics is associated with the role of information in organizations and the market place. Various interesting phenomena arise as a result of imperfect or incomplete information in person-to-person interactions. The purpose of this paper is to attempt to weave a common thread among these three apparently unrelated subjects: team theory, market signaling, and information theory. While no particularly significant new results are obtained, we believe the conceptual unity displayed here is new and hopefully will lead to much future cooperative efforts among researchers in these different fields.

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2. Fundamentals of Information Structure and Decentralized Decision Making

There are five basic ingredients of decision theory.

(1) The state of the world $\xi \equiv [\xi_1,\ldots,\xi_n] \in \Omega$ which can be thought of as a vector of random variables defined on a probability space having a density (or distribution or measure) $p(\xi)$. ξ represents all the uncertainties in the problem under consideration, e.g. unknown initial conditions, measurement noise, uncertain parameters, etc.

(2) A set of decision variables $u \equiv [u_1,\ldots,u_m] \in U$ each representing one decision maker (DM). One person making two decisions at different times is regarded as two DMs in this setup.

(3) A loss (payoff) function which is a measurable function of u and ξ i.e. $L(u,\xi)$. We assume L is expressed in appropriate utility units.

(4) A set of information functions $z \equiv \eta(\xi) \in Z \equiv [\eta_1(\xi),\ldots,\eta_m(\xi)]$, one for each DM. In other words $z_i \equiv \eta_i(\xi)$, what DM_i knows, is in general different from z_j, what DM_j knows. Alternatively, in place of $\eta(\xi)$, we can be given subalgebras induced by the η's on the underlying probability space. The set of η's or the subalgebras are known as the information structure of the problem.

(5) A set of strategies $\gamma \equiv [\gamma_1,\ldots,\gamma_m] \in \Gamma$, one for each DM, where γ_i is a mapping from the z_i-space to the u_i-space. Thus, each DM must choose actions $u_i = \gamma_i(z_i)$ based on different information. It is in this sense the problem is decentralized.

Since for fixed γ, $E[L(u = \gamma(\eta(\xi)),\xi)]$ is well-defined* and depends on γ, we can state the decision problem as

$$\underset{\gamma\in\Gamma}{\text{Min}}\; J(\gamma) = \underset{\gamma\in\Gamma}{\text{Min}}\; E[L(u=\gamma(\eta(\xi)),\xi)]$$

a deterministic optimization problem in the Γ-space which is usually taken to be the space of all measurable functions from $U \equiv \prod_{i=1}^{m} U_i$ to $Z = \prod_{i=1}^{m} Z_i$.

*provided, of course, γ and η are appropriately measurable functions

This problem is known as the static team problem. It is static in the sense that information z_i available to DM_i depends only on ξ. The evaluation of posterior probability such as $p(\xi/z_i)$ can be separately carried out from the problem of choosing the actions u_i. However, in general when different DMs act at different times, information z_i received later by DM_i may be dependent upon the action u_j of DM_j, who acted earlier. Thus, in general decision problems, we must consider

(4)' $z = \eta(\xi,u) \equiv [\eta_1(\xi,u),\ldots,\eta_m(\xi,u)]$

where η must satisfy some causality conditions [1]. When the team problem is characterized by the information structure (4)' instead of (4), it is called a dynamic team problem [5]. The word dynamic is used to indicate the presence of order of actions of the DMs.

A superficially simple example which we shall use throughout this paper is now stated below.

Let $\xi = [x,v]$ where $x \sim N(0,1)$ and $v \sim N(0,\sigma^2)$, x,v independent.

(6) $L(u,\xi) = \frac{1}{2}(x + au_1 + bu_2)^2 + \frac{1}{2}cu_1^2$

$a, b, c \geq 0$

$z_1 = x$

$z_2 = gx + hu_1 + v \quad g,h \geq 0, \quad h = ga$

One interpretation of the example is that x is the initial condition; the state after DM_1 acts is $x_1 \equiv x + au_1$; similarly $x_2 \equiv x_1 + bu_2$; z_1 is the measurement of the initial state by DM_1, and z_2 is a noisy measurement of a linear transformation of $x_1 = x + au_1$ by DM_2. The objective is to minimize the final state x_2 and the energy, or power, $\frac{1}{2}cu_1^2$ of DM_1, a control-theoretic problem. Note that this information structure is dynamic and that DM_1 can signal or control the knowledge of x to DM_2 through his action $u_1 = \gamma_1(x)$. A rather different interpretation can be given if we take $a = g = 0$, $b = -1$, $h = 1$. In this case, DM_1, knowing x, is trying to transmit a decision u_1, subject to energy constraints, through a noisy media so that DM_2 can act based on z_2 in order to minimize the difference (distortion) between x and u_2. If DM_1 is called the "encoder" and DM_2 the "decoder", then the information-theoretic significance of this interpretation is obvious [7]. In any case, this example appears to be the simplest type of team decision problem which incorporates dynamic information structure*[2] and all its attendant complexities.

With regard to the general team problem, the conditions for optimality are (letting m = 2 for simplicity)

(P-1) $\begin{cases} \text{Find } \gamma_1^*,\gamma_2^* \ni \\ J(\gamma_1^*,\gamma_2^*) \leq J(\gamma_1,\gamma_2) \quad \forall\, \gamma_1,\gamma_2 \in \Gamma \end{cases}$

A necessary condition for γ_1^* and γ_2^* to satisfy (P-1) is that they solve

(P-2) $\begin{cases} \text{Find } \gamma_1^*,\gamma_2^* \ni \\ J(\gamma_1^*,\gamma_2^*) \leq J(\gamma_1,\gamma_2^*) \ \forall \gamma_1 \in \Gamma \\ J(\gamma_1^*,\gamma_2^*) \leq J(\gamma_1^*,\gamma_2) \ \forall \gamma_2 \in \Gamma \end{cases}$

which is known as person-by-person optimality (pbpo) or equilibrium solutions. The reason for the latter terminology becomes clear if we realize that, in general, DM_1 need not necessarily have the same loss function or criterion of performance as DM_2. For i=1,2, let J_i be the criterion of DM_i, and let $J_1 \neq J_2$. No conceptual difficulties are involved if we extend (P-2) to

(P-2)' $\begin{cases} \text{Find } \gamma_1^*,\gamma_2^* \ni \\ J_1(\gamma_1^*,\gamma_2^*) \leq J_1(\gamma_1,\gamma_2^*) \ \forall\, \gamma_1 \in \Gamma \\ J_2(\gamma_1^*,\gamma_2^*) \leq J_2(\gamma_1^*,\gamma_2) \ \forall\, \gamma_2 \in \Gamma \end{cases}$

This is known as Nash equilibrium in the parlance of game theory. If $J_1 \neq J_2$, the problem is called a nonzero-sum (NZS) game. If $J_1 = -J_2 \triangleq J$, the problem is a zero-sum (ZS) game because $J_1 + J_2 = 0$. (P-2)' becomes

(P-3) $\begin{cases} \text{Find } (\gamma_1^*,\gamma_2^*) \ni \\ J(\gamma_1^*,\gamma_2) \leq J(\gamma_1^*,\gamma_2^*) \leq J(\gamma_1,\gamma_2^*) \end{cases}$

the saddle point condition. With this condition, the example problem now admits a game-theoretic interpretation. DM_1 wishes to act to cancel out x without using too much energy, but his action reveals the knowledge of x to DM_2 through z_2: DM_2 wishes to maximize the terminal state $x + au_1 + bu_2$ which he can do if he knows x well.

More will be said about (P-2)' and (P-3) later on in sec. 4 and elsewhere [3]. For the moment let us return to (P-1) and (P-2). The principal difficulties introduced by dynamic information structure (4)' are twofold:

(i) The observation z_2 is not a well-defined random variable until the strategy γ_1 is specified. This makes the various probability measures required in the solution process solution-dependent. There is a vicious circle and the problem of estimation is no longer separable from that of control.

(ii) The optimization problem $\underset{\gamma_1,\gamma_2}{\text{Min}}\ J(\gamma_1,\gamma_2)$ is not necessarily convex in γ_1. This is because γ_1 enters in $J(\gamma_1,\gamma_2(\gamma_1))$ also through $\gamma_2(z_2) = \gamma_2(gx + h\gamma_1(x) + v)$. Since there is no reason

*dynamic in the sense of (4)'.

to expect γ_2 to be convex, there is no assurance that J is convex in γ_1 even though L may be convex in u_1.

Both difficulties were fully investigated by Witsenhausen [4] for the case of (P-1) with a = g = h = 1. Since his seminal work, other efforts have been made to isolate cases where these difficulties can be circumvented [5]. In fact, it can be argued that whatever success we have in optimal stochastic control theory is based on the existence result under the special information structure of perfect memory which bypasses the above mentioned difficulties [6, p. 461].

3. Signaling and Information Theory

In view of the difficulties mentioned above and in [4], it is somewhat surprising that, in fact, something can be done for (P-1). Suppose we take instead the information-theoretic interpretation of the problem (6) with a = g = 0, b = -1, h = 1 (the problem in [4] is the same except that a = g = 1), but with average signal power constrained to be less that or equal to 1 (that is, choose c in (6) appropriately). Then the problem becomes*

$$(A)\begin{cases} \min\limits_{\gamma_1,\gamma_2} E[(x-\gamma_2(\gamma_1(x)+v))^2] \\ \text{s.t. } E[\gamma_1(x)]^2 \leq 1 \\ \text{with } z_1 = x \\ z_2 = u_1 + v \end{cases}$$

with optimal solutions $\gamma_1^*(x) = x$ and $\gamma_2^*(z_2) = E(x/z_2) = \frac{1}{1+\sigma^2}\, z_2$. But (A) is recognized as a special case of Shannon's information-theoretic problem involving a memoryless Gaussian source and an additive memoryless Gaussian channel where the source rate and channel rate are equal. In the language of (A), this means we require the dimensions of x and u_1, if regarded as a vector, to be equal (and in the case of (A), equal to 1). The encoder $\gamma_1(z_1)$, and the decoder $\gamma_2(z_2)$, are instantaneous and linear. Here is a situation where information theory provided a solution to a dynamic team problem which, in the absence of this knowledge, would have been most difficult to solve.+

This interesting connection between team and information theory can be further exploited by considering several variants of (A). First, consider the general case with dimension (x) = n and dim (u_1) = m. In other words, the source has block length n and the channel block length m. Then (A) becomes

$$(A)'\begin{cases} \min\limits_{\gamma_1,\gamma_2} \frac{1}{n} E[(x-\gamma_2(\gamma_1(x)+v))^T(x-\gamma_2(\gamma_1(x)+v))] \\ \text{s.t. } \frac{1}{m} E[\gamma_1(x)^T\gamma_1(x)] \leq 1 \\ z_1 = x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \triangleq \begin{bmatrix} z_{11} \\ \vdots \\ z_{1n} \end{bmatrix}, \; x_i \sim N(0,1),\ x_i \text{ and } x_j \text{ independent for } i\neq j \\ z_2 = u_1+v = \begin{bmatrix} u_{11}+v_1 \\ \vdots \\ u_{1m}+v_m \end{bmatrix} \triangleq \begin{bmatrix} z_{21} \\ \vdots \\ z_{2m} \end{bmatrix}, \; v_i \sim N(0,\sigma^2),\ v_i \text{ and } v_j \text{ independent for } i\neq j \end{cases}$$

The variants of (A)' that we will consider are as follows:

$$(A\text{-}1)\begin{cases} \text{general } n \text{ and } m \\ \text{let } n\to\infty, \text{ but } \frac{n}{m} = \text{constant} \end{cases}$$

(A-2) $n=m$

(A-3) $n=1,\ m=2$

(A-4) $n=2,\ m=1$

$$(A\text{-}5)\begin{cases} n=1,\ m=2, \text{ but } z_1 = \begin{bmatrix} x \\ z_{21} \end{bmatrix} \\ \text{Then } u_{11} = \gamma_{11}(x) \\ \qquad\ u_{12} = \gamma_{12}(x, z_{21}) \end{cases}$$

Problem (A-1) is a statement of the well-known Shannon information theory problem [8, p. 911] where one is only concerned with the rate of information transmission (n/m = constant) but initial delay is acceptable ($n\to\infty$). The optimum performance (minimum distortion) J* is known and is obtained when one equates $R_{eq}(\beta)$, the equivalent source rate for a given distortion β, to the equivalent channel rate $C_{eq}(\alpha)$ for a given signal power level α [8]. That is, $J^*=\beta^*$, where β^* is defined by $R_{eq}(\beta^*) = C_{eq}(\alpha)$. However, the encoder and decoder (γ_1^*, γ_2^*) pair to realize J* is still unknown. Problem (A-1) can be generalized considerably by allowing memory in both the source and the channel. But even with memory, within the linear (memory structure)-quadratic (distortion)-Gaussian (source and channel) setup (LQG), the minimum distortion can still be obtained from Shannon theory (see Whittle and Rudge [9]).

Once we leave problem (A-1), it may be argued that we have entered the realm of "real-time information theory." In problems (A-2) through (A-5), we are not allowed to encode a large number of messages (x's) together before transmission. Arbitrary delay is not permitted. Equivalently, the block length is fixed. The emphasis here is more decision-theoretic. However, much information-theoretic insight can still be borrowed to provide solutions or partial solutions to these problems, as the following discussion will show.

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*A similar setup has been proposed by Witsenhausen [7], Whittle and Rudge [9], and Wyner [8].

+There exist no general sufficiency conditions to verify the optimality of a solution besides the Shannon bounds.

As mentioned earlier, Problem (A-2) corresponds to the situation where the channel rate and source rate are equal. The asymptotic results are the same as if the vector, or block, lengths are fixed: the optimal encoder and decoder are linear [9], [21]. This is not true if $n \neq m$, as will now be discussed.

Problem (A-3) is the prototype of situations where one is allowed to signal more than once for each piece of information he wishes to send. In the language of communication, we are allowed to trade bandwidth for performance. Both of the following are optimal linear strategies: $u = \begin{bmatrix} x \\ x \end{bmatrix}$ and $u = \begin{bmatrix} \sqrt{2}\, x \\ 0 \end{bmatrix}$. The latter clearly shows that the only gain is in increasing power and not in making use of the expanded bandwidth. Hence, far better nonlinear strategies must exist, and a construction of a near-optimal strategy in the small noise case can be obtained by using Shannon's twisted modulation idea [22], [23]. It should be pointed out that the optimum J^* is not even known in this case, although a lower bound is possible via the Shannon theory. Whether or not a better bound is possible with a different definition of mutual information in the spirit of Ziv and Zakai [10] is an open question.

Problem (A-4) is the opposite of (A-3) and is representative of source coding, where data compression is desired. In many respects, it is similar to a problem in optimal quantization. From topological considerations, we know that if the mapping is to be invertible in the absence of noise, then it cannot be continuous. Hence, even without considering the effect of noise in detail, we know that an optimal mapping must be nonlinear*. Similar remarks on J^* apply here as in (A-3).

Problem (A-5) is the same as (A-3) except that noiseless feedback is allowed. DM_1 is allowed to send the second signal based on x <u>and</u> z_{21}. It turns out that the solution to this problem is known. The best (γ_1^*, γ_2^*) is linear for (A-5) and realizes the Shannon bound in real time [11, 12, 13]. The solution consists of sending x as the first signal, then sending an amplified innovation term $x - E(x/z_{21}) \equiv \nu$ which is independent of z_{21} as the second signal, resulting in $z_{22} = k\nu + v_2$, where $k = (\frac{1+\sigma^2}{2})^{1/2}$ is the amplification factor such that $E(k^2 \nu^2) = 1$. Heuristically, this result can be understood in terms of our knowledge of (A) where dim x = 1 = dim u. Since the innovation term is independent of the first received signal z_{21}, the sender, in sending the second signal, essentially faces a new (A) type of problem which is known to possess linear solutions. Roughly speaking, we have transformed via noiseless feedback a problem <u>of unequal source and channel</u> rates to that of equal rates.

*See the Appendix for examples of (A-3) and (A-4) where explicit nonlinear schemes are illustrated and which are better than the best linear schemes.

4. Zero-and Nonzero-Sum Versions of Signaling

While none of the results in the previous sections taken by themselves are particularly significant, taken together they do provide considerable insight into the relationship between nonclassical decision and control theory on the one hand and Shannon's information theory on the other. We see how knowledge in one subject provides solutions in another. In fact, results in (A-1) - (A-5) form a significant portion of <u>all</u> the nontrivial knowledge concerning explicit solutions to dynamic information structure problems. Information-theoretic results play a crucial role in the solution or partial solution of these problems. On the other hand, viewed in this light, we also realize that the information theory problem is a very special kind of problem in dynamic teams. In a sense, it is the simplest kind of such problem: only two DMs are involved and are explicitly and exclusively concerned with signaling. As we have mentioned briefly in section 2, a natural generalization to other classes of dynamic information structure problem exist. This development will be pursued now.

Once we consider the nonzero- or zero-sum version of the signaling problem, (P-2)' and (P-3) become the governing conditions of optimality. This permits a considerable simplification. Either one of the two inequalities of (P-2)' or (P-3) defines a <u>one-person</u> decision problem for <u>fixed</u> strategy of the other DM. The difficulties of solution-dependent convexity discussed in section 2 are largely ameliorated. One still has to solve a pair of implicit equations in (γ_1, γ_2). But this is a much simpler task as the discussion below will show.

A current problem of interest in economics is that of market signaling by Spence [14]. In terms of our basic formulation, the problem can be stated as follows. An employer must hire someone for a job without knowing how productive that individual will be. In other words, the employer has imperfect information about an individual's ability. Spence suggests that the employer can improve his information by looking on the job application for some signal, such as educational level. The employer offers wages based on the signal he sees; that is, a person with more education is offered higher wages, because the employer believes that the higher education indicates higher ability. The individual applying for the job, on the other hand, knowing he will receive wages based on his educational level, must decide how much education to get, taking into consideration that education is costly. Let DM_1 = all potential employees considered together, DM_2 = the employer, x = an individual's ability (known to that individual, but not to the employer), u_1 = educational level, and u_2 (or u_2 + noise) = wages. The payoff or loss function of DM_2 is $[x-u_2]^2$; he does not wish to overpay or underpay with respect to x. The payoff of DM_1 is simply the net profit $u_2 - c(u_1, x)$ where c is the cost of signaling. Thus, we have precisely the following example, where the appropriate optimality conditions are (P-2)'.

$$(B)\begin{cases} \max J_1 = E[\gamma_2(z_2) - c(\gamma_1(z_1),x)] \\ \min J_2 = E[(\gamma_2(z_2) - x)^2] \\ \text{where } z_1 = x \\ \qquad z_2 = u_1 \text{ or } u_1 + v \\ p(x,v) \text{ Given} \end{cases}$$

A reasonable special case of (B) is for $c(u_1,x) = \frac{u_1}{x}$, $p(x,v) = p(x)p(v)$ each being a uniform distribution, U_1 = a discrete set, and $U_2 = R^+$. Under this and other similar set-ups, equilibrium solutions γ_1^*,γ_2^* can be obtained. The details and economic interpretations are available elsewhere [3], [21]. Two noteworthly features of the solution are worth mentioning.

(i) There are multiple equilibrium solutions (γ_1^*,γ_2^*). This is a phenomenon that seems to occur only with dynamic information structure. Essentially, the equilibrium conditions are not sufficiently constraining, so that a large number of (γ_1,γ_2) pairs can satisfy them. It is for the same reason that team solutions satisfying (P-2) do not usually produce solutions which also satisfy (P-1) in the case $J_1 = J_2$. (P-2) is far from sufficient a condition. On the other hand, in static team problems, (P-2), under reasonable conditions on J, often turns out to be necessary <u>and</u> sufficient [2].

(ii) There are threshold phenomena in market signaling. If the cost of signaling is too high, or the signaling channel too noisy, or the underlying signal x itself too predictable, then signaling will suddenly cease altogether, i.e. $u_1 = 0$. This phenomenon may be due to the nonzero-sum nature of the problem. In the cooperative case of information theory, it is always worthwhile to send some message, at least in the Gaussian case.

Finally, we can consider the case of $J_1 = -J_2$. As described earlier, we have a situation of "anti-signaling". A prototype problem can be formulated by slight modification of (6)

$$(C)\begin{cases} \text{Find the saddle point pair } (\gamma_1^*,\gamma_2^*) \text{ for} \\ L(u,\xi) = \frac{1}{2}(x+au_1+bu_2)^2 + \frac{c}{2}u_1^2 - \frac{d}{2}u_2^2 \\ z_1 = x \\ z_2 = gx + hu_1 + v \\ a, b, g, h \geq 0 \qquad c, d > 0 \\ x \sim N(0,1) \qquad v \sim N(0,\sigma^2),\ x,v \text{ independent} \end{cases}$$

(C) is different from (6) only in the addition of the $-\frac{d}{2}u_2^2$ term in $L(u,\xi)$ and maximization with respect to $u_2 = \gamma_2(z_2)$ instead of minimization. In addition to the advantage of solving only for equilibrium solutions, we have the added structure of $J_1 = -J_2$. Any saddle point solution is as good globally as any other solution on the product set of admissible (γ_1,γ_2) solutions by virtue of interchangability [15, p. 66]. Linear or affine saddle point strategies can be obtained for (C). In fact, (C) can be generalized considerably to include state (as well as information) dynamics resulting in a stochastic differential game problem and solved similarly [16, 17]. Other non-LQG setups are also possible [18, 19]. The underlying idea of solutions is apparently a tradeoff between "revealing knowledge of x through u_1" vs. "achieving some desirable payoff through u_1".

In terms of information-theoretic ideas, a possible further tie with discussion in this paper is through the problem of crytography [20] which clearly embodies the concept of "anti-signaling". However, we shall leave the formulation and unification of these ideas to future work of interested parties.

5. <u>Conclusion</u>

The previous discussions can be summarized in Figure 1. Considerable obvious and easy generalizations of the results to (A-1) - (A-5) are possible. However, nothing conceptually new is added.

Several conclusions can be drawn from this study:

(i) Simple two-person decision problems with dynamic information structure have many interesting areas of application, such as real-time (fixed block length) encoder and decoder design, economic theory, and (possibly) cryptography.

(ii) Dynamic information structure leads to a new kind of deterministic optimization problem in which composition of functions is involved, namely, $J(\gamma_1,\gamma_2(\gamma_1))$. No reasonable algorithm seems to exist for this class of problems.

Figure 1 follows the references.

References

[1] H. S. Witsenhausen, "On Information Structure, Feedback, and Causality," <u>SIAM J. Control</u>, Vol. 9, No. 2, May 1971, pp. 149-160.

[2] Y. C. Ho and K. C. Chu, "Information Structure in Many-Person Optimization Problems," <u>Automatica</u>, Vol. 10, 1974, pp. 341-351.

[3] Y. C. Ho and M. P. Kastner, "Market Signaling: An Example of a Two-Person Decision Problem with Dynamic Information Structure," <u>IEEE Trans. AC</u>, Vol. AC-23, No. 2.

[4] H. S. Witsenhausen, "A Counterexample in Stochastic Optimum Control," SIAM Journal Control, Vol. 6, No. 1, 1968, pp. 131-147.

[5] Y. C. Ho and K. C. Chu, "Team Decision Theory and Information Structures in Optimal Control Problems -- Part I," IEEE Trans. AC, Vol. AC-17, No. 1, Feb. 1972, pp. 15-22.

[6] V. E. Benes, "Existence of Optimal Stochastic Control Laws," SIAM J. Control, Vol. 9, No. 3, August, 1971, pp. 446-472.

[7] H. S. Witsenhausen, "The Intrinsic Model for Discrete Stochastic Control: Some Open Problems," Lecture Notes in Econ. and Math. Systems, Vol. 107, 1974, pp. 322-335.

[8] A. D. Wyner, "Another Look at the Coding Theorem of Information Theory - A Tutorial," Proc. IEEE, Vol. 58, No. 6, June 1970, pp. 894-913.

[9] P. Whittle and J. F. Rudge, "Simultaneous Communication and Control," Adv. Appl. Prob., Vol. 8, 1976, pp. 365-384.

[10] J. Ziv and M. Zakai, "On Functionals Satisfying a Data-Processing Theorem," IEEE Trans. Inform. Theory, Vol. IT-9, No. 3, May 1973, pp. 275-283.

[11] T.T. Kadota, M. Zakai, and J. Ziv, "Mutual Information of the White Gaussian Channel with and Without Feedback," IEEE Trans. Inform. Theory, Vol. IT-17, No. 4, July 1971, pp. 368-371.

[12] A. N. Shiryayev, "Statistics of Diffusion Type Processes," Lecture Notes on Mathematics, Springer-Verlag, No. 330, 1973, pp. 397-411.

[13] S. Ihara, "Optimal Coding in White Gaussian Channel with Feedback," Lectures Notes on Mathematics, Springer-Verlag, No. 330, 1973, pp. 120-123.

[14] M. Spence, Market Signaling: Information Transfer in Hiring and Related Screening Processes, Cambridge, MA: Harvard University Press, 1974.

[15] R. D. Luce and H. Raiffa, Games and Decisions, New York: Wiley, 1957.

[16] Y. C. Ho, "On the Minimax Principle and Zero-Sum Stochastic Differential Games," J. Optimization Theory and Applications, Vol. 13, No. 3, March 1974, pp. 343-361.

[17] G. Hexner, private communication, and "A Differential Game of Incomplete Information," submitted J. Optimization Theory and Applications, 1976.

[18] R. J. Aumann and M. Maschler, "Repeated Games with Incomplete Information: The Zero-Sum Extensive Case," Final Report on Contract ACDA/ST-143, prepared by Mathematica, Chapter II, pp. 25-108, Princeton, N.J., 1968.

[19] E. Kohlberg, "Optimal Strategies in Repeated Games with Incomplete Information," Intl. J. of Game Theory, Vol. 4, Issue 1, 1975, pp. 7-24.

[20] W. Diffie and M. E. Hellman, "New Directions in Cryptography," IEEE Trans. Inform. Theory, Vol. IT-22, No. 6, Nov. 1976, pp. 644-654.

[21] M. P. Kastner, "Information and Signaling in Decentralized Decision Problems," Harvard University, Division of Applied Sciences Technical Report No. 669, September 1977.

[22] C. E. Shannon, "Communication in the Presence of Noise," Proc. IRE, Vol. 37, Jan. 1949, pp. 10-21.

[23] J. M. Wozencraft and L. M. Jacobs, Principles of Communication Engineering, New York: Wiley, 1965.

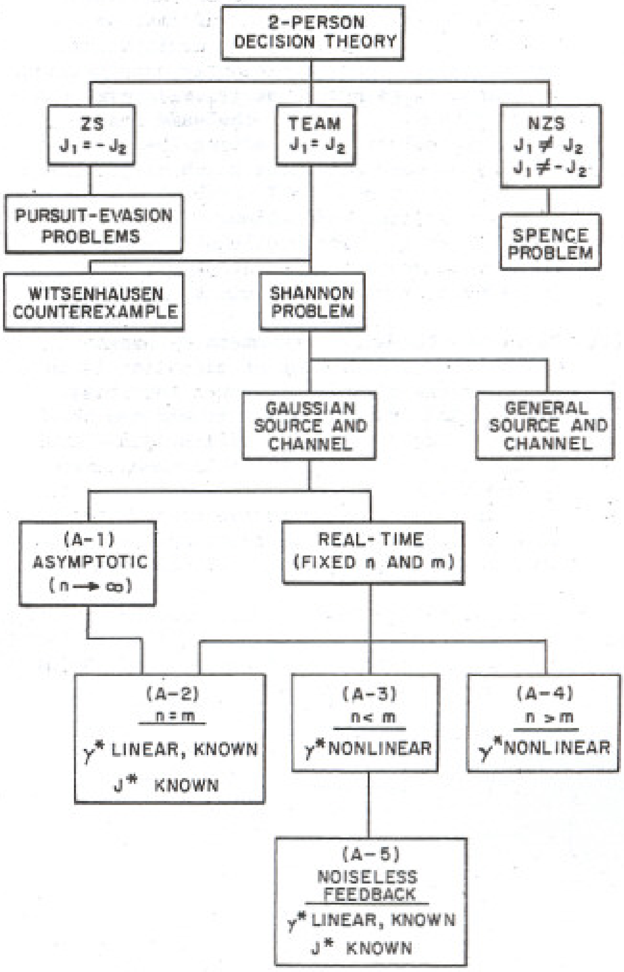

Figure 1
Teams, Signaling, and Information Theory

Appendix*

I. (A-3): One Sample to Two Signals Encoding and Decoding

$$x \sim N(0,1) \qquad v \sim N(0,\sigma^2 I_2)$$

Divide x into four equiprobable regions, as shown in Figure 2. For the encoder, let u_{11} represent the region r(x) that x is from and u_{12} be a linear transformation of x in a stretched out version of this region (see Figure 2). More precisely, $u_{11} = cr(x)$ and

$$u_{12} = B(\frac{2x}{A} + 5 - 2r(x)),$$

where c and B are chosen so that the power constraint is satisfied. Let the estimates be:

$$\hat{r} = \arg\max_{r} p(r/z_{21}), \; z_{21} = cr + v_1$$

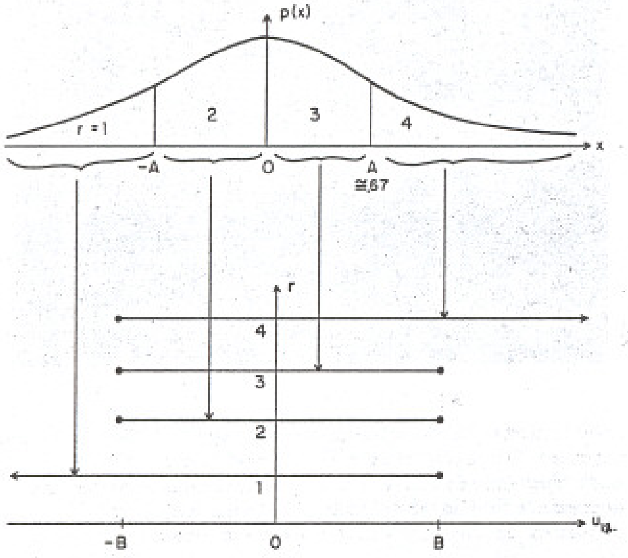

FIG. 2 STRETCHED MAPPING OF x TO TWO DIMENSIONS

$$\hat{u}_{12} = z_{22} = u_{12} + v_2$$

Then inverting the expression for u_{12} yields

$$\hat{x} = \frac{A}{2}(\frac{\hat{u}_{12}}{B} - 5 + 2\hat{r})$$

For small noise, the mean square error distortion with this scheme is better than with any linear scheme.

II. (A-4): Two Samples to One Signal Encoding and Decoding

$$x \sim N(0,I_2) \qquad v \sim N(0,\sigma^2)$$

Transform x_i to $\theta_i = \frac{2}{\pi}\tan^{-1} x_i$, i=1,2. Then $\theta_i \in [-1,1]^2$.

*See Appendices III-B and III-C in [21] for details.

Map all points in the resulting square to the dotted lines r = 1, 2, 3, 4, as shown in Figure 3, where

$$r(\theta_2) = \begin{cases} 1 & -1 \le \theta_2 \le -\frac{1}{2} \\ 2 & -\frac{1}{2} < \theta_2 \le 0 \\ 3 & 0 < \theta_2 \le \frac{1}{2} \\ 4 & \frac{1}{2} < \theta_2 \le 1 \end{cases}$$

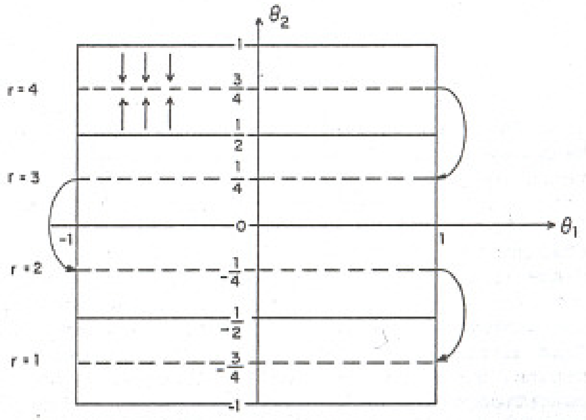

FIG. 3 TRANSFORMATION OF SQUARE TO DOTTED LINE

Straighten out the dotted line and compress it to fit into the interval [-1,1], and call the variable u_1, as shown in Figure 4. Then it can be shown that

$$u_1 = \frac{1}{4}[(-1)^r\theta_1 + 5 - 2r].$$

Let the estimates be $\hat{u}_1 = z_1 = u_1 + v$ and

$$\hat{r} = \begin{cases} 1 \text{ if } \frac{1}{2} < \hat{u}_1 \\ 2 \text{ if } 0 < \hat{u}_1 \le \frac{1}{2} \\ 3 \text{ if } -\frac{1}{2} < \hat{u}_1 \le 0 \\ 4 \text{ if } \hat{u}_1 \le -\frac{1}{2} \end{cases}$$

$$\hat{\theta}_1 = \begin{cases} (-1)^{\hat{r}}(4\hat{u}_1 - 5 + 2r), & -1 \le \hat{u}_1 \le 1 \\ -1 & , \hat{u}_1 < -1 \\ 1 & , \hat{u}_1 > 1 \end{cases}$$

so that $\hat{\theta}_1 \in [-1,1]$. Then let $\hat{\theta}_2 = (5-2\hat{r})/4$ and $\hat{x}_i = \pi/2 \tan \hat{\theta}_i$, i=1,2. Again, for small noise, the mean square error distortion with this scheme is better than with any linear scheme.

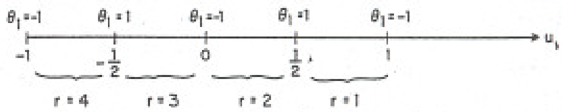

FIG. 4 TRANSFORMATION OF DOTTED LINE TO u_1