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The Oscillation of Stochastic Integrals

By

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1. Introduction

Let $D_n = (a = t_0^{(n)} < t_1^{(n)} < t_2^{(n)} \cdots < t_{k(n)}^{(n)} = b)$, $n = 1, 2, \ldots$ be a partition of the interval $[a, b]$, let $\tau^{(n)} = \max_{0 \leq i \leq k(n)-1} (t_{i+1}^{(n)} - t_i^{(n)})$. It will be assumed throughout this paper that $\lim_{n\to\infty} \tau^{(n)} = 0$. Let $y(t, \omega)$, $t \geqq 0$, be the Brownian motion process (with $E\{y(t, \omega)\} = 0$, $E\{y^2(t, \omega)\} = t$, $t \geqq 0$). It is a well known result of P. Levy [7] that

$$\lim_{n\to\infty} \sum_{i=0}^{k(n)-1} [y(t_{i+1}^{(n)}, \omega) - y(t_i^{(n)}, \omega)]^2 = (b - a) \tag{1}$$

in the mean and if also (from some n_0 on) D_{n+1} is a refinement of D_n then (1) holds with probability 1.

A similar result, for a class of stationary Gaussian processes was given by Baxter [1] and later generalized to a class of non-stationary Gaussian processes by Gladyshev [4]. An extension to processes with stationary independent increments was given by Kozin [6]. Berman [2] considered limits of similar forms for time-homogeneous diffusion processes.

In this paper, a result of type (1) will be shown to hold for a general class of processes $F(t, \omega)$ which can be represented as stochastic integrals (theorems 1, 1a) (as in Berman's paper, the limits are random variables). It then follows that the sample functions of such processes are of unbounded variation (Corollary 1). These results are applied to obtain corresponding results for a general class of diffusion processes (Theorems 5, 5a) and to the mutual singularity of the probability measures induced in function space by a pair of diffusion processes.

2. The oscillation of stochastic integrals

Let $y(t, \omega)$ and D_n be as defined above. Let $\mathfrak{B}_t$ be the σ-field induced by $y(s, \omega) - y(a, \omega)$, $a \leqq s \leqq t$. Let $f(t, \omega)$ satisfy:

I_1: $f(t, \omega)$ is real valued and measurable in ω and $t\,(a \leqq t \leqq b)$.

I_2: for each t in $[a, b]$, $f(t, \cdot)$ is measurable with respect to $\mathfrak{B}_t$.

I_3: $\int_a^b E\{f^4(t, \omega)\}\, dt < \infty$.

Theorem 1. *Let $f(t, \omega)$ satisfy I_1, I_2, I_3 and $F(t, \omega)$ be defined by the stochastic integral*

$$F(t, \omega) = \int_a^t f(s, \omega)\, dy(s, \omega); \qquad a \leqq t \leqq b.$$

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Then

$$\lim_{n\to\infty} \sum_{i=0}^{k(n)-1} [F(t_{i+1}^{(n)}, \omega) - F(t_i^{(n)}, \omega)]^2 = \int_a^b f^2(t, \omega)\, dt \tag{2}$$

in the mean. If, furthermore, for some $\delta > 0$

$$\lim_{n\to\infty} \tau^{(n)} n^{2+\delta} = 0 \tag{3}$$

then (2) *holds with probability one. Alternatively, if* I_3 *and* (3) *are replaced by* I_3' *and* (3′):

I_3': $E\{f^4(t, \omega)\} \leqq M^4, \qquad a \leqq t \leqq b$

$$\lim_{n\to\infty} \tau^{(n)} n^{1+\delta} = 0 \qquad \text{for some } \delta > 0\,, \tag{3'}$$

then (2) *also holds with probability one.*

The *proof* will be based on the following result of ITÔ [5]: Let $f(t, \omega)$ and $g(t, \omega)$ satisfy I_1, I_2 and also $\int_a^b f^2(t, \omega) dt < \infty$, $\int_a^b g^2(t, \omega) dt < \infty$ for almost all ω. Let

$$G(t, \omega) = \int_a^t g(s, \omega)\, dy(s, \omega)\,, \quad F(t, \omega) = \int_a^t f(s, \omega)\, dy(s, \omega)$$

then, for almost all ω:

$$\int_a^t g(s, \omega)\, dy(s, \omega) \int_a^t f(s, \omega)\, dy(s, \omega) \tag{4}$$

$$= \int_a^t g(s, \omega)\, F(s, \omega)\, dy(s, \omega) + \int_a^t f(s, \omega)\, G(s, \omega)\, dy(s, \omega) + \int_a^t g(s, \omega)\, f(s, \omega)\, ds\,.$$

Form now on, when there is no confusion, we will omit the probability parameter ω. Before proving Theorem 1, we state and prove the following lemma.

Lemma 1. *If* $f(t, \omega)$ *satisfies* I_1, I_2, I_3 *then for* t *in* $[a, b]$:

$$E\{F^4(t)\} \leqq C(t-a) \int_a^t E\{f^4(s)\}\, ds \tag{5}$$

where $C = (4 + \sqrt{18})^2$.

We first prove the lemma under the additional restriction $|f(t)| \leqq M$ for all t in $[a, b]$ and almost all ω.

By (4):

$$F^4(t) = \left(2 \int_a^t f(s)\, F(s)\, dy(s) + \int_a^t f^2(s)\, ds\right)^2$$

$$\leqq 8 \left(\int_a^t f(s)\, F(s)\, dy(s)\right)^2 + 2\left(\int_a^t f^2(s)\, ds\right)^2.$$

Since $E\{f^2(s) F^2(s)\} \leqq M^2 E\{F^2(s)\} = M^2 \int_a^t E\{f^2(s)\} ds < \infty$, it follows that

$$E\{F^4(t)\} \leqq 8 \int_a^t E\{f^2(s) F^2(s)\}\, ds + 2(t-a) \int_a^t E\{f^4(s)\}\, ds$$

therefore $E\{F^4(t)\} < \infty$ and

$$E\{F^4(t)\} \leqq 8 \int_a^t E^{1/2}\{f^4(s)\} \cdot E^{1/2}\{F^4(s)\} ds + 2(t-a) \int_a^t E\{f^4(s)\} ds .$$

Since $F(t)$ is a martingale, $F^4(t)$ is a semi-martingale and for $a \leqq s \leqq t \leqq b$ $E\{F^4(s)\} \leqq E\{F^4(t)\}$, hence

$$E\{F^4(t)\} \leqq 8\, E^{1/2}\{F^4(t)\}(t-a)^{1/2} \left(\int_a^t E\{f^4(s)\} ds\right)^{1/2} + 2(t-a) \int_a^t E\{f^4(s)\} ds .$$

Let $E\{F^4(t)\} = x^2$, $(t-a)\int_a^t E\{f^4(s)\} ds = \mu^2$ then the last inequality becomes $x^2 \leqq 8\mu x + 2\mu^2$, $x \geqq 0$, $\mu \geqq 0$, hence $x \leqq \mu(4 + \sqrt{18})$ and (5) has been proved for $|f(t)| \leqq M$. If $f(t)$ is not uniformly bounded, let $f_M(t, \omega) = f(t, \omega)$ for $|f(t, \omega)| \leqq M$, t in $[a, b]$, and $f_M(t, \omega) = M$ for $f(t, \omega) > M$ and $f_M(t, \omega) = -M$ for $f(t, \omega) < -M$. By (5)

$$E\{F_M^4(t)\} \leqq C(t-a) \int_a^t E\{f_M^4(s)\} ds \leqq C(t-a) E\left\{\int_a^t f^4(s)\, ds\right\}; F_M(t) = \int_a^t f_M(s)\, dy . \tag{6}$$

For any fixed t in $[a, b]$, $F_M(t)$ converges in the mean to $F(t)$ as $M \to \infty$, hence there is a sequence M_j such that $F_{M_j}(t) \to F(t)$ almost surely; the result follows from (6) by FATOU's theorem.

We turn now to the proof of the "in the mean" part of Theorem 1. Let

$$Q_n = \sum_{i=0}^{k(n)-1} [F(t_{i+1}^{(n)}, \omega) - F(t_i^{(n)}, \omega)]^2 - \int_a^b f^2(s, \omega)\, ds$$
$$= \sum_{i=0}^{k(n)-1} \left\{\left(\int_{t_i^{(n)}}^{t_{i+1}^{(n)}} f(s, \omega)\, dy(s, \omega)\right)^2 - \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} f^2(s, \omega)\, ds\right\} .$$

By (4):

$$Q_n = 2 \sum_{i=0}^{k(n)-1} \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} f(s) \left(\int_{t_i^{(n)}}^{s} f(u)\, dy(u)\right) dy(s)$$

$$E\{Q_n^2\} = 4 \sum_{i=0}^{k(n)-1} \int_{t_i^{(n)}}^{t_i^{(n)}} E\left\{f^2(s) \left(\int_{t_i^{(n)}}^{s} f(u)\, dy(u)\right)^2\right\} ds$$
$$\leqq 4 \sum_{i=0}^{k(n)-1} \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} E^{1/2}\{f^4(s)\} \cdot E^{1/2}\left\{\left(\int_{t_i^{(n)}}^{s} f(u)\, dy(u)\right)^4\right\} ds$$

By (5):

$$E\{Q_n^2\} \leqq 4\sqrt{C} \sum_{i=0}^{k(n)-1} \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} E^{1/2}\{f^4(s)\} \cdot (s - t_i^{(n)})^{1/2} \left(\int_{t_i^{(n)}}^{t_{i+1}^{(n)}} E\{f^4(u)\}\, du\right)^{1/2} ds \tag{7}$$

$$E\{Q_n^2\} \leqq 4\sqrt{C}\, (\tau^{(n)})^{1/2} \left(\int_a^b E f^4(u)\, du\right)^{1/2} \cdot \int_a^b E^{1/2}\{f^4(s)\}\, ds . \tag{7'}$$

Hence $E\{Q_n^2\}$ converges to zero as $n \to \infty$ and the "in the mean" part of Theorem 1 is proved.

By (7′) and the Tchebichev inequality,

$$\text{Prob.}\{|Q_n| \geqq (\tau^{(n)} n^{2+\delta})^{1/4}\} \leqq \frac{K_1 (\tau^{(n)})^{1/2}}{(\tau^{(n)} n^{2+\delta})^{1/2}} = K_1 \frac{1}{n^{1+\delta/2}}.$$

Since $\sum_{n=1}^{\infty} n^{-(1+\delta/2)} < \infty$, $(\delta > 0)$, it follows by the Borel-Cantelli Lemma that the set of ω for which $Q_n(\omega) \geqq (\tau^{(n)} n^{2+\delta})^{1/4}$ for infinitely many n has probability zero. Therefore, for almost all ω, $|Q_n(\omega)| \geqq (\tau^{(n)} n^{2+\delta})^{1/4}$ is satisfied only finitely many times. Since we assumed that $\lim_{n\to\infty} \tau^{(n)} n^{2+\delta} = 0$, it follows that $|Q_n| \to 0$ almost surely.

If also I_3' is satisfied then it follows from (7) that $E\{Q_n^2\} \leqq K_2 \tau^{(n)}$. Hence

$$\text{Prob.}\{|Q_n| \geqq (\tau^{(n)} n^{1+\delta})^{1/2}\} \leqq \frac{K_2}{n^{1+\delta}}.$$

Since $\sum_{1}^{\infty} n^{-(1+\delta)} < \infty$, $(\delta > 0)$, the almost sure convergence of (2) follows by (3′) and the Borel-Cantelli Lemma.

Corollary 1. *Under the conditions of theorem 1, let $F(t)$ be the continuous version of $\int_a^t f(s)\,dy(s)$. Then with probability 1, either $F(\cdot, \omega) \equiv 0$ in $[a, b]$ or $F(\cdot, \omega)$ is of unbounded variation in $[a, b]$.*

Proof:

$$\sum_{i=0}^{k(n)-1} [F(t_{i+1}^{(n)}) - F(t_i^{(n)})]^2 \leqq \max_{0 \leqq i \leqq k(n)-1} |F(t_{i+1}^{(n)}) - F(t_i^{(n)})| \cdot \sum_{i=0}^{k(n)-1} |F(t_{i+1}^{(n)}) - F(t_i^{(n)})|. \tag{8}$$

Consider only the continuous sample functions $F(\cdot, \omega)$. Because of the continuity of $F(\cdot, \omega)$; $\max_{0 \leqq i \leqq k(n)-1} |F(t_{i+1}^{(n)}) - F(t_i^{(n)})|$ converges to zero as $\tau^{(n)} \to 0$. Let D_n be a sequence of partitions such that the "almost sure" part of Theorem 1 holds. If for a given $\omega f(t, \omega) = 0$ for almost all t in $[a, b]$ then $F(\cdot, \omega) = 0$, if not then the right hand side of (2), hence the left hand side of (8) are strictly positive. Hence it follows from (8) that $\sum_{i=0}^{k(n)-1} |F(t_{i+1}^{(n)}) - F(t_i^{(n)})|$ must diverge to ∞ as $n \to \infty$ and $F(\cdot)$ is of unbounded variation.

We consider now the following generalization of Theorem 1. Let $Y(t, \omega) = (y^1(t, \omega), \ldots, y^q(t, \omega))$, $t \geqq 0$, be a q-dimensional Brownian motion (that is, the components $y^p(t, \omega)$, $1 \leqq p \leqq q$, of $Y(t, \omega)$ are one dimensional Brownian motions and $y^p(t, \omega)$, $1 \leqq p \leqq q$, are independent processes). Let $\mathfrak{B}_t^q$ be the σ-field induced by $Y(s, \omega) - Y(a, \omega)$, $a \leqq s \leqq t$.

Theorem 1a. *Let each $f_p(t, \omega)$ satisfy I_1, I_2 (with respect to $\mathfrak{B}_t^q$), I_3 and let $F(t, \omega)$ be defined as*

$$F(t, \omega) = \sum_{p=1}^{q} \int_a^t f_p(s, \omega)\, dy^p(s, \omega); a \leqq t \leqq b.$$

Then

$$\lim_{n\to\infty} \sum_{i=0}^{k(n)-1} [F(t_{n+1}^{(n)}, \omega) - F(t_i^{(n)}, \omega)]^2 = \sum_{p=1}^{q} \int_a^b f_p^2(t, \omega)\, dt \tag{2a}$$

in the mean. If, also, for some $\delta > 0$

$$\lim_{n\to\infty} \tau^{(n)} n^{2+\delta} = 0 \tag{3a}$$

then (2a) *holds with probability one. Furthermore, if for some finite M*

$$I_3'':\quad E\{f_p^4(t)\} \leqq M\,,\quad a \leqq t \leqq b\,,\quad 1 \leqq p \leqq q$$

and if for some $\delta > 0$

$$\lim_{n\to\infty} \tau^{(n)} n^{1+\delta} = 0 \tag{3'a}$$

then (2a) *holds with probability one.*

Proof:

$$\sum_{i=0}^{k(n)-1} \left\{ \left[\sum_{p=1}^{q} \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} f_p(t)\, dy^p(t) \right]^2 - \sum_{p=1}^{q} \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} f_p^2(t)\, dt \right\}$$

$$= \sum_{i=0}^{k(n)-1} \left\{ \sum_{p=1}^{q} \left[\left(\int_{t_i^{(n)}}^{t_{i+1}^{(n)}} f_p(t)\, dy^p(t) \right)^2 - \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} f_p^2(t)\, dt \right\} + \right.$$

$$+ \sum_{i=0}^{k(n)-1} \sum_{r \neq p}^{q} \sum^{q} \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} f_r(t)\, dy^r(t) \cdot \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} f_p(t)\, dy^p(t) = Q_n^{(q)} + R_n^{(q)}\,.$$

From equation (7′) it follows that $E\{(Q_n^{(q)})^2\} \leqq K_a (\tau^{(n)})^{1/2}$ (where K_a is independent of n). From (7) it follows that under I_3'', $E\{(Q_n^{(q)})^2\} \leqq \tau^{(n)} \cdot K_b$.

In order to evaluate $E\{(R_n^{(q)})^2\}$ we use the following result of Itô [5]: for $r \neq s$

$$\int_a^b f_r(t) \cdot dy^r(t) \cdot \int_a^b f_s(\theta)\, dy^s(\theta)$$

$$= \int_a^b f_r(t) \left(\int_a^t f_s(u)\, dy^s(u) \right) dy^r(t) + \int_a^b f_s(\theta) \left(\int_a^\theta f_r(u)\, dy^r(u) \right) dy^s(\theta)\,.$$

Following the steps that lead to equations (7) and (7′) we obtain: $E\{(R_n^{(q)})^2\} \leqq (\tau^{(n)})^{1/2} K_c$ and under I_3'', $E\{(R_n^{(q)})^2\} \leqq \tau^{(n)} K_d$; K_d independent of n. The rest of the proof is the same as the proof of Theorem 1.

3. Some related results

Theorem 2. *Let* $f(t, \omega)$ *and* $F(t, \omega)$ *be as in Theorem 1. Let* D_n *be a sequence of partitions such that: (a) from some* n_0 *on* D_{n+1} *is a refinement of* D_n, *(b)* (2) *holds with probability one. Let* $g(t, \omega)$ *satisfy* I_1, I_2 *and* I_3 *and let the sample functions* $g(\cdot, \omega)$ *be continuous in* $[a, b]$ *with probability one, then*

$$\lim_{n\to\infty} \sum_{i=0}^{k(n)-1} g(t_i^{(n)})\, |F(t_{i+1}^{(n)}) - F(t_i^{(n)})|^2 = \int_a^b g(t) f^2(t)\, dt \tag{9}$$

with probability one.

Berman [2] considered the special case $f(t) = 1$ and $g(t) = \varphi(y(t))$ where $\varphi(\cdot)$ is continuous. The proof of Theorem 2 is precisely the same as the proof given by

Berman and is, therefore, omitted. It follows by the same arguments that if $\Phi(\cdot)$ has a continuous derivative then, a. s.,

$$\lim_{n\to\infty}\sum_{i=0}^{k(n)-1}(\Phi(F(t_{i+1}^{(n)}))-\Phi(F(t_i^{(n)})))^2=\int_a^b(\Phi'(F(t)))^2 f^2(t)\,dt\,.$$

Theorem 3. *Let $g(t,\omega)$ satisfy I_1, I_2 and let $E\{g^2(t,\omega)\}$ be continuous in $[a,b]$. Then*

$$\lim_{n\to\infty}\sum_{i=0}^{k(n)-1}g(t_i^{(n)})(y(t_{i+1}^{(n)})-y(t_i^{(n)}))^2=\int_a^b g(t)\,dt \tag{10}$$

in the mean.

Proof: $\sum_{i=0}^{k(n)-1}g(t_i^{(n)})(t_{i+1}^{(n)}-t_i^{(n)})$ converges in the mean to $\int_a^b g(t)\,dt$.

Let
$$\tau_i^{(n)}=t_{i+1}^{(n)}-t_i^{(n)},\quad \eta_i^{(n)}=y(t_{i+1}^{(n)})-y(t_i^{(n)})\,.$$

$$E\left\{\left(\sum_{i=0}^{k(n)-1}g(t_i^{(n)})[(\eta_i^{(n)})^2-\tau_i^{(n)}]\right)^2\right\}$$
$$=E\left\{\sum_{i\neq j}\sum g(t_i^{(n)})\,g(t_j^{(n)})[(\eta_i^{(n)})^2-\tau_i^{(n)}][(\eta_j^{(n)})^2-\tau_j^{(n)}]\right\}+$$
$$+E\left\{\sum_{i=0}^{k(n)-1}g^2(t_i^{(n)})[(\eta_i^{(n)})^2-\tau_i^{(n)}]^2\right\}.$$

Consider a typical term in the double sum, assume $j>i$. Then $\eta_j^{(n)}$ is independent of all the other random variables in this term and $E\{(\eta_j^{(n)})^2-\tau_j^{(n)}\}=0$. Therefore the expectation of the double sum is zero. Since $\eta_i^{(n)}$ is independent of $g(t_i^{(n)})$ we have

$$E\left(\sum_{i=0}^{k(n)-1}g(t_i^{(n)})[(\eta_i^{(n)})^2-\tau_i]^2\right)^2=\sum_{i=0}^{k(n)-1}E\{g^2(t_i^{(n)})\}\,E\{(\eta_i^{(n)})^2-\tau_i^{(n)})^2\}$$
$$=2\sum_{i=0}^{k(n)-1}E\{g^2(t_i^{(n)})\}(\tau_i^{(n)})^2\leqq 2\,\tau^{(n)}\sum_{i=0}^{k(n)-1}E\{g^2(t_i^{(n)})\,\tau_i^{(n)}\}$$

which converges to zero.

Theorem 4. *Let $g(t,\omega)$ satisfy I_1, I_2 (for almost all t in $[a,b]$ and $\int_a^b E\{g^2(t)\}\,dt<\infty$. Let $g_n(t,\omega)$ be a sequence of approximations to $g(t,\omega)$ with the following properties [reference 3, p. 439]: (1) For each n exists a partition $a=t_0^{(n)}<t_1^{(n)}<\cdots<t_{k(n)}^{(n)}=b$ (independent of ω) such that $g_n(t,\omega)=g_n(t_\nu^{(n)},\omega)$, $t_\nu^{(n)}\leqq t<t_{\nu+1}^{(n)}$ and $\tau^{(n)}\to 0$ as $n\to\infty$. (2) $g_n(t,\omega)$ satisfies I_1 and I_2. (3):*

$$\lim_{n\to\infty}\int_a^b E\{(g(t)-g_n(t))^2\}\,dt=0\,.$$

Then

$$\operatorname*{l.i.m}_{n\to\infty}\sum_{i=1}^{k(n)-1}g_n(t_i^{(n)})(t_{i+1}^{(n)}-t_i^{(n)})=\int_a^b g(t)\,dt$$

and

$$\operatorname*{l.i.m}_{n\to\infty}\sum_{i=0}^{k(n)-1}g_n(t_i^{(n)})(y(t_{i+1}^{(n)})-y(t_i^{(n)}))^2=\int_a^b g(t)\,dt;$$

Proof:

$$E\left\{\left(\sum_{i=0}^{k(n)-1}\int_{t_i^{(n)}}^{t_{i+1}^{(n)}}(g(t)-g_n(t))\,dt\right)^2\right\} \leqq (b-a)\sum_{i=0}^{k(n)-1}\int_{t_i^{(n)}}^{t_{i+1}^{(n)}}E\{(g(t)-g_n(t))^2\}\,dt\,.$$

This proves the first assertion. From the proof of Theorem 3 it follows that

$$E\left\{\left(\sum_{i=0}^{k(n)-1}g_n(t_i)\,[(y(t_{i+1}^{(n)})-y(t_i^{(n)}))^2-(t_{i+1}^{(n)}-t_i^{(n)})]\right)^2\right\} \leqq$$
$$\leqq 2\,\tau^{(n)}\sum_{i=0}^{k(n)-1}E\{g_n^2(t_i^{(n)})\}\,(t_{i+1}^{(n)}-t_i^{(n)})$$

which converges to zero since $\tau^{(n)}\to 0$ and

$$\sum_{i=0}^{k(n)-1}E\{g_n^2(t_i^{(n)})\,(t_{i+1}^{(n)}-t_i^{(n)})\} = \int_a^b E\{g_n^2(t)\}\,dt \leqq 2\int_a^b E\{g^2(t)\}\,dt +$$
$$+\,2\int_a^b E\{(g_n(t)-g(t))^2\}\,dt$$

REMARK: The results of this section can be written in the symbolic form:

$$\int_a^b g(t)\,(dy(t))^2 = \int_a^b g(t)\,dt$$
$$\int_a^b g(t)\,(dF(t))^2 = \int_a^b g(t)\,f^2(t)\,dt\,.$$

4. Application to the sample functions of diffusion processes

Lemma 2. *Let $f_p(t,\omega)$, $1\leqq p\leqq q$, satisfy the conditions of Theorem 1a. Let $g(t,\omega)$ satisfy I_1 and I_3. Let*

$$G(t) = \int_a^t g(s)\,ds$$

and

$$H(t) = G(t) + F(t)$$

then

$$\lim_{n\to\infty}\sum_{i=0}^{k(n)-1}[H(t_{i+1}^{(n)})-H(t_i^{(n)})]^2 = \sum_{p=1}^{q}\int_a^b f_p^2(t)\,dt \qquad (11)$$

in the mean. Furthermore, if for a given sequence of partitions D_n, (2) *converges with probability 1 then* (11) *also converges with probability 1.*

Proof:

$$\sum_{i=0}^{k(n)-1}(H(t_{i+1}^{(n)})-H(t_i^{(n)}))^2 = \sum_{i=0}^{k(n)-1}(G(t_{i+1}^{(n)})-G(t_i^{(n)}))^2 + \sum_{i=0}^{k(n)-1}(F(t_{i+1}^{(n)})-F(t_i^{(n)}))^2 +$$
$$+\,2\sum_{i=0}^{k(n)-1}(G(t_{i+1}^{(n)})-G(t_i^{(n)}))\,(F(t_{i+1}^{(n)})-F(t_i^{(n)}))\,.$$

The expectation of the square of the last sum is dominated by

$$4\,E^{1/2}\left\{\left(\sum_{i=0}^{k(n)-1}(F(t_{i+1}^{(n)})-F(t_i^{(n)}))^2\right)^2\right\}\cdot E^{1/2}\left\{\left(\sum_{i=0}^{k(n)-1}\left(\int_{t_i^{(n)}}^{t_{i+1}^{(n)}}g(t)\,dt\right)^2\right)^2\right\}.$$

Therefore, by Theorem 1a, it is sufficient to show that

$$\sum_{i=0}^{k(n)-1} \left(\int_{t_i^{(n)}}^{t_{i+1}^{(n)}} g(t)\, dt \right)^2 \tag{12}$$

converges to zero in the mean and with probability 1. Since

$$\left[\sum_{i=0}^{k(n)-1} \left(\int_{t_i^{(n)}}^{t_{i+1}^{(n)}} g(s)\, ds \right)^2 \right]^2 \leqq \left[\sum_{i=0}^{k(n)-1} (t_{i+1}^{(n)} - t_i^{(n)}) \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} g^2(s)\, ds \right]^2$$

$$\leqq \sum_{i=0}^{k(n)-1} (t_{i+1}^{(n)} - t_i^{(n)})^2 \cdot \sum_{i=0}^{k(n)-1} \left(\int_{t_i^{(n)}}^{t_{i+1}^{(n)}} g^2(s)\, ds \right)^2$$

$$\leqq (\tau^{(n)})^2 (b-a) \int_a^b g^4(s)\, ds .$$

It follows that (12) converges in the mean and with probability one, which completes the proof of the lemma.

Theorem 5. *Let $x(t, \omega)$ be the solution to the stochastic differential equation*

$$x(t) = x(a) + \int_a^t m(x(s), s)\, ds + \int_a^t \sigma(x(s), s)\, dy(s) \tag{13}$$

where

$$|m(x,t) - m(\xi,t)| \leqq k\,|x - \xi|\,; |m(x,t)| \leqq k(1+x^2)^{1/2}$$
$$|\sigma(x,t) - \sigma(\xi,t)| \leqq k\,|x - \xi|\,; |\sigma(x,t)| \leqq k(1+x^2)^{1/2} \tag{14}$$

and let

$$E\{x^4(a)\} < \infty .$$

Then

$$\lim_{n\to\infty} \sum_{i=0}^{k(n)-1} (x(t_{i+1}^{(n)}) - x(t_i^{(n)}))^2 = \int_a^b \sigma^2(x(t), t)\, dt \tag{15}$$

in the mean, and if $\lim_{n\to\infty} \tau^{(n)} n^{1+\varepsilon} = 0$ *for some* $\varepsilon > 0$ *then* (15) *holds with probability one**.

Proof: If $E\{x^4(t)\}$ is bounded in $[a, b]$ then the result follows by applying Lemma 2 to the right hand side of equation 13. It remains only to show that under the assumptions of the theorem, $E\{x^4(t)\}$ is bounded in $[a, b]$. The proof of this will follow closely the proof of existence and uniqueness of the solution to (13) by successive approximations (reference 3 p. 281).

Let $x_0(t)$ be any process satisfying I_1, I_2 and

$$E\{x_0^4(t)\} \leqq M_0, \quad a \leqq t \leqq b . \tag{16}$$

Consider the sequence of successive approximations

$$x_{n+1}(t) = x(a) + \int_a^t m(x_n(s), s)\, ds + \int_a^t \sigma(x_n(s), s)\, dy(s) .$$

* After this paper was submitted for publication we learned that a similar theorem has been proved by S. BERMAN: Sign-invariant random variables and stochastic processes with sign invariant increments. Theorem 4.1. To be published in Trans. Amer. math. Soc.

Then $x_n(t)$ converges with probability one to $x(t)$ which is the solution to (13). It follows from (14) and Lemma 1 that if $E\{x_n^4(t)\}$ is bounded in $[a, b]$, so are

$$E\{(m(x_n(t), t))^4\},\ E\{(\sigma(x_n(t), t))^4\} \quad \text{and} \quad E\{x_{n+1}^4(t)\}.$$

Let

$$\Delta_n x(t) = x_n(t) - x_{n-1}(t)$$

$$\Delta_n m(t) = m(x_n(t), t) - m(x_{n-1}(t), t);\ \Delta_n \sigma(t) = \sigma(x_n(t), t) - \sigma(x_{n-1}(t), t).$$

by (14): $|\Delta_n m(t)| \leqq K|\Delta_n x(t)|;\ |\Delta_n \sigma(t)| \leqq K|\Delta_n x(t)|$.

$$E\{(\Delta_n x(t))^4\} \leqq 16E\left\{\left(\int_a^t \Delta_{n-1} m(s)\,ds\right)^4\right\} + 16E\left\{\left(\int_a^t \Delta_{n-1}\sigma(s)\,dy(s)\right)^4\right\}$$

$$\leqq 16(t-a)^3 K^4 \int_a^t E\{(\Delta_{n-1}x(s))^4\,ds\} + 16(t-a)CK^4\int_a^t E\{(\Delta_{n-1}x(s))^4\,ds$$

$$\leqq K_1 \int_a^t E\{(\Delta_{n-1}x(s))^4\}\,ds.$$

By (16)

$$E\{(\Delta_n x(t))^4\} \leqq K_2 \frac{K_1^n (t-a)^n}{n!} \leqq K_2 \frac{K_3^n}{n!}; \quad a \leqq t \leqq b.$$

Now

$$\left(\sum_{j=1}^m \Delta_j x(t)\right)^4 \leqq \left(\sum_{j=1}^m 2^{-i} \sum_{j=1}^m 2^j (\Delta_j x)^2\right)^2$$

$$\leqq \left(\sum_{j=1}^m 2^{-i}\right)^3 \sum_{j=1}^m 2^{2j} (\Delta_j x)^4$$

Hence

$$E\{(x_m(t) - x_0(t))^4\} \leqq K_2 \sum_1^m \frac{(2^2 K_3)^n}{n!} \leqq K_2 e^{4K_3}.$$

Hence, for some $K_4 < \infty$ $E\{x_m^4(t)\} < K_4$ for all t in $[a, b]$ and all m. Since $x_m(t) \to x(t)$ with probability one, it follows from FATOU's theorem that $E\{x^4(t)\}$ is bounded in $[a, b]$.

Theorem 5a. *Let* $Y(t, \omega) = (y^1(t), \ldots, y^q(t)),\ t \geqq 0,$ *be the q-dimensional Brownian motion. Let* $X(t, \omega) = (x^1(t), \ldots, x^q(t))$ *be the solution to the vector stochastic differential equation:*

$$x^p(t) = x^p(a) + \int_a^t m^p(X(s), s)\,ds + \sum_{r=1}^q \int_a^t \sigma^{pr}(X(s), s)\,dy^r(s) \quad p = 1, 2, \ldots, q;$$

where

$$\|X(t)\|^2 = \sum_{p=1}^q (x^p(t))^2$$

$$|m^p(X_1, t) - m^p(X_2, t)| \leqq K\|X_1 - X_2\|;\ 1 \leqq p \leqq q$$

$$|\sigma^{pr}(X_1, t) - \sigma^{pr}(X_2, t)| \leqq K\|X_1 - X_2\|;\ 1 \leqq p, r \leqq q$$

and let

$$E\{\|X(a)\|^4\} < \infty.$$

Then

$$\lim_{n\to\infty} \sum_{i=1}^{k(n)-1} (x^p(t_{i+1}^{(n)}) - x^p(t_i^{(n)}))^2 = \sum_{r=1}^q \int_a^b (\sigma^{pr}(X(t), t))^2\,dt \tag{17}$$

in the mean, and if $\lim \tau^{(n)} n^{1+\delta} = 0$ *for some* $\sigma > 0$ *then* (17) *holds with probability one. The proof is the same as the proof of Theorem 5 and is therefore omitted.*

Baxter's result [1] was applied by Slepian [8] to the problem of detection of noise-type signals. A similar application of Theorem 5 is the following. Consider the processes $x_\alpha(t, \omega)$ and $x_\beta(t, \omega)$ defined by

$$x_\alpha(t, \omega) = x_\alpha(a) + \int_a^t m_\alpha(x_\alpha(s), s)\, ds + \int_a^t \sigma_\alpha(x_\alpha(s), s)\, dy(s), \qquad a \leqq t \leqq b$$

$$x_\beta(t, \omega) = x_\beta(a) + \int_a^t m_\beta(x_\beta(s), s)\, ds + \int_a^t \sigma_\beta(x_\beta(s), s)\, dy(s), \qquad a \leqq t \leqq b\,,$$

where $x_\alpha(a)$, $x_\beta(a)$, m_α, m_β, σ_α, σ_β satisfy the conditions of Theorem 5. Let σ_α, σ_β be such that there exists a fixed t_0 in $[a, b]$ for which $\int_a^{t_0} \sigma_\alpha^2(x(s), s)\, ds \neq \int_a^{t_0} \sigma_\beta^2(x(s), s)\, ds$ for all continuous $x(s)$. A sample $x_0(t)$, $a \leqq t \leqq b$, is available to an observer and it is known to him that $x_0(t)$ is either a sample from the system with subscript α or with subscript β. The observer is to decide whether $x_0(t)$ came from α or from β. Let $p_\alpha(p_\beta)$ be the probability that the observer decides that the sample came from $\alpha(\beta)$ when indeed it came from $\beta(\alpha)$. If the result of Theorem 5 is used as a test (in the interval $[a, t_0]$), then $p_\alpha = p_\beta = 0$; namely, the probability measures induced by $x_\alpha(t)$ and $x_\beta(t)$ on the space of functions are mutually singular.

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