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TWO-DIMENSIONAL RANDOM FIELDS AND REPRESENTATION OF IMAGES*

E. WONG†

Abstract. This paper considers some aspects of two-dimensional random fields with a view toward application in information processing problems involving two-dimensional data. In particular, we call attention to two possible properties which have important implications in terms of representations. They are (a) second order homogeneity with respect to some groups of transformation; (b) the Markovian property. The most interesting results of this paper are those concerning Gaussian random fields which are both homogeneous and Markov.

1. Introduction. In an increasing number of information processing situations, one encounters data which are most naturally presented in two-dimensional form. Most frequently, these two-dimensional data are in the form of optical images, i.e., fluctuations of optical amplitude in two dimensions. In such situations time series or stochastic processes with a one-dimensional time parameter are no longer suitable abstractions for the signals and noise that are encountered. The suitable framework for studying the random phenomena arising in these image processing problems is the theory of random functions with a multidimensional parameter space, i.e., random fields. The purpose of this paper is twofold; first, to call attention to certain known results which may have important implications in image representations, and secondly, to present some new results concerning two-dimensional random fields. The most interesting of these results concern the characterization of Markovian random fields.

Consider a family of complex-valued random variables $\{\xi_z(\omega),\ z \in E^2\}$ defined on some fixed but unspecified probability space. Here, the parameter space E^2 is the Euclidean plane. In this paper we shall deal only with second order random fields, i.e., finite first and second moments, and assume the mean to be zero hereafter. Furthermore, we suppose that the covariance function $R(z, z_0)$ is continuous on $E^2 \times E^2$. Then, there is a version of ξ_z which is separable, Lebesgue measurable, and locally integrable, and we assume that such a version is always chosen.

In one dimension a zero-mean stochastic process $x(t)$ is said to be *wide-sense stationary* if its covariance function is a function of only the time

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† Department of Electrical Engineering and Computer Sciences, Electronics Research Laboratory, University of California, Berkeley, California 94720. This research was supported in part by the United States Army Research Office (Durham) under Grant DA-ARO-D-31-124-G576, after a summer of initial research at IBM Thomas J. Watson Research Center, Yorktown Heights, New York.

difference, i.e., $Ex(t)x^*(s) = R(t - s)$.[1] By Bochner's theorem $R(\tau)$ can always be written as

$$(1) \qquad R(\tau) = \int_{-\infty}^{\infty} e^{i\lambda\tau} F(d\lambda),$$

where $F(\cdot)$ is bounded and nondecreasing. The sample functions of $x(t)$ also admit a spectral representation,

$$(2) \qquad x(t) = \int_{-\infty}^{\infty} e^{i\lambda t} \hat{x}(d\lambda),$$

where $\hat{x}(\cdot)$ is a random measure with

$$(3) \qquad E\hat{x}(\Lambda)\hat{x}^*(\Lambda') = F(\Lambda \cap \Lambda').$$

The integral in (2) is a stochastic integral to be interpreted in a standard manner, e.g., as the limit in quadratic mean of a sequence arising from approximating $e^{i\lambda t}$ by simple functions.

Recognizing that the spectral representation formulas arise from the translation-invariance property of a stationary covariance function, we see immediately that these formulas can be generalized in a number of different ways for a random field with a higher dimensional parameter space. For a two-dimensional random field the most straightforward generalizations of the spectral representation theorems are associated with invariance of the covariance function under translation. The resulting formulas are precisely the same as (1) through (3), except that the integrals are now over E^2. The more interesting results, and possibly more useful, are those associated with a rotational symmetry, especially when it is coupled with additional invariance properties.

For one-dimensional stochastic processes one of the most fruitful ideas is that of a Markov process. There is not only a rich theory associated with Markov processes, but the Markovian properties also play an important role in applications of stochastic processes. The idea of Markovianness can also be generalized to two dimensions (and higher dimensions). The generalization is a rather subtle one due to Lévy. From the point of view of applications, a study of two-dimensional random fields is motivated by the same considerations as in one dimension. Because of its simplicity, a Markovian model (of some degree) is always to be preferred, provided that such a model is compatible with basic requirements of the problem, e.g., continuity. Our results on two-dimensional Markovian random fields are concerned with relating the Markovian character of a Gaussian random field to its second order properties. Some of these results are surprising.

[1] Complex conjugate is denoted by an asterisk.

For example, with some obvious and natural qualifications, the following statement is true: There is no continuous Gaussian random field of two dimensions (or higher dimension) which is both homogeneous (invariant with respect to all rigid body motions) and Markov (degree 1).

2. Isotropic random fields. Let $\{\xi_z(\omega), z \in E^2\}$ be a second order random field (with zero mean as usual). It is said to be *isotropic* if its covariance function is invariant under all rotations about a fixed point. This can be made more explicit by choosing a polar coordinate system (r, ϕ) with the fixed point of the rotations as the origin. Isotropy then means

$$E\xi(r, \phi)\xi^*(r_0, \phi_0) = E\xi(r, \phi + \delta)\xi^*(r_0, \phi_0 + \delta) \tag{4}$$

for all δ. It is always assumed that angular additions are modulo 2π. By setting $\delta = -\phi_0$ in (4) it becomes obvious that the covariance function depends only on $\phi - \phi_0$, i.e.,

$$E\xi(r, \phi)\xi^*(r_0, \phi_0) = R(r, r_0, \phi - \phi_0). \tag{5}$$

Equation (5) immediately implies that the Fourier coefficients

$$\xi_n(r) = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\phi}\,\xi(r, \phi)\,d\phi \tag{6}$$

are orthogonal, in fact,

$$E\xi_n(r)\xi_m^*(r_0) = \delta_{mn}\frac{1}{2\pi}\int_0^{2\pi} R(r, r_0, \phi)\,e^{-in\phi}\,d\phi. \tag{7}$$

The orthogonality of the Fourier coefficients suggests that the Fourier series representation for $\xi(r, \phi)$ is an advantageous one, i.e.,

$$\xi(r, \phi) \overset{\text{q.m.}}{=} \sum_{-\infty}^{\infty} e^{in\phi}\,\xi_n(r) \tag{8}$$

with orthogonal coefficients, where q.m. stands for quadratic mean.

Suppose now that the covariance function of ξ_z is invariant under translation as well as rotation; then clearly the covariance function can only depend on the Euclidean distance, i.e.,

$$E\xi(r, \phi)\xi^*(r_0, \phi_0) = R([r^2 + r_0^2 - 2rr_0 \cos(\phi - \phi_0)]^{1/2}). \tag{9}$$

In this case it is well known [1] that $R(\cdot)$ admits a spectral representation of the form

$$R(r) = \int_0^\infty J_0(\lambda r)F(d\lambda), \tag{10}$$

where $J_0(\cdot)$ is the Bessel function and $F(\cdot)$ is bounded and nondecreasing.

The sample functions of ξ_z also admit a spectal representation

$$(11) \qquad \xi(r, \phi) \overset{\text{q.m.}}{=} \sum_{n=-\infty}^{\infty} e^{in\phi} \int_0^{\infty} J_n(\lambda r)\, \hat{\xi}_n(d\lambda),$$

where the $\hat{\xi}_n(\cdot)$ are random measures with

$$(12) \qquad E\hat{\xi}_m(\Lambda)\hat{\xi}_n^*(\Lambda') = \delta_{mn} \int_{\Lambda \cap \Lambda'} F(d\lambda).$$

By comparing (8) and (11) we have

$$(13) \qquad \xi_n(r) = \int_0^{\infty} J_n(\lambda r)\hat{\xi}_n(d\lambda)$$

with

$$(14) \qquad E\xi_n(r)\xi_m^*(r_0) = \delta_{mn} \int_0^{\infty} J_n(\lambda r) J_n(\lambda r_0) F(d\lambda).$$

Of course, (14) can also be obtained from (10) by an expansion of $J_0(\lambda|z - z_0|)$.

3. Homogeneous random fields. A second order random field $\{\xi_z, z \in E^2\}$ is said to be *homogeneous* if its covariance function is invariant under all Euclidean motions.[2] We have seen that such random fields have the feature that their second order properties are characterizable in terms of a single one-dimensional spectral distribution. In this sense, a homogeneous random field is no more complicated than a one-dimensional stationary process. Of course, random fields which are not homogeneous but easily transformable into homogeneous fields also have this property. This question arises as whether there are other classes of random fields in two dimensions which can be so simply described. There is indeed a natural generalization of the notion "homogeneous". Under this generalization formulas (9) through (14) will appear as special cases. These formulas were given by Yaglom [1], who generalized the concept much further than we shall here.

Consider a two-dimensional space V_2 in which a Riemannian metric is defined. Such a metric is given by a symmetric quadratic form (first fundamental form)

$$(15) \qquad ds^2 = g_{ij}(x_1, x_2)\, dx_i\, dx_j \qquad \text{(sum over repeated indices)},$$

which relates the differential arc length ds to a given coordinate system. The element of length is independent of the coordinate system, hence so

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[2] Our definition of a homogeneous random field is not entirely standard. In the literature homogeneity often refers to just translation invariance.

are all properties derivable from it. In particular, the metric defines at every point of V_2 a scalar function, called the Gaussian curvature, which is independent of the coordinate system. Two-dimensional spaces of *constant* Gaussian curvature include the Euclidean plane E^2 as a special case (zero Gaussian curvature) and constitute a particularly suitable generalization of the Euclidean plane for the purpose of studying two-dimensional isotropic random fields.

For spaces with constant Gaussian curvature, (15) takes on a simple form in terms of a polar coordinate system (r, ϕ) with respect to a fixed point (origin):

$$ds^2 = dr^2 + g^2(r)\,d\phi^2. \tag{16}$$

The Gaussian curvature K is given by

$$K = -\frac{1}{g(r)}\frac{d^2}{dr^2}g(r). \tag{17}$$

With the requirement that $g(0) = 0$ and K be a constant, there are basically only three solutions to (17), namely,

$$g(r) = r, \quad \sinh r, \quad \sin r, \tag{18}$$

representing spaces with Gaussian curvature 0, -1, $+1$, respectively. The first case $g(r) = r$ is clearly the Euclidean plane. The third case $g(r) = \sin r$ is geometrically equivalent to a sphere S_2 in 3-space.

The distance between two points (r_0, ϕ_0) and (r, ϕ) can be obtained by integrating ds along a geodesic connecting the two points. Since the rotation $(r, \phi) \to (r, (\phi + \delta) \bmod 2\pi)$ preserves the metric, the distance must be a periodic function of $(\phi - \phi_0)$. For the three cases corresponding to (18), we have

$$d(r, r_0, \phi - \phi_0) = \begin{cases} \sqrt{r^2 + r_0^2 - 2rr_0\cos(\phi - \phi_0)}, \\ \cosh^{-1}[\cosh r\cosh r_0 - \cos(\phi - \phi_0)\sinh r\sinh r_0], \\ \cos^{-1}[\cos r\cos r_0 + \cos(\phi - \phi_0)\sin r\sin r_0]. \end{cases} \tag{19}$$

Consider now a random field $\{\xi_z, z \in V_2\}$ with a covariance function which is invariant under rotation. We call such fields isotropic, thus generalizing our earlier definition. For all isotropic random fields formulas (4) through (8) require no change. It is well known [2, pp. 226–227] that a space V_2 with constant Gaussian curvature admits a 3-parameter group G_3 of transformations which preserves all metric properties and acts transitively on the space, i.e., takes any point into any other point. We call a random field ξ_z with parameter space V_2 homogeneous, if its covariance function is invariant under all transformations of G_3. Rotations being a subgroup of G_3, a homogeneous random field is necessarily isotropic.

Since G_3 acts transitively on V_2, there exists a transformation which takes (r_0, ϕ_0) into the origin and, simultaneously, (r, ϕ) into $(d(r, r_0, \phi - \phi_0), 0)$. Therefore, the covariance function of a homogeneous random field must be a function of the distance only, i.e.,

$$E\xi(r, \phi)\xi^*(r_0, \phi_0) = R(d(r, r_0, \phi - \phi_0)). \tag{20}$$

It is clear that (20) is a generalization of (9). Corresponding to (10) and (11), we now have

$$R(r) = \int_0^\infty \psi_0(r, \nu)F(d\nu) \tag{21}$$

and

$$\xi(r, \phi) \overset{\text{q.m.}}{=} \sum_{n=-\infty}^{\infty} e^{in\phi} \int_{\Lambda_n} \psi_n(r, \nu)\hat{\xi}_n(d\nu), \tag{22}$$

where $F(\cdot)$ and $\hat{\xi}_n(\cdot)$ satisfy the same conditions as in (10) and (11). The functions $\psi_n(r, \nu)$ are eigenfunctions, and Λ_n, the spectrum, of

$$\frac{1}{g(r)} \frac{d}{dr}\left[g(r) \frac{d\psi_n(r, \nu)}{dr}\right] - \frac{n^2}{g^2(r)} \psi_n(r, \nu) = -\nu\psi_n(r, \nu). \tag{23}$$

For the three cases corresponding to $g(r) = r, \sinh r, \sin r$, (21) and (22) can be written explicitly as

$$R(r) = \begin{cases} \int_0^\infty J_0(\lambda r)F(d\lambda), \\ \int_0^\infty P_{\lambda(\nu)} (\cosh r)F(d\nu), \qquad \lambda(\nu) = -\frac{1}{2} + \sqrt{\frac{1}{4} - \nu}, \\ \sum_{l=0}^\infty F_l P_l (\cos r), \end{cases} \tag{24}$$

$$\xi(r, \phi) = \sum_{n=-\infty}^{\infty} e^{in\phi} \begin{cases} \int_0^\infty J_n(\lambda r)\hat{\xi}_n(d\nu), \\ \int_0^\infty \gamma_{\lambda(\nu)}^n P_{\lambda(\nu)}^n (\cosh r) \hat{\xi}_n(d\nu), \\ \sum_{l \geq |n|}^\infty \gamma_l^n P_l^n (\cos r) \hat{\xi}_{nl}. \end{cases} \tag{25}$$

The Legendre functions $P_\lambda^n(x)$ are defined by the generating function

$$[x + (x^2 - 1)^{1/2} \cos \theta]^\lambda = 2\Gamma(\lambda + 1) \sum_{n=0}^\infty \frac{\cos n\theta}{i^n\Gamma(\lambda + n + 1)} P_\lambda^n(x); \tag{26}$$

and the normalizing constants γ_λ^n are given by

$$\gamma_\lambda^n = \left[(-1)^n \frac{\Gamma(\lambda - n + 1)}{\Gamma(\lambda + n + 1)}\right]^{1/2}. \tag{27}$$

It is clear from (24) that every homogeneous random field is characterized (up to second order properties) by a one-dimensional spectral distribution. They form a natural generalization of wide-sense stationary processes and may be said to be the simplest random fields in two dimensions. By suitably mapping $V_2 \to E^2$, a large class of isotropic random fields on E^2 can be generated.

4. Gauss-Markov random fields. For a two-dimensional space V_2 of constant curvature with a metric given by (12), consider a smooth simply connected closed curve ∂G separating V_2 into a bounded region G^-, which includes the origin, and G^+. Using the language of time series, we shall call ∂G the present, G^- the past and G^+ the future. Following Lévy [3], we shall call a real random field $\{\xi_z, z \in V_2\}$ Markovian of degree $\leqq p + 1$ if for any $\tilde{\xi}_z$ (defined on a neighborhood of ∂G, which approximates ξ_z so well that near ∂G

$$|\xi_z - \tilde{\xi}_z| = o(\delta^p), \qquad \delta = \text{distance } (z, \partial G)) \tag{28}$$

the knowledge of $\tilde{\xi}$ makes the past $\{\xi_z, z \in G^-\}$ and the future $\{\xi_z, z \in G^+\}$ independent. The random field ξ_z is Markovian of degree $p + 1$ if it is of degree $\leqq p + 1$ but not $\leqq p$. If ξ_z has continuous sample functions, then the definition of a simple Markovian field (of degree 1) reduces to the usual definition: (Future independent of Past | Present). It may be useful to recall that a sufficient condition for sample continuity is provided by the Kolmogorov condition (see, e.g., [4, p. 519]):

(a) $z, z_0 \in C$ compact,

(b) for some $\alpha > 0$,

$$E|\xi_z - \xi_{z_0}|^\alpha = O(\delta^{1+\beta}), \qquad \delta = d(z, z_0), \quad \beta > 0. \tag{29}$$

Suppose ξ_z is real and Gaussian with zero mean. Then, whether ξ_z is Markovian or not must be determinable by examining its covariance function. Furthermore, suppose that ξ_z is isotropic; then it must be possible to relate the Markovian character of ξ_z to the properties of its Fourier coefficients $\{\xi_n(r)\}$. Since the real and imaginary parts of the Fourier coefficients constitute a set of independent Gaussian processes in one dimension, this simplifies its analysis considerably. The results obtained here are in this spirit.

THEOREM 1. *Let $\xi(r, \phi)$ be a real zero-mean, Gaussian, isotropic, sample-continuous, and Markov process (of degree 1). Then*

$$\begin{aligned} \xi_n(r) &= \frac{1}{\pi}\int_0^{2\pi} \cos n\phi\, \xi(r, \phi)\, d\phi, \quad n = 0, 1, 2, \cdots, \quad r \geqq 0, \\ \eta_n(r) &= \frac{1}{\pi}\int_0^{2\pi} \sin n\phi\, \xi(r, \phi)\, d\phi, \quad n = 0, 1, 2, \cdots, \quad r \geqq 0, \end{aligned} \tag{30}$$

constitute a family of independent zero-mean Gaussian Markov processes.

COROLLARY. *Under the conditions of Theorem* 1, *the covariance function of* $\xi(r, \phi)$ *must be of the form*

$$\begin{aligned} R(r, r_0, \phi - \phi_0) &= E\xi(r, \phi)\xi(r_0, \phi_0) \\ &= \sum_{n=0}^{\infty} \cos n(\phi - \phi_0) f_n(\min(r, r_0)) h_n(\max(r, r_0)), \end{aligned} \tag{31}$$

where $f_n(r)$, $h_n(r)$, $r \geqq 0$, *are continuous real-valued functions.*

Proof. That $\{\xi_n(r)\}$, $\{\eta_n(r)\}$ are Gaussian zero-mean is obvious. Independence follows from

$$\begin{aligned} E\xi_n(r)\xi_m(r_0) &= E\eta_n(r)\,\eta_m(r_0) \\ &= \delta_{mn}\int_0^{2\pi} R(r, r_0, \phi) e^{-in\phi}\, d\phi; \end{aligned} \tag{32}$$

$$E\xi_m(r)\eta_n(r_0) = 0. \tag{33}$$

To prove that they are Markovian, consider $r > c > r_0$. Then clearly $\xi_n(r)$ and $\xi_n(r_0)$ are independent given $\{\xi(c, \phi), 0 \leqq \phi < 2\pi\}$. But given $\{\xi(c, \phi), 0 \leqq \phi < 2\pi\}$ is the same as given $\{\xi_m(c), \eta_m(c), \text{for all } m\}$. For a fixed n the joint distribution of $\xi_n(r)$ and $\xi_n(r_0)$ given $\{\xi_m(c), \eta_m(c), \text{for all } m\}$ must be the same as that given $\xi_n(c)$, because of (32) and (33). Hence, $\xi_n(r)$ and $\xi_n(r_0)$ are independent given $\xi_n(c)$. The same proof applies to $\eta_n(r)$.

The corollary is easily proved by noting that

$$R(r, r_0, \phi - \phi_0) = \sum_{n=0}^{\infty} \cos n(\phi - \phi_0) R_n(r, r_0) \tag{34}$$

and

$$\begin{aligned} R_n(r, r_0) = E\xi_n(r)\xi_n(r_0) &= E\eta_n(r)\eta_n(r_0) \qquad \text{(except for } n = 0) \\ &= f_n(\min(r, r_0)) h_n(\max(r, r_0)) \end{aligned} \tag{35}$$

must have the product form because $\xi_n(r)$ and $\eta_n(r)$ are Gauss-Markov [5, pp. 233–234].

Equation (31) provides a simple necessary condition for an isotropic Gaussian random field to be Markovian. It is not likely to be a sufficient condition. However, a sufficient condition can be stated as follows.

THEOREM 2. *Let $f_n(r)$ and $h_n(r)$ satisfy*

$$\Delta_n f_n(r) = \frac{1}{g(r)}\frac{d}{dr}\left[g(r)\frac{df_n(r)}{dr}\right] - \frac{n^2}{g^2(r)}f_n(r)$$

$$(36) \qquad = K(r)f_n(r), \qquad r > 0,$$

$$\Delta_n h_n(r) = K(r)h_n(r), \qquad r > 0,$$

where $K(r)$ is bounded and nonnegative,[3] *and $g(r)$ is one of the three forms in (18). It is well known that the Wronskian of f_n, g_n must be of the form*

$$(37) \qquad f_n'(r)h_n(r) - f_n(r)h_n'(r) = \frac{\alpha_n}{g(r)}.$$

Let $\alpha_n = \alpha$ be nonnegative and independent of n. Finally, let the sum

$$\sum_{n=0}^{\infty} f_n(r)h_n(r)$$

converge uniformly on every compact set in $[0, \infty)$. Then,

$$(38) \quad R(r, r_0, \phi - \phi_0) = \sum_{n=0}^{\infty} \epsilon_n \cos n(\phi - \phi_0) f_n(\min(r, r_0))h_n(\max(r, r_0))$$

is the covariance function of an isotropic Gauss-Markov random field, where $\epsilon_n = 1$ for $n = 0$ and $\epsilon_n = 2$ for $n \neq 0$.

Proof. For an arbitrary smooth ∂G, we need to prove that ξ_z, $z \in G^+$, and ξ_{z_0}, $z_0 \in G^-$, are independent given ξ_z, $z \in \partial G$, or, what is the same thing, that $E\xi_{z_0}\{\xi_z - E(\xi_z \mid \xi_{z'}, z' \in \partial G)\} = 0$. Now, let

$$R_n(r, r_0, \phi - \phi_0) = \cos n(\phi - \phi_0)f_n(\min(r, r_0))h_n(\max(r, r_0)).$$

Then, because of (36) and (37), we have

$$\frac{1}{g(r)}\frac{\partial}{\partial r}\left[g(r)\frac{\partial}{\partial r}R_n(r, r_0, \phi - \phi_0)\right] + \frac{1}{g^2(r)}\frac{\partial^2}{\partial \phi^2}R_n(r, r_0, \phi - \phi_0)$$

$$(39) \qquad = K(r)R_n(r, r_0, \phi - \phi_0) + \delta(r - r_0)\cos n(\phi - \phi_0)\frac{\alpha}{g(r)},$$

where $\delta(r - r_0)$ is the Dirac delta function and hence the second derivative $\partial^2/\partial r^2$ in (39) must be interpreted as a generalized derivative in the distribution sense. Since it is easy to verify that

$$\sum_{n=0}^{\infty} \epsilon_n \cos n(\phi - \phi_0)$$

[3] $K(r) \geq 0$ can probably be relaxed. It is imposed here to insure that the exterior Dirichlet problem associated with (41) is always well-posed.

converges to $2\pi\delta(\phi - \phi_0)$ as a sequence of distributions, we have

$$\frac{1}{g(r)} \frac{\partial}{\partial r}\left[g(r) \frac{\partial}{\partial r} R(r, r_0, \phi - \phi_0)\right] + \frac{1}{g^2(r)} \frac{\partial^2}{\partial \phi^2} R(r, r_0, \phi - \phi_0) = K(r)R(r, r_0, \phi - \phi_0) + \frac{2\pi\alpha}{g(r)} \delta(r - r_0)\delta(\phi - \phi_0). \tag{40}$$

Hence, for a fixed $(r_0, \phi_0) \in G^-$,

$$\frac{1}{g(r)} \frac{\partial}{\partial r}\left[g(r) \frac{\partial R}{\partial r}\right] + \frac{1}{g^2(r)} \frac{\partial^2 R}{\partial \phi^2} = K(r)R, \qquad (r, \phi) \in G^+. \tag{41}$$

Treating (41) as an exterior Dirichlet problem with boundary conditions on ∂G, we find that R can be expressed as

$$R(r, r_0, \phi - \phi_0) = \int_{\partial G} H(r, \phi \mid r(s))R(r(s), r_0, \phi(s) - \phi_0)\, ds, \qquad (r, \phi) \in G^+, \quad (r_0, \phi_0) \in G^-, \tag{42}$$

where $H(z' \mid z) = \partial_n F(z' \mid z)$, ∂_n = outward normal derivative, and $F(z' \mid z)$ is the Green's function uniquely defined by the following properties:

(a) for each fixed $z' \in G^+$, $F(z' \mid z)$ satisfies (38) on $G^+ - \{z'\}$;

(b) for each fixed $z' \in G^+$, $F(z' \mid z) = 0$, $z \in \partial G$;

(c) if $C(\epsilon, z') = \{z \mid d(z, z') = \epsilon\}$, then

$$\lim_{\epsilon \to 0} \int_{C(\epsilon, z')} \partial_n F(z' \mid z)\Big|_{z=z(s)} ds = -1.$$

We have assumed that $C(\xi, z')$ and ∂G are similarly oriented and both parameterized by arc length. Therefore, for every $z_0 \in G^-$ (and by continuity for every $z_0 \in \partial G$),

$$E\xi_{z0}\left[\xi_z - \int_{\partial G} H(z \mid z(s))\xi_{z(s)}\, ds\right] = 0. \tag{43}$$

Since $E(\xi_z \mid \xi_{z'}, z' \in \partial G)$ is the unique linear combination of $\{\xi_{z'}, z' \in \partial G\}$, whose difference with ξ_z is independent of ξ_{z_0} for every $z_0 \in \partial G$, we must have

$$E(\xi_z \mid \xi_{z'}, z' \in \partial G) = \int_{\partial G} H(z \mid z(s))\xi_{z(s)}\, ds. \tag{44}$$

Equations (44) and (43) complete the proof of the theorem.

5. Homogeneous Gauss-Markov fields. For a homogeneous random field the Fourier coefficients are interrelated through (22). For the processes

$\{\xi_n(r)\}$, $\{\eta_n(r)\}$ defined by (30) we have

$$E\xi_n(r)\xi_n(r_0) = E\eta_n(r)\eta_n(r_0) = \int_{\Lambda_n} \psi_n(r, \nu)\psi_n(r_0, \nu)F(d\nu), \tag{45}$$

where $F(\cdot)$ is bounded nondecreasing and independent of n. It follows that once $E\xi_0(r)\xi_0(r_0)$ is given, $F(\cdot)$ can be found, and through (45) the covariance functions for the remaining components are already specified. This suggests that we can sharpen the conditions for being Markovian considerably. Indeed, the conditions of Theorem 2, suitably modified, become both necessary and sufficient. Specifically, we have the following theorem.

THEOREM 3. *Let $\xi(r, \phi)$ be a real homogeneous Gaussian random field with continuous sample functions. In order for $\xi(r, \phi)$ to be Markov, it is both necessary and sufficient that its covariance function satisfies*

$$\frac{1}{g(r)}\frac{d}{dr}\left[g(r)\,\frac{dR(r)}{dr}\right] = KR(r), \qquad r > 0, \tag{46}$$

where K is a finite constant.

At the very outset we should note that the content of this theorem is somewhat empty in the sense that all the cases that satisfy the conditions of this theorem are rather degenerate. However, it is a remarkable assertion in another sense, because it states that there are no Gauss-Markov fields in two dimensions (or any greater dimensions) which are also homogeneous. A sketch of the proof will now be given.

Proof. We note from (35) and (45) that

$$E\xi_0(r)\xi_0(r_0) = f_0(r_0)h_0(r) = \int_{\Lambda_0} \psi_0(r, \nu)\psi_0(r_0, \nu)F(d\nu), \quad r \geqq r_0. \tag{47}$$

For the three cases under consideration $\Lambda_0 = [0, \infty)$ or $\{0, 1, 2, \cdots\}$ (cf. (24)). Consider the continuous spectrum case, $\Lambda_0 = [0, \infty)$ (the discrete case requires only trivial modifications in the arguments). Now, let $\mathfrak{D}(0, T)$, $\mathfrak{D}(T, \infty)$ denote subspaces of the Schwartz space $\mathfrak{D}$ with supports contained in $(0, T)$ and (T, ∞) respectively. Then,

$$\begin{aligned}\int_0^\infty g(r_0)f_0(r_0)\phi_1(r_0)\,dr_0 \int_0^\infty g(r)h_0(r)\phi_2(r)\,dr \\ = \int_0^\infty \hat{\phi}_1(\nu)\hat{\phi}_2(\nu)F(d\nu), \qquad \phi_1 \in \mathfrak{D}(0, T), \quad \phi_2 \in \mathfrak{D}(T, \infty),\end{aligned} \tag{48}$$

where

$$\hat{\phi}(\nu) = \int_0^\infty g(r)\psi_0(r, \nu)\phi(r)\,dr. \tag{49}$$

Now, let Δ denote the differential operator

$$\Delta = \frac{1}{g(r)} \frac{d}{dr}\left[g(r) \frac{d}{dr}\right]. \tag{50}$$

Then, since $\Delta\psi_0(r, \nu) = -\nu\psi_0(r, \nu)$, we have

$$\begin{aligned} &\int_0^\infty g(r)f_0(r)\Delta\phi_1(r)\, dr \int_0^\infty g(r)h_0(r)\phi_2(r)\, dr \\ &\quad = \int_0^\infty g(r)f_0(r)\phi_1(r)\, dr \int_0^\infty g(r)h_0(r)\Delta\phi_2(r)\, dr \\ &\quad = -\int_0^\infty \nu F(d\nu)\hat{\phi}_1(\nu)\hat{\phi}_2(\nu), \quad \phi_1 \in \mathfrak{D}(0, T), \quad \phi_2 \in \mathfrak{D}(T, \infty), \end{aligned} \tag{51}$$

or

$$\frac{\int_0^\infty g(r)f_0(r)\Delta\phi_1(r)\, dr}{\int_0^\infty g(r)f_0(r)\phi_1(r)\, dr} = \frac{\int_0^\infty g(r)h_0(r)\Delta\phi_2(r)\, dr}{\int_0^\infty g(r)h_0(r)\phi_2(r)\, dr}. \tag{52}$$

Since (52) is to hold for arbitrary $\phi_1 \in \mathfrak{D}(0, T)$, $\phi_2 \in \mathfrak{D}(T, \infty)$, we must have both sides equal to a constant. Hence,

$$\begin{aligned} \int_0^\infty g(r)f_0(r)\Delta\phi_1(r)\, dr &= K\int_0^\infty g(r)f_0(r)\phi_1(r)\, dr, \quad \phi_1 \in \mathfrak{D}(0, T), \\ \int_0^\infty g(r)h_0(r)\Delta\phi_2(r)\, dr &= K\int_0^\infty g(r)h_0(r)\phi_2(r)\, dr, \quad \phi_2 \in \mathfrak{D}(T, \infty). \end{aligned} \tag{53}$$

Since $R(r) = f_0(0)h_0(r)$ and T in (53) is arbitrary, we have

$$\int_0^\infty g(r)R(r)\Delta\phi(r)\, dr = K\int_0^\infty g(r)R(r)\phi(r)\, dr, \qquad \phi \in \mathfrak{D}(0, \infty), \tag{54}$$

or

$$-\int_0^\infty \nu F(d\nu)\hat{\phi}(\nu)\, d\nu = K\int_0^\infty F(d\nu)\hat{\phi}(\nu).$$

But it follows by a standard approximation argument that

$$-\int_0^\infty \nu F(d\nu)\psi_0(r, \nu) = K\int_0^\infty F(d\nu)\psi_0(r, \nu), \tag{55}$$

whence it follows that $\Delta R(r)$, $r > 0$, not only exists but is equal to $KR(r)$. This proves that (46) is necessary. To prove sufficiency we can make use of Theorem 2. But it is easier to exhaust all solutions of (46), and show that if a solution of (46) is the covariance function of a Gaussian random field satisfying the stated conditions, then the random field in question must be Markov. Now, consider first $K > 0$; then for $r \approx 0$ either $R(r) \sim \ln r$, which violates the continuity condition, or $R(r) \approx R(0) \cdot [1 + Kr^2/4]$ which cannot be a covariance function. For $K = 0$, the only solution bounded at the origin is a constant, for which $\xi(r, \phi)$ would be merely a single random variable. For $K < 0$, the only solutions bounded at the origin are of the form

$$R(r) = A\psi_0(r, |K|), \tag{56}$$

for which $F(\nu)$ is simply a step at $\nu = |K|$, and $\xi(r, \phi)$ can be written as

$$\xi(r, \phi) = \sum_{-\infty}^{\infty} \xi_n e^{in\phi} \psi_n(r, |K|). \tag{57}$$

Equation (57) indicates that if $\xi(r, \phi)$ is homogeneous and Gauss-Markov, then its Fourier components take on very special forms,

$$\xi_n(r, \omega) = \xi_n(\omega)\psi_n(r, |K|), \tag{58}$$

which represent a rather degenerate situation. The degeneracy is clearly revealed by the fact that $\xi(r, \phi)$ is perfectly predictable by its value on any nondegenerate closed contour.

6. Multiple Gauss-Markov fields. Since in a very real sense all homogeneous Gauss-Markov fields (of degree 1) are degenerate cases, homogeneous Markovian fields of degree more than one assume even greater interest. We begin with an example. Consider a real homogeneous Gaussian random field ξ_z with zero mean and a covariance function of the form

$$R(|z - z_0|) = \int_0^\infty \frac{\lambda}{(1 + \lambda^2)^2} J_0(\lambda |z - z_0|) \, d\lambda. \tag{59}$$

It is easy to show that

$$(\nabla^2 - 1)^2 R(|z - z_0|) = \delta(z - z_0). \tag{60}$$

Heuristically, this means that ξ_z should satisfy

$$(\nabla^2 - 1)\xi_z = \eta_z, \tag{61}$$

where $E\eta_z\eta_{z_0} = \delta(z - z_0)$; hence η_z is a two-dimensional white noise. However, as it stands, (61) has no meaning since ξ_z is not even once differentiable. Even though (61) is formal, its parallel with the Langevin equation in the one-dimensional case is clear. Somehow one suspects that ξ_z is indeed Markovian of degree 2. To show this is not too difficult. The main idea here is due to McKean [6]. Let ∂D be a smooth curve separating R^2 into D^- and

D^+. Consider the following boundary value problem:

$$(\nabla^2 - 1)^2 G_{z'}(z) = \delta(z - z'), \qquad z, z' \in D^+, \tag{62}$$

$$G_{z'}(z) = \partial_n G_{z'}(z) = 0, \qquad z \in \partial D, \quad z' \in D^+. \tag{63}$$

Because of (63) and the continuity of ξ_z, we can write

$$\xi_{z_1} = \int_{D^+} \xi_z (\nabla^2 - 1)^2 \, G_{z_1}(z) \, dz. \tag{64}$$

If ξ_z had continuous second partials (which it does not), we would be able to write

$$\xi_{z_1} = \int_{D^+} F_{z_1}(z)(\nabla^2 - 1) \, \xi_z \, dz + \int_{\partial D} [\xi_z \, \partial_n \, F_{z_1}(z) - \partial_n \, \xi_z \, F_{z_1}(z)] \, dl_z, \tag{65}$$

where $F_{z_1}(z) = (\nabla^2 - 1) G_{z_1}(z)$. Although (65) is completely formal, we can now write, using (61),

$$\xi_{z_1} = \int_{D^+} F_{z_1}(z) \mu(dz) + \int_{\partial D} [\xi_z \, \partial_n \, F_{z_1}(z) - F_{z_1}(z) \partial_n \, \xi_z] \, dl_z, \tag{66}$$

where $\mu(\cdot)$ is a Gaussian random measure with

$$E\mu(\Lambda)\mu(\Lambda') = \text{area } (\Lambda \cap \Lambda'). \tag{67}$$

Equation (66) now admits a precise interpretation. First of all, it is easy to show that

$$\int_{D^+} F_{z_1}^2(z) \, dz < \infty; \tag{68}$$

hence the first integral is well-defined. Secondly, we can interpret

$$\int_{\partial D} F_{z_1}(z) \partial_n \, \xi_z = \frac{\partial}{\partial t} \int_{\partial D} F_{z_1}(z) \xi_{z+tn} \, dz \bigg|_{t=0}. \tag{69}$$

The Markovian character of ξ_z is immediately deducible from (65) or (66). Specifically, for any $z_0 \in D^-$,

$$\begin{aligned} E\xi_{z_0} \int_{D^+} F_{z_1}(z)(\nabla^2 - 1)\xi_z \, dz &= \int_{D^+} F_{z_1}(z)(\nabla^2 - 1) R(|\, z - z_0 \,|) \, dz \\ &= \int_{D^+} G_{z_1}(z)(\nabla^2 - 1) R(|\, z - z_0 \,|) \, dz = 0. \end{aligned} \tag{70}$$

In other words,

$$E\xi_{z_0}\left[\xi_{z_1} - \int_{\partial D}[\xi_z\,\partial_n F_{z_1}(z) - F_{z_1}(z)\partial_n\,\xi_z]\,dl_z\right] = 0, \quad z_1 \in D^+, \quad z_0 \in D^-. \tag{71}$$

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