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Riemann-Stieltjes Approximations of Stochastic Integrals*

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Summary. We consider the space $C[0, 1]$ together with its Borel σ-algebra $\mathscr{A}$ and a Wiener measure $\mathscr{P}$. Let ω denote a point in $C[0, 1]$ and let $x(\omega, t)$ denote the coordinate process. Then, $\{x(\omega, t), t \in [0, 1]\}$ is a Wiener process, and stochastic integrals of the form $\int_0^1 \varphi(\omega, t)\, dx(\omega, t)$ can be defined for a suitable class of φ. In this paper we consider a sequence of Stieltjes integrals of the form

$$I_n = \int_0^1 \varphi(\omega^n(\omega), t)\, dx(\omega^n(\omega), t)$$

where $\{\omega^n(\omega)\}$ is a sequence of polygonal approximations to ω. Conditions are found which ensure the quadratic-mean convergence of $\{I_n\}$, and the limit is expressed as the sum of the stochastic integral $\int_0^1 \varphi(\omega, t)\, dx(\omega, t)$ and a "correction term".

1. Introduction

Let $x(\omega, t)$ $t \geqq 0$ be a separable Brownian motion defined on a fixed, but as yet unspecified, probability space $(\Omega, \mathscr{A}, \mathscr{P})$. Because a Brownian motion is almost surely of unbounded variation, integrals of the form

$$I(\Phi) = \int_0^1 \Phi(\omega, t)\, d_t\, x(\omega, t) \tag{1}$$

require special definition. One definition, and until recently the only definition, is that due to Ito, and will be referred to as the *stochastic integral* in this paper. The definition of a stochastic integral proceeds as follows: [1, Chap. 9; 2, Chap. 7].

Let $\Phi(\cdot, \cdot)$ satisfy

(A) Φ is a (ω, t) function measurable with respect to $\mathscr{A} \times \mathscr{B}$ and for each t $\Phi(\cdot, t)$ is $\mathscr{A}_t$ measurable, where $\mathscr{A}_t$ is the smallest sub-σ-algebra of ω sets with respect to which $\{x(\omega, s), s \leqq t\}$ are all measurable, and $\mathscr{B}$ is the σ-algebra of one-dimensional Lebesgue measurable sets.

(B) $\qquad \int_0^1 |\Phi(\omega, t)|^2\, dt < \infty \qquad$ for almost all ω

or

(B') $\qquad \int_0^1 E\,|\Phi(\omega, t)|^2\, dt < \infty.$

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The stochastic integral is first defined for Φ functions which are step functions in t for almost all ω by the Riemann sum

$$I(\Phi)=\sum_{\nu=1}^{N}\Phi_\nu(\omega)\big(x(\omega,t_{\nu+1})-x(\omega,t_\nu)\big). \tag{2}$$

For more general Φ, let Φ_n be a sequence of step functions such that

$$\int_0^1 |\Phi(\omega,t)-\Phi_n(\omega,t)|^2\,dt \xrightarrow[n\to\infty]{} 0 \qquad \text{almost all } \omega$$

or

$$\int_0^1 E\,|\Phi(\omega,t)-\Phi_n(\omega,t)|^2\,dt \xrightarrow[n\to\infty]{} 0$$

according to whether (B) or (B′) is satisfied. The stochastic integral $I(\Phi)$ is then defined as the limit in probability (resp. limit in quadratic mean) of $I(\Phi_n)$.

While the definition of a stochastic integral is entirely self-consistent, it need not have any connection with ordinary integrals. Indeed, as is shown by the familiar example [1, p. 444]

$$\int_0^1 x(\omega,t)\,d_t\,x(\omega,t)=\tfrac{1}{2}[x^2(\omega,1)-x^2(\omega,0)]-\tfrac{1}{2}, \tag{3}$$

a calculus based on the stochastic integral cannot be entirely compatible with that corresponding to ordinary integrals which must surely yield $\int_0^1 x(t)\,dx(t)=\frac{1}{2}[x^2(1)-x^2(0)]$. These considerations motivated Stratonovich [3] to suggest a symmetrized definition for (1), which resulted in a calculus compatible with ordinary calculus. In a similar vein we have suggested in earlier papers [4, 5] that in applications one is frequently concerned with the limit of a sequence of Riemann-Stieltjes integrals resembling a stochastic integral but with a sequence of "smooth" approximations $\{x_n(\omega,t)\}$ replacing the Brownian motion $x(\omega,t)$. It was found that this limit, when it existed, differed in general from the stochastic integral having the same form. For example, if $\{x_n(\omega,t)\}$ have piecewise continuous t derivatives, then clearly

$$\int_0^1 x_n(\omega,t)\,d_t\,x_n(\omega,t)=\tfrac{1}{2}[x_n^2(\omega,1)-x_n^2(\omega,0)]\xrightarrow[n\to\infty]{}\tfrac{1}{2}[x^2(\omega,1)-x^2(\omega,0)]$$

which differs from (3) by a "correction term" equal to $\frac{1}{2}$. These earlier papers [4, 5] established the relationship between the limits of such sequences of Riemann-Stieltjes integrals and the corresponding stochastic integrals. However, these results as well as those of Stratonovich [3] were restricted to two special cases:

(a) $\Phi(\omega,t)=F(x(\omega,t),t)$,

(b) $\Phi(\omega,t)=F(y(\omega,t),t)$, and $y(\omega,t)$ is a diffusion process related to $x(\omega,t)$ through a stochastic differential equation.

This paper extends the results of [4, 5] in considering more general integrands $\Phi(\omega, t)$, while retaining the idea of approximating the Brownian motion by differentiable processes. It will be shown that the "correction term" between the limit of a sequence of Riemann-Stieltjes integrals and the corresponding stochastic integral can be expressed in terms of the Frechét differential of $\Phi(\cdot, t)$. In those special cases where the earlier results [3, 4, 5] apply, results of this paper reduce accordingly.

2. A Statement of the Problem

For integrands of the form $\Phi(\omega, t) = F(x(\omega, t), t)$ or $\Phi(\omega, t) = F(y(\omega, t), t)$, an approximation of $x(\omega, t)$ by $x_n(\omega, t)$ induces automatically an approximation $\Phi^{(n)}(\omega, t) = F(x_n(\omega, t), t)$ or $\Phi^{(n)}(\omega, t) = F(y_n(\omega, t), t)$. One of the difficulties in extending our earlier results [4, 5] is that it is unclear as how $\Phi(\omega, t)$ is to be affected in general by an approximation of the Brownian motion. Roughly speaking, the dependence of $\Phi(\omega, t)$ on the sample function $x(\omega, \cdot)$ must be kept the same, while $x(\omega, \cdot)$ undergoes an approximation. The approach taken here in overcoming this difficulty is to choose the basic space Ω in such a way that approximating the sample functions of the Brownian motion is equivalent to approximating elements of Ω, thus inducing an approximation of $\Phi(\omega, t)$ in a natural way.

Let $\Omega = C[0, 1]$ be the space of all continuous real valued functions defined on $[0, 1]$, and denote by $x(\omega, t)$ the value of ω at t. Let $\mathscr{A}$ be the σ-algebra of Borel (= Baire) sets with respect to the (uniform) topology induced by the norm

$$\|\omega\| = \max_{0 \le t \le 1} |x(\omega, t)|. \tag{4}$$

It is well known [6, 7] that the finite dimensional distributions of a standard Brownian motion (Gaussian, zero-mean, $\operatorname{cov}(s, t) = \min(s, t)$) can be uniquely extended to a measure $\mathscr{P}$ on $(\Omega, \mathscr{A})$, and this is the Wiener measure. Defined in this way, $x(\omega, t)$ is necessarily separable. In what follows, we denote by $\mathscr{B}$ the class of Lebesgue measurable sets and $\mu(\cdot)$ the Lebesgue measure. Almost surely (a.s.) shall mean either for all (ω, t) except a set of $\mathscr{P} \times \mu$ measure zero, or for all ω except a set of $\mathscr{P}$ measure zero; which one it is is always clear from the context. Now, let $\Phi(\omega, t)$ satisfy the following hypotheses.

H_1: *Φ is a complex valued (ω, t) function measurable with respect to $\mathscr{A} \times \mathscr{B}$ and for each t $\Phi(\cdot, t)$ is $\mathscr{A}_t$ measurable, where $\mathscr{A}_t \subset \mathscr{A}$ is the smallest σ-algebra with respect to which $\{x(\omega, s), s \le t\}$ are all measurable.*

H_2: *For each $(\omega, t) \in \Omega \times [0, 1]$, there exists a unique continuous linear functional $F(\cdot, \omega, t)$ on Ω such that*

$$|\Phi(\omega + \omega', t) - \Phi(\omega, t) - F(\omega'; \omega, t)| \le K \|\omega'\|^{1+\alpha} (1 + \|\omega\|^{\beta} + \|\omega'\|^{\beta}) \tag{5}$$

where K, α, β are finite positive constants independent of ω, ω', t.

The linear functional $F(\cdot, \omega, t)$, which is necessarily the Frechét differential of $\Phi(\cdot, t)$ at ω, admits the Riesz representation

$$F(\omega'; \omega, t) = \int_0^1 x(\omega', s)\, d_s f(s; \omega, t) \tag{6}$$

where $f(\cdot, \omega, t)$ has bounded variation.

H_3: *We assume that f and Φ satisfy*

$$\int_0^1 |d_s f(s; 0, t)| \leqq K < \infty$$
$$|\Phi(0, t)| \leqq K < \infty$$

where K may be assumed to be the same as that in (5) with no loss of generality.

A function $\Phi(\cdot,\cdot)$ which satisfies H_1, H_2 and H_3 can be shown to satisfy conditions (A) and (B′) of the introduction. Hence, the stochastic integral $\int_0^1 \Phi(\omega, t)\, d_t\, x(\omega, t)$ is well defined as a quadratic-mean limit. Furthermore, a sequence $\omega^n(\omega) \in \Omega$ can be so chosen that

P_1: $\|\omega^n - \omega\| \xrightarrow[n\to\infty]{} 0$

P_2: $x(\omega^n, t)$ has piecewise continuous t-derivative

and

P_3: $\int_0^1 \Phi(\omega^n(\omega), t)\, d_t\, x(\omega^n(\omega), t) \xrightarrow[n\to\infty]{\text{q.m.}} \int_0^1 \Phi(\omega, t)\, d_t\, x(\omega, t) + \frac{1}{2}\int_0^1 \Psi(\omega, t)\, dt$.

In P_3, the integral $\int_0^1 \Phi(\omega, t)\, d_t\, x(\omega, t)$ is a stochastic integral, but $\int_0^1 \Phi\, d_t\, x(\omega^n(\omega), t)$ is an ordinary integral because of P_2. The function $\Psi(\omega, t)$ is defined by

$$\Psi(\omega, t) = f(t^+; \omega, t) - f(t^-; \omega, t). \tag{7}$$

Proposition P_3 is the main result of this paper and extends the results of [4, 5], especially [4].

The details of the proof of our main result are not particularly illuminating as to how the correction term $\frac{1}{2}\int_0^1 \Psi(\omega, t)\, dt$ arises. It may be worthwhile to give a heuristic explanation for it. The Ito definition of a stochastic integral is basically one involving forward difference approximation, i.e.,

$$\int_t^{t+\Delta} \Phi(\omega, t')\, d_{t'}\, x(\omega, t') \sim \Phi(\omega, t)[x(\omega, t+\Delta) - x(\omega, t)].$$

Suppose we consider instead a backward approximation

$$\int_t^{t+\Delta} \Phi(\omega, t')\, d_{t'}\, x(\omega, t') \sim \Phi(\omega, t+\Delta)[x(\omega, t+\Delta) - x(\omega, t)],$$

the difference between the two is $[\Phi(\omega, t+\Delta) - \Phi(\omega, t)][x(\omega, t+\Delta) - x(\omega, t)]$. For a $\Phi(\cdot,\cdot)$ satisfying H_1, H_2, H_3, $\Phi(\omega, t+\Delta) - \Phi(\omega, t) \sim [x(\omega, t+\Delta) - x(\omega, t)]\, \Psi(\omega, t) + 0(\Delta)$, hence the difference between a forward approximation and a backward approximation is $\sim \Psi(\omega,t)[x(\omega,t+\Delta)-x(\omega,t)]^2+o(\Delta)\sim\Psi(\omega,t)\Delta$. The factor $\frac{1}{2}$ in P_3 represents an average of these two approximations.

3. Proof of the Main Result

First, some simply verifiable consequences of H_1, H_2 and H_3 are stated below.

(a) $$|\Phi(\omega,t)|\leqq K\{1+\|\omega\|+\|\omega\|^{1+\alpha}(1+\|\omega\|^{\beta})\}\leqq 3K(1+\|\omega\|^{1+\alpha+\beta}). \tag{8}$$

(b) Since $x(\omega,t)$ has independent increments and $x(\omega,0)=0$ for almost all ω, it follows that [1, p. 363]

$$E\|\omega\|^{\gamma}\leqq 8E|x(\omega,1)|^{\gamma}, \qquad \gamma\geqq 1. \tag{9}$$

(c) Hence,

$$\left.\begin{matrix} E|\Phi(\omega,t)|^2 \\ \int_0^1 E|\Phi(\omega,t)|^2\,dt \end{matrix}\right\}\leqq M<\infty. \tag{10}$$

(d) Therefore, (see [1, pp. 440–441]), there exists a sequence of partitions $\{t_\nu^{(n)}\}$ of [0, 1] such that

$$\max_\nu [t_{\nu+1}^{(n)}-t_\nu^{(n)}]\xrightarrow[n\to\infty]{}0 \tag{11}$$

and if we define

$$\begin{aligned}\alpha_n(t)&=\max_\nu\{t_\nu^{(n)},\ t_\nu^{(n)}\leqq t\}\\ \beta_n(t)&=\min_\nu\{t_\nu^{(n)},\ t_\nu^{(n)}>t\}\end{aligned} \tag{12}$$

then

$$\int_0^1 E|\Phi(\omega,t)-\Phi(\omega,\alpha_n(t))|^2\,dt\xrightarrow[n\to\infty]{}0. \tag{13}$$

(e) Because $\Phi(\omega,t)$ is $\mathscr{A}_t$ measurable, $x(\omega',s)=x(\omega,s)$ $s\leqq t$ implies $\Phi(\omega',t)=\Phi(\omega,t)$. Hence (6) can be written

$$F(\omega',\omega,t)=\int_0^t x(\omega',s)\,d_s f(s;\omega,t) \tag{14}$$

provided that $f(s;\omega,t)$ is made continuous from the right.

(f) Let $\{\varphi_n^t\}$ be any sequence from $\Omega=C[0,1]$ satisfying

$$1\geqq\|\varphi_n^t\|=x(\varphi_n^t,t)\xrightarrow[n\to\infty]{}0, \tag{15}$$

$$\frac{1}{\|\varphi_n^t\|}x(\varphi_n^t,s)\xrightarrow[n\to\infty]{}0, \qquad s<t \tag{16}$$

then for every $\omega\in\Omega$

$$\begin{aligned}\frac{1}{\|\varphi_n^t\|}[\Phi(\omega+\varphi_n^t,t)-\Phi(\omega,t)]&=\int_0^t x(\varphi_n^t/\|\varphi_n^t\|,s)\,d_s f(s;\omega,t)\\ &\quad+0(\|\varphi_n^t\|^2)\xrightarrow[n\to\infty]{}\Psi(\omega,t).\end{aligned} \tag{17}$$

(g) Since

$$\frac{1}{\|\varphi_n^t\|}|\Phi(\omega+\varphi_n^t,t)-\Phi(\omega,t)| \leqq \left|\int_0^t x(\varphi_n^t/\|\varphi_n^t\|,s)\,d_s f(s;\omega,t)\right| + K\|\varphi_n^t\|^\alpha(1+\|\varphi_n^t\|^\beta+\|\omega\|^\beta) \tag{18}$$

$$\leqq |\Phi(\varphi_n^t/\|\varphi_n^t\|+\omega,t)-\Phi(\omega,t)|+2K(2+\|\omega\|^\beta)$$

$$\leqq 9K\,2^{1+\alpha+\beta}(1+\|\omega\|^{1+\alpha+\beta}),$$

it follows by dominated convergence that

$$\left.\begin{array}{r} E|\Psi(\omega,t)|^2 \\ \int_0^1 E|\Psi(\omega,t)|^2\,dt \end{array}\right\} \leqq M<\infty. \tag{19}$$

(h) For some sequence of partitions $\{t_\nu^{(n)}\}$, which can be assumed to be the same one as in (d),

$$\int_0^1 E|\Psi(\omega,t)-\Psi(\omega,\alpha_n(t))|^2\,dt \xrightarrow[n\to\infty]{} 0. \tag{20}$$

$\alpha_n(t)$ being defined by (11).

Given a sequence of partitions $\{0=t_0^{(n)}<t_1^{(n)}\cdots<t_{N_n}^{(n)}=1\}$ and defining $\alpha_n(t)$, $\beta_n(t)$ as before, we can define a corresponding sequence of polygonal approximations to the Brownian motion as follows [8]: For every $\omega\in\Omega=C[0,1]$ define $\omega^n(\omega)$ by

$$x(\omega^n(\omega),t)=x(\omega,\alpha_n(t))+\frac{t-\alpha_n(t)}{\beta_n(t)-\alpha_n(t)}\left[x(\omega,\beta_n(t))-x(\omega,\alpha_n(t))\right]. \tag{21}$$

Now, if, as is the case for (d) and (h) above,

$$\max_{1\leqq t\leqq 1}[\beta_n(t)-\alpha_n(t)] \xrightarrow[n\to\infty]{} 0$$

then

$$\|\omega^n(\omega)-\omega\| \leqq 2 \operatorname*{Sup}_{0\leqq t\leqq 1}|x(\omega,t)-x(\omega,\alpha_n(t))| \xrightarrow[n\to\infty]{} 0 \qquad \text{a.s.} \tag{22}$$

Our main result can now be stated as

Theorem. *Let $\Phi(\omega,t)$ satisfy H_1, H_2 and H_3. Then, there exist a sequence of partitions of $[0,1]$ and a corresponding sequence of polygonal approximations $\omega^n(\omega)$ defined by (21) such that*

$$\int_0^1 \Phi(\omega^n(\omega),t)\,d_t\,x(\omega^n(\omega),t) \xrightarrow[n\to\infty]{\text{q.m.}} \int_0^1 \Phi(\omega,t)\,d_t\,x(\omega,t)+\tfrac{1}{2}\int_0^1 \Psi(\omega,t)\,dt \tag{23}$$

where the first integral on the right hand side is a stochastic integral (but because of (21) the left hand side is an ordinary integral).

Proof. According to (d) and (h) we can always choose a sequence of partitions so that (12), (13) and (20) are satisfied. Because of (13) and the definition of a stochastic integral

$$\sum_{\nu=1}^{N_n} \Phi(\omega, t_{\nu-1}^{(n)})[x(\omega, t_\nu^{(n)}) - x(\omega, t_{\nu-1}^{(n)})] \\ = \int_0^1 \Phi(\omega, \alpha_n(t))\, d_t\, x(\omega^n(\omega), t) \xrightarrow[n\to\infty]{\text{q.m.}} \int_0^1 \Phi(\omega, t)\, d_t\, x(\omega, t). \tag{24}$$

Hence, we only need to prove

$$F_n(\omega) = \int_0^1 [\Phi(\omega^n(\omega), t) - \Phi(\omega, \alpha_n(t))]\, d_t\, x(\omega^n(\omega), t) \xrightarrow[n\to\infty]{\text{q.m.}} \tfrac{1}{2}\int_0^1 \Psi(\omega, t)\, dt. \tag{25}$$

Now, let $\xi_n(\omega, t) \in \Omega$ be defined by

$$x(\xi_n(\omega, t), s) = x(\omega^n(\omega), \min(s, \alpha_n(t))), \qquad 0 \leqq s \leqq 1 \tag{26}$$

and rewrite (25) as

$$F_n(\omega) = \int_0^1 [\Phi(\omega^n(\omega), t) - \Phi(\xi_n(\omega, t), t)]\, d_t\, x(\omega^n(\omega), t) \\ + \int_0^1 [\Phi(\xi_n(\omega), t) - \Phi(\omega, \alpha_n(t))]\, d_t\, x(\omega^n(\omega), t). \tag{27}$$

The integrand of the second integral is $\mathscr{A}_{\alpha_n(t)}$ measurable and

$$E\{[x(\omega, \beta_n(t)) - x(\omega, \alpha_n(t))]^k | \mathscr{A}_{\alpha_n(t)}\} = \begin{cases} 0, & k=1 \\ \beta_n(t) - \alpha_n(t), & k=2. \end{cases} \tag{28}$$

Therefore,

$$E\left|\int_0^1 [\Phi(\xi_n(\omega, t), t) - \Phi(\omega, \alpha_n(t))]\, d_t\, x(\omega^n(\omega), t)\right|^2$$

$$= E\left\{\sum_\nu \sum_\mu \left[\frac{x(\omega, t_\nu) - x(\omega, t_{\nu-1})}{t_\nu - t_{\nu-1}}\right]\left[\frac{x(\omega, t_\mu) - x(\omega, t_{\mu-1})}{t_\mu - t_{\mu-1}}\right]\right.$$

$$\left. \cdot \int_{t_{\nu-1}}^{t_\nu} dt \int_{t_{\mu-1}}^{t_\mu} ds [\Phi(\xi_n(\omega, t), t) - \Phi(\omega, t_{\nu-1})][\Phi(\xi_n(\omega, s), s) - \Phi(\omega, t_{\mu-1})]\right\}$$

$$= E\{\sum_\nu \sum_\mu E[\cdot | \mathscr{A}_{\max(t_{\nu-1}, t_{\mu-1})}]\} \tag{29}$$

$$= E\left\{\sum_\nu \frac{1}{(t_\nu - t_{\nu-1})}\left|\int_{t_{\nu-1}}^{t_\nu} [\Phi(\xi_n(\omega, t), t) - \Phi(\omega, t_{\nu-1})]\, dt\right|^2\right\}$$

$$\leqq \int_0^1 E|\Phi(\xi_n(\omega, t), t) - \Phi(\omega, \alpha_n(t))|^2\, dt$$

$$\leqq 4\left\{\int_0^1 E|\Phi(\xi_n(\omega, t), t) - \Phi(\omega, t)|^2\, dt + \int_0^1 E|\Phi(\omega, t) - \Phi(\omega, \alpha_n(t))|^2\, dt\right\} \xrightarrow[n\to\infty]{} 0$$

by virtue of dominated convergence and (13). Thus, (25) reduces to

$$\int_0^1 [\Phi(\omega^n(\omega), t) - \Phi(\xi_n(\omega, t), t)]\, d_t\, x(\omega^n(\omega), t) \xrightarrow[n\to\infty]{\text{q.m.}} \tfrac{1}{2}\int_0^1 \Psi(\omega, t)\, dt. \tag{30}$$

From H_2, (14) and (26) we can write

$$\begin{aligned}
&\Phi(\omega^n(\omega), t) - \Phi(\xi_n(\omega, t), t)\\
&\quad = \left[\frac{x(\omega, \beta_n(t)) - x(\omega, \alpha_n(t))}{\beta_n(t) - \alpha_n(t)}\right] \int_{\alpha_n(t)}^{t} (s - \alpha_n(t))\, d_s f(s; \xi_n(\omega, t), t)\\
&\qquad + |x(\omega, \beta_n(t) - x(\omega, \alpha_n(t))|^{1+\alpha} G_n(\omega, t)\\
&\quad = \left[\frac{x(\omega, \beta_n(t)) - x(\omega, \alpha_n(t))}{\beta_n(t) - \alpha_n(t)}\right] \int_{\alpha_n(t)}^{t} [f(t; \xi_n(\omega, t), t) - f(s; \xi_n(\omega, t), t)]\, ds \qquad (31)\\
&\qquad + |x(\omega, \beta_n(t)) - x(\omega, \alpha_n(t))|^{1+\alpha} G_n(\omega, t)\\
&\quad = \left[\frac{x(\omega, \beta_n(t)) - x(\omega, \alpha_n(t))}{\beta_n(t) - \alpha_n(t)}\right] (t - \alpha_n(t))\, \Psi(\xi_n(\omega, t), t)\\
&\qquad + [x(\omega, \beta_n(t)) - x(\omega, \alpha_n(t))]\, H_n(\omega, t) + |x(\omega, \beta_n(t)) - x(\omega, \alpha_n(t))|^{1+\alpha} G_n(\omega, t)
\end{aligned}$$

where $|G_n(\omega, t)|$, $|H_n(\omega, t)|$ are both dominated by $K'(1 + \|\omega\|^{1+\gamma})$, $\gamma > 0$, $H_n(\omega, t)$ is $\mathscr{A}_{\alpha_n(t)}$ measurable and $\xrightarrow[n\to\infty]{} 0$ a.s. Hence, it is easy to show that (30) reduces to

$$\int_0^1 \left[\frac{x(\omega, \beta_n(t)) - x(\omega, \alpha_n(t))}{\beta_n(t) - \alpha_n(t)}\right]^2 (t - \alpha_n(t))\, \Psi(\xi_n(\omega, t), t)\, dt \xrightarrow[n\to\infty]{\text{q.m.}} \frac{1}{2} \int_0^1 \Psi(\omega, t)\, dt \qquad (32)$$

or

$$\begin{aligned}
&\int_0^1 \left\{\frac{[x(\omega, \beta_n(t)) - x(\omega, \alpha_n(t))]}{\beta_n(t) - \alpha_n(t)} - 1\right\} \left[\frac{t - \alpha_n(t)}{\beta_n(t) - \alpha_n(t)}\right] \Psi(\xi_n(\omega, t), t)\, dt\\
&\quad + \int_0^1 \left[\frac{t - \alpha_n(t)}{\beta_n(t) - \alpha_n(t)}\right] [\Psi(\xi_n(\omega, t), t) - \Psi(\omega, \alpha_n(t))]\, dt \qquad (33)\\
&\quad + \frac{1}{2} \int_0^1 [\Psi(\omega, \alpha_n(t)) - \Psi(\omega, t)]\, dt \xrightarrow[n\to\infty]{\text{q.m.}} 0.
\end{aligned}$$

Denoting the three integrals in (33) by I_1, I_2 and I_3, we find that because $\Psi(\xi_n(\omega, t), t)$ is $\mathscr{A}_{\alpha_n(t)}$ measurable and

$$E\left\{\frac{[x(\omega, \beta_n(t)) - x(\omega, \alpha_n(t))]^2}{\beta_n(t) - \alpha_n(t)} - 1 \,\middle|\, \mathscr{A}_{\alpha_n(t)}\right\} = 0 \qquad (34)$$

by using arguments similar to those of (29), we can show that

$$E\, I_1^2 \leqq 2 \max_{0 \leqq t \leqq 1} [\beta_n(t) - \alpha_n(t)] \int_0^1 E\, |\Psi(\xi_n(\omega, t), t)|^2\, dt \xrightarrow[n\to\infty]{} 0.$$

The last integral I_3 in (33) converges to zero in quadratic mean because of (20). Thus, it only remains to prove

$$\int_0^1 \left[\frac{t - \alpha_n(t)}{\beta_n(t) - \alpha_n(t)}\right] [\Psi(\xi_n(\omega, t), t) - \Psi(\omega, \alpha_n(t))]\, dt \xrightarrow[n\to\infty]{\text{q.m.}} 0 \qquad (35)$$

which can be further reduced to

$$\int_0^1 \left[\frac{t-\alpha_n(t)}{\beta_n(t)-\alpha_n(t)}\right]\left[\Psi(\xi_n(\omega,t),t)-\Psi(\omega,t)\right]dt \xrightarrow[n\to\infty]{\text{q.m.}} 0. \tag{36}$$

To prove (36), we note that from (f) we can find for every t in $[0,1]$ a sequence $\{\varphi_n^t\}$ satisfying (15) and (16) and in addition

$$\|\varphi_n^t\| \geqq \operatorname*{Sup}_{0\leqq s\leqq 1} |\beta_n(s)-\alpha_n(s)|^{\frac{1}{3}} \tag{37}$$

so that for almost all ω

$$\Psi(\xi_n(\omega,t),t)-\left[\frac{\Phi(\xi_n(\omega,t)+\varphi_n^t,t)-\Phi(\xi_n(\omega,t),t)}{\|\varphi_n^t\|}\right]\xrightarrow[n\to\infty]{} 0, \tag{38}$$

$$\Psi(\omega,t)-\frac{\Phi(\omega+\varphi_n^t,t)-\Phi(\omega,t)}{\|\varphi_n^t\|}\xrightarrow[n\to\infty]{} 0. \tag{39}$$

Further, because $x(\omega,s)$ is a Brownian motion, we have

$$\begin{aligned}\frac{\operatorname*{Max}_{0\leqq s\leqq t}|x(\xi_n(\omega,t),s)-x(\omega,s)|}{\|\varphi_n^t\|} &\leqq \frac{\operatorname*{Max}_{0\leqq s\leqq 1}|x(\omega^n(\omega),s)-x(\omega,s)|}{\operatorname*{Max}_{0\leqq t\leqq 1}|\beta_n(t)-\alpha_n(t)|^{\frac{1}{3}}}\\ &\leqq \frac{2\operatorname*{Sup}_{0\leqq s\leqq 1}|x(\omega,s)-x(\omega,\alpha_n(s))|}{\operatorname*{Max}_{0\leqq t\leqq 1}|\beta_n(t)-\alpha_n(t)|^{\frac{1}{3}}}\\ &\leqq 2\operatorname*{Sup}_{0\leqq s\leqq 1}\left\{\frac{|x(\omega,s)-x(\omega,\alpha_n(s))|}{|s-\alpha_n(s)|^{\frac{1}{3}}}\right\}\xrightarrow[n\to\infty]{} 0 \quad \text{a.s.}\end{aligned} \tag{40}$$

In the last step we made use of the modulus of continuity of Brownian motion [9, p. 547]. Thus, for all t and almost all ω

$$\frac{\Phi(\xi_n(\omega,t)+\varphi_n^t,t)-\Phi(\omega+\varphi_n^t,t)}{\|\varphi_n^t\|}\xrightarrow[n\to\infty]{} 0, \tag{41}$$

$$\frac{\Phi(\xi_n(\omega,t),t)-\Phi(\omega,t)}{\|\varphi_n^t\|}\xrightarrow[n\to\infty]{} 0. \tag{42}$$

Whence

$$\Psi(\xi_n(\omega,t),t)-\Psi(\omega,t)\xrightarrow[n\to\infty]{} 0 \quad \text{a.s.} \tag{43}$$

and (36) follows by dominated convergence (using the bounds provided by (18)). The proof for the theorem is now complete.

Corollary. *Under the hypothesis of the theorem, a sequence of partitions exists for which*

$$\int_0^1 \Phi(\omega^n(\omega),t)\,d_t x(\omega^n(\omega),t)\xrightarrow[n\to\infty]{\text{a.s.}}\int_0^1 \Phi(\omega,t)\,d_t x(\omega,t)+\frac{1}{2}\int_0^1 \Psi(\omega,t)\,dt. \tag{44}$$

Proof. This result is obvious since every q.m. convergent sequence has an a.s. convergent subsequence with the same limit.

4. Examples and Applications

First, consider a class of examples corresponding more or less to the situation in [3, 4, 5]. Let

$$\Phi(\omega, t) = M(y(\omega, t), t) \tag{45}$$

where

$$y(\omega, t) = \int_0^t v(\omega, s)\, d_s x(\omega, s) \tag{46}$$

is a stochastic integral and $M(y, t)$ is twice y-differentiable. It is easy to show that if $v(\cdot, \cdot)$ satisfies H_1, H_2 and H_3 then so does $\Phi(\cdot, \cdot)$. Furthermore, by virtue of (17)

$$\begin{aligned} \Psi(\omega, t) &= \lim_{n\to\infty} \frac{1}{\|\varphi_n^t\|} [\Phi(\omega+\varphi_n^t, t) - \Phi(\omega, t)] \\ &= v(\omega, t)\, M'(y(\omega, t), t) \qquad \left(M'(y, t) = \frac{\partial M(y, t)}{\partial y}\right). \end{aligned} \tag{47}$$

Much weaker conditions on $v(\cdot, \cdot)$ also suffice to yield (47), but this fact would require a more lengthy discussion. Applying the main theorem to the example considered earlier (see (3)), we find

$$\begin{aligned} \int_0^1 x(\omega^n(\omega), t)\, d_t x(\omega^n(\omega, t)) \xrightarrow[n\to\infty]{\text{q.m.}} & \int_0^1 x(\omega, t)\, d_t x(\omega, t) + \frac{1}{2}\int_0^1 dt \\ &= \frac{1}{2}[x^2(\omega, t) - x^2(\omega, 0)] \end{aligned} \tag{48}$$

as it should.

From the point of view of many physical problems, application of stochastic integral to differential equations is important. It is well known [1, pp. 273–291] that under suitable conditions on $\sigma(\cdot, \cdot)$ and $m(\cdot, \cdot)$, the following stochastic differential equation has a unique solution:

$$d_t y(\omega, t) = m(y(\omega, t), t)\, dt + \sigma(y(\omega, t), t)\, d_t x(\omega, t). \tag{49}$$

Here, a solution $y(\cdot, t)$ is interpreted as an $\mathscr{A}_t$ measurable function satisfying

$$y(\omega, t) = y(\omega, 0) + \int_0^t m(y(\omega, s), s)\, ds + \int_0^t \sigma(y(\omega, s), s)\, d_s x(\omega, s) \tag{50}$$

where the last integral is a stochastic integral. Let

$$\begin{aligned} \Phi_t(\omega, s) &= \sigma(y(\omega, s), s), \qquad s \leqq t \\ &= 0 \qquad\qquad\qquad\quad s > t \end{aligned} \tag{51}$$

then in view of our discussion preceding (47), we can expect that under suitable conditions on $\sigma(\cdot, \cdot)$

$$\begin{aligned} \int_0^1 \Phi_t(\omega^n(\omega), s)\, d_s x(\omega^n(\omega), s) \xrightarrow[n\to\infty]{\text{q.m.}} & \int_0^t \sigma(y(\omega, s), s)\, d_s x(\omega, s) \\ & + \frac{1}{2}\int_0^t \sigma'(y(\omega, s), s)\, \sigma(y(\omega, s), s)\, ds. \end{aligned} \tag{52}$$

This was the basic motivation of the results given in [4, 5]. If, as in the references [4, 5], we define

$$y_n(\omega, t) = y(\omega, 0) + \int_0^t m(y_n(\omega, s), s)\, ds + \int_0^t \sigma(y_n(\omega, s), s)\, d_s x(\omega^n(\omega), s) \tag{53}$$

where $\omega^n(\omega)$ is defined by (21), then even if $y_n(\omega, t)$ has a limit as $n \to \infty$, the limit is not the solution of (50). Rather, we expect the limit $\hat{y}(\omega, t)$ to satisfy

$$\begin{aligned} \hat{y}(\omega, t) = y(\omega, 0) &+ \int_0^t m(\hat{y}(\omega, s), s)\, ds + \int_0^t \sigma(\hat{y}(\omega, s), s)\, d_s x(\omega, s) \\ &+ \frac{1}{2} \int_0^t \sigma(\hat{y}(\omega, s), s)\, \sigma'(\hat{y}(\omega, s))\, ds. \end{aligned} \tag{54}$$

Our main theorem can be used to prove (54). However, the conditions given in [4] on $\sigma(\cdot, \cdot)$ need to be strengthened to accomodate H_2.

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References

1. Doob, J. L.: Stochastic processes. New York: John Wiley & Sons 1953.
2. Dynkin, E. B.: Markov processes — I. Berlin: Springer and New York: Academic Press 1965.
3. Stratonovich, R. L.: A new representation for stochastic integrals. Vestnik Moskov Univ., Ser. I, **1**, 3–12 (1964). Reprinted in SIAM J. Control **4**, 362–371 (1966).
4. Wong, E., and M. Zakai: On the relationship between ordinary and stochastic differential equations. Internat. J. Engin. Sci. **3**, 213–229 (1965).
5. — — On the convergence of ordinary integrals to stochastic integrals. Ann. math. Statistics **36**, 1560–1564 (1965).
6. Prokhorov, Yu V.: Convergence of random processes and limit theorems in probability theory. Theor. Probab. Appl. **1**, 157–213 (1956).
7. Nelson, Edward: Regular probability measures on function space. Ann. of Math., II. Ser. **69**, 630–643 (1959).
8. Lévy, P.: Processus stochastique et mouvement Brownien. Paris: Gauthier-Villars 1948.
9. Loève, M.: Probability theory, 3rd ed. Princeton: Van Nostrand 1963.

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