THE CONSTRUCTION OF A CLASS OF STATIONARY MARKOFF PROCESSES

BY

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Introduction. We take a forward, or Fokker–Planck, equation of diffusion theory

$$\text{(1)} \qquad \frac{\partial^2}{\partial x^2}[B(x)p] - \frac{\partial}{\partial x}[A(x)p] = \frac{\partial p}{\partial t}, \qquad p = p(x|x_0, t), \qquad 0 < t < \infty,$$

on an interval I with end points x_1 and x_2, and consider the solution $p(x|x_0, t)$ corresponding to an initial value $\delta(x - x_0)$, i.e., the principal solution. The function $B(x)$ can be interpreted as a variance [15], and is therefore non-negative on (x_1, x_2). In general, we take, as boundary conditions, those corresponding to reflecting barriers [9],

$$\text{(2)} \qquad \frac{\partial}{\partial x}[B(x)p] - A(x)p = 0, \qquad x = x_1, x_2.$$

The unique solution $p(x|x_0, t)$ is the density function for the transitional probability of a stationary Markoff process $X(t)$ in one dimension,

$$\text{(3)} \qquad \Pr\{X(t+\tau) \in E | X(\tau) = x_0\} = \int_E p(x|x_0, t)\, dx, \qquad x_0 \in I, \qquad E \subseteq I.$$

Next, we consider the class of first-order probability density functions $W(x)$ which satisfy the Pearson equation [4]

$$\text{(4)} \qquad \frac{dW(x)}{dx} = \frac{ax + b}{cx^2 + dx + e} W(x).$$

In §1 we shall show that by suitably identifying the functions $A(x)$ and $B(x)$ in (1) with the polynomials $ax + b$ and $cx^2 + dx + e$ in (4), a class of stationary Markoff processes is constructed for which

$$\text{(5)} \qquad \lim_{t \to \infty} p(x|x_0, t) = \int_{x_1}^{x_2} W(x_0) p(x|x_0, t)\, dx_0 = W(x),$$

where $W(x)$ satisfies (4). The identification scheme is such that the Pearson equation (4) uniquely specifies the Fokker–Planck equation (1). In a sense, the Pearson equation serves a rather natural role in bridging the gap between the first-order statistical description characterized by $W(x)$ and the transitional properties represented by $p(x|x_0, t)$.

In §2 we shall exhibit some specific processes of this class. These examples include several well-known ones, as well as some that appear not to have been investigated previously.

In §3 we shall examine some properties of processes of this class and give some physical interpretation to processes in §2. Finally, the distribution of some functionals involving processes of this class are discussed in §4.

1. **Construction.** We note that a straightforward application of separations of variables to (1) yields an equation of the Sturm–Liouville type

$$\frac{d}{dx}\left[B(x)\rho(x)\frac{d\varphi(x)}{dx}\right] + \lambda\rho(x)\varphi(x) = 0. \tag{6}$$

The boundary conditions (2) imply that

$$B(x)\rho(x)\frac{d\varphi(x)}{dx} = 0, \qquad x = x_1, x_2. \tag{7}$$

The function $\rho(x)$ is taken to be non-negative on (x_1, x_2) and satisfies

$$\frac{d}{dx}[B(x)\rho(x)] - A(x)\rho(x) = 0. \tag{8}$$

A comparison of (8) and (4) shows that they become identical if we make the following identifications:

$$\rho(x) = W(x), \tag{9}$$

$$B(x) = \beta(cx^2 + dx + e) \tag{10}$$

and

$$A(x) = \frac{dB(x)}{dx} + \beta(ax + b), \tag{11}$$

subject to the condition that $B(x)$ be non-negative on (x_1, x_2). It is clear that through (10) and (11), $A(x)$ and $B(x)$ are uniquely specified by the Pearson equation (4) up to a common positive multiplicative constant β, which represents a scaling factor in the variable t.

With the substitution of $W(x)$ for $\rho(x)$, (6) and (7) become

$$\frac{d}{dx}\left[B(x)W(x)\frac{d\varphi(x)}{dx}\right] + \lambda W(x)\varphi(x) = 0, \tag{12}$$

and

$$B(x)W(x)\frac{d\varphi(x)}{dx} = 0, \qquad x = x_1, x_2. \tag{13}$$

From the classical Sturm–Liouville theory it is known that the spectrum of (12) is discrete if x_1 and x_2 are finite. If one or both of the boundaries are infinite, then a continuous range of eigenvalues may be present. In terms of the solutions of (12) the principal solution $p(x|x_0, t)$ of (1) can be written as

$$p(x|x_0, t) = W(x)\left\{\sum_n e^{-\lambda_n t}\varphi_n(x_0)\varphi_n(x) + \int e^{-\lambda t}\varphi(\lambda, x_0)\varphi(\lambda, x)\, d\lambda\right\}, \tag{14}$$

where the summation is taken over all discrete eigenvalues and the integral is taken over the continuous range of eigenvalues. The eigenfunctions $\varphi(x)$ in (14) are assumed to be normalized so that corresponding to discrete eigenvalues we have

$$\int_{x_1}^{x_2} W(x)\varphi_m(x)\varphi_n(x) = \delta_{mn}, \tag{15}$$

and corresponding to continuous eigenvalues

$$\int_{x_1}^{x_2} W(x)\varphi(\lambda, x)\varphi(\lambda', x)\, dx = \delta(\lambda - \lambda'). \tag{16}$$

It can be verified directly that corresponding to boundary conditions (13), the Sturm–Liouville equation (12) has at least one discrete eigenvalue, namely, $\lambda = 0$ with corresponding eigenfunction $\varphi(x) = 1$. In addition, if we assume that the spectrum is discrete, then

$$\begin{aligned} \lambda_n &= -\int_{x_1}^{x_2} \varphi_n(x) \frac{d}{dx}\left[B(x)W(x)\frac{d\varphi_n(x)}{dx}\right] dx \\ &= \int_{x_1}^{x_2} B(x)W(x)\left[\frac{d\varphi_n(x)}{dx}\right]^2 dx, \end{aligned} \tag{17}$$

where we have made use of the boundary conditions (13). Since $B(x)W(x)[d\varphi(x)/dx]^2$ is non-negative on (x_1, x_2), it follows from (17) that

$$\lambda_n \geqq 0. \tag{18}$$

Furthermore, the case where a continuous range of eigenvalues is present can be arrived at by taking cases with finite boundaries, thus with discrete spectra, and considering the limiting situation when one or both of the boundaries becomes infinite. It is clear that, in general, we have

$$\lambda \geqq 0. \tag{19}$$

Further, even in the limiting cases $\lambda = 0$ must remain a discrete eigenvalue, since the corresponding eigenfunction $\varphi = 1$ is always square-integrable with respect to the density function $W(x)$. It follows, from the fact that $\lambda = 0$, $\varphi = 1$ is always a discrete solution, that

$$\int_{x_1}^{x_2} p(x|x_0, t)W(x_0)\, dx_0 = W(x). \tag{20}$$

Further, from the fact that $\lambda = 0$ is also the minimum eigenvalue, it follows that

$$\lim_{t\to\infty} p(x|x_0, t) = W(x). \tag{21}$$

The construction procedure is seen to consist of two parts. First, starting with a first-order probability density function $W(x)$ and a corresponding Pearson equation, we identify the functions $A(x)$ and $B(x)$ in (1) with the polynomials in the Pearson equation according to (10) and (11). Secondly, the density function $p(x|x_0, t)$ of the transitional probability is obtained as the principal solution to (1). (For a brief discussion along similar lines, see Kolmogorov [13].)

2. **Some specific processes.** If we consider the roots of the equation $B(x) = 0$, five distinct situations are possible, namely: (a) no root, in which case $B(x)$ is a constant, (b) a single real root, (c) two unequal real roots, (d) two equal real roots, and (e) a pair of complex-conjugate roots. In this section we shall consider six specific processes which, while by no means exhausting all the possibilities, include at least one example of each of the five situations corresponding to the roots of $B(x) = 0$. In every case except one, the choice of boundaries will be a natural one in that each boundary is either at a real root of "$B(x) = 0$" or is infinite. Without loss of generality we shall set $\beta = 1$, thus normalizing the time scale.

A. Consider the first-order density function

$$W(x) = e^{-x}, \qquad 0 \leqq x < \infty, \tag{22}$$

with a corresponding Pearson equation

$$\frac{dW(x)}{dx} = -\frac{1}{1} W(x).$$

The resulting Fokker–Planck equation is

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial p}{\partial x} = \frac{\partial p}{\partial t}$$

with boundary condition

$$\frac{d}{dx} p(x|x_0, t) = 0, \qquad x = 0.$$

The principal solution $p(x|x_0, t)$ is given by

$$p(x|x_0, t) = e^{-x}\left\{1 + e^{-t/4}\int_0^\infty e^{-\mu^2 t}\psi(\mu, x_0)\psi(\mu, x)\, d\mu\right\}, \tag{23}$$

where

$$\psi(\mu, x) \equiv \sqrt{\frac{2}{\pi}}\, \frac{2\mu}{\sqrt{1 + 4\mu^2}}\left[\cos \mu x - \frac{1}{2\mu}\sin \mu x\right] e^{x/2}.$$

The function $p(x|x_0, t)$ can also be written as

$$\begin{aligned} p(x|x_0, t) = {} & \frac{1}{2\sqrt{\pi t}} \exp\left[-\tfrac{1}{2}(x - x_0)\right] \exp\left[-\tfrac{1}{4}t\right]\left\{\exp\left[-\frac{(x - x_0)^2}{4t}\right]\right. \\ & \left. + \exp\left[-\frac{(x + x_0)^2}{4t}\right]\right\} + \frac{1}{\sqrt{\pi}}\, e^{-x}\int_{(x+x_0-t)/2\sqrt{t}}^\infty e^{-z^2}\, dz. \end{aligned} \tag{24}$$

B. Let the first-order density function be given by

(25) $$W(x) = \frac{1}{\sqrt{2\pi}} \exp\left[-\tfrac{1}{2}x^2\right], \qquad -\infty < x < \infty,$$

with an associated Pearson equation

$$\frac{dW(x)}{dx} = -\frac{x}{1} W(x).$$

The resulting Fokker–Planck equation can be written as

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial}{\partial x}(xp) = \frac{\partial p}{\partial t}.$$

The density function $p(x|x_0, t)$ is given by

(26) $$p(x|x_0, t) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \sum_{n=0}^{\infty} \frac{e^{-nt}}{n!} H_n(x_0)H_n(x),$$

where $H_n(x)$ are the Hermite polynomials **[14]**

$$H_n(x) = (-1)^n e^{-x^2/2} \frac{d^n}{dx^n}(e^{-x^2/2}).$$

By the use of Mehler's formula we can write $p(x|x_0, t)$ as

(27) $$p(x|x_0, t) = \frac{1}{\sqrt{2\pi(1 - e^{-2t})}} \exp\left[-\frac{1}{2(1 - e^{-2t})}(x - x_0 e^{-t})^2\right].$$

C. Consider the first-order density function

(28) $$W(x) = \frac{x^\alpha}{\Gamma(\alpha + 1)} e^{-x}, \qquad \alpha > -1, 0 \leqq x < \infty$$

with corresponding Pearson equation

$$\frac{dW(x)}{dx} = \left(\frac{\alpha - x}{x}\right) W(x).$$

The associated Fokker–Planck equation is

$$\frac{\partial^2}{\partial x^2}(xp) - \frac{\partial}{\partial x}[(\alpha + 1 - x)p] = \frac{\partial p}{\partial t}.$$

The principal solution $p(x|x_0, t)$ is given by

(29) $$p(x|x_0, t) = x^\alpha e^{-x} \sum_{n=0}^{\infty} \frac{n!}{\Gamma(n + \alpha + 1)} e^{-n\beta t} L_n^\alpha(x_0)L_n^\alpha(x),$$

where

$$L_n^\alpha(x) = \frac{1}{n!} x^{-\alpha} e^x \frac{d^n}{dx^n}(x^\alpha e^{-x}),$$

are the Laguerre polynomials [14]. The function $p(x|x_0, t)$ can also be written as

$$(30)\qquad p(x|x_0, t) = \frac{1}{1 - e^{-t}} \left(\frac{x}{x_0 \exp[-t]}\right)^{\alpha/2} \cdot \exp\left[-\frac{1}{(1 - e^{-t})}(x + x_0 e^{-t})\right] I_\alpha\left(\frac{2 e^{-t/2}\sqrt{x_0 x}}{1 - \exp[-t]}\right),$$

where $I_\alpha(z)$ is the modified Bessel function.

D. Corresponding to the situation when "$B(x) = 0$" has two unequal real roots, we take $W(x)$ to be

$$(31)\qquad W(x) = \frac{\Gamma(\alpha + \gamma + 2)}{\Gamma(\alpha + 1)\Gamma(\gamma + 1)} \frac{(1 + x)^\alpha (1 - x)^\gamma}{2^{\alpha+\gamma+1}}, \quad \alpha, \gamma \geqq -1, \; -1 \leqq x \leqq +1,$$

with associated Pearson equation

$$\frac{dW(x)}{dx} = \frac{(\alpha - \gamma) - (\alpha + \gamma)x}{(1 - x^2)} W(x).$$

The resulting Fokker–Planck equation becomes

$$\frac{\partial^2}{\partial x^2}[(1 - x^2)p] + (\alpha + \gamma + 2)\frac{\partial}{\partial x}[xp] - (\alpha - \gamma)\frac{\partial p}{\partial x} = \frac{\partial p}{\partial t}.$$

The solution $p(x|x_0, t)$ is given by

$$(32)\qquad p(x|x_0, t) = \frac{(1 + x)^\alpha (1 - x)^\gamma}{2^{\alpha+\gamma+1}} \sum_{n=0}^{\infty} e^{-n(n+\alpha+\gamma+1)t} A_n P_n^{\alpha,\gamma}(x_0) P_n^{\alpha,\gamma}(x),$$

where $P_n^{\alpha,\gamma}(x)$ are the Jacobi polynomials[1] [14]

$$P_n^{\alpha,\gamma}(x) = \frac{(-1)^n}{2^n}(1 + x)^\alpha (1 - x)^\gamma \frac{d^n}{dx^n}[(1 + x)^{\alpha+n}(1 - x)^{\gamma+n}],$$

and

$$A_n = \frac{(2n + \alpha + \gamma + 1)\Gamma(n + \alpha + \gamma + 1)}{\Gamma(n + \alpha + 1)\Gamma(n + \gamma + 1)n!}.$$

E. Corresponding to a pair of complex conjugate-roots of "$B(x) = 0$," we take

$$(33)\qquad W(x) = \frac{\Gamma(\alpha + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\alpha)}(1 + x^2)^{-(\alpha+1/2)}, \qquad \alpha > 0, \; -\infty < x < \infty,$$

and associate with it a Pearson equation

$$\frac{dW(x)}{dx} = -\frac{(2\alpha + 1)x}{(1 + x^2)} W(x).$$

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[1] The normalization for $P_n^{\alpha,\gamma}$ here is not the conventional one.

The Fokker–Planck equation for this case becomes

$$\frac{d^2}{dx^2}[(1+x^2)p] + (2\alpha - 1)\frac{d}{dx}(xp) = \frac{\partial p}{\partial t}.$$

The Sturm–Liouville equation (12) in this case has $N+1$ discrete eigenvalues ($\alpha - 1 \leqq N < \alpha$), and a continuous range of eigenvalues. More precisely, we have

$$\lambda_n = n(2\alpha - n), \qquad n = 0, 1, 2, \cdots, N,$$

and

$$\lambda = \alpha^2 + \mu^2, \qquad \mu \geqq 0.$$

The solution $p(x|x_0, t)$ can be written as

$$p(x|x_0, t) = (1+x^2)^{-(\alpha+1/2)}\left\{\frac{1}{\pi}\sum_{n=0}^{N}\frac{(\alpha - n)}{n!\,\Gamma(2\alpha + 1 - n)} e^{-n(2\alpha-n)t}\theta_n(x_0)\theta_n(x)\right.$$

$$\left. + \frac{1}{2\pi}\int_0^\infty e^{-(\alpha^2+\mu^2)t}[\psi(\mu, x_0)\psi(-\mu, x) + \psi(-\mu, x_0)\psi(\mu, x)]\, d\mu\right\}, \tag{34}$$

where

$$\theta_n(x) = 2^{\alpha-n}\Gamma(\alpha - n + \tfrac{1}{2})(-1)^n(1+x^2)^{\alpha+1/2}\frac{d^n}{dx^n}[(1+x^2)^{n-\alpha-1/2}], \tag{35}$$

are polynomials of degree n, and $\psi(\mu, x)$ is given by

$$\psi(\mu, x) = (x + \sqrt{1+x^2})^{i\mu}(1+x^2)^{1/2}{}_2F_1\left(-\alpha, \alpha+1; 1+i\mu; \frac{1}{2} + \frac{1}{2}\frac{x}{\sqrt{1+x^2}}\right),$$

${}_2F_1$ being the Gauss hypergeometric series [7].

For $\alpha = K$, a positive integer, $p(x|x_0, t)$ can be written somewhat more explicitly as

$$p(x|x_0, t) = \frac{1}{(1+x^2)^{K+1/2}}\left\{[(1+x_0^2)(1+x^2)]^{K/2}\frac{1}{2\sqrt{\pi t}} e^{-K^2 t} e^{-u^2}\right.$$

$$\left. + \frac{1}{\pi}\sum_{n=0}^{K-1}\frac{(K-n)}{n!\,\Gamma(2K+1-n)} e^{-n(2K-n)t}\theta_n(x_0)\theta_n(x) f_n(x_0, x, t)\right\}, \tag{36}$$

where

$$u = u(x_0, x, t) = \frac{\operatorname{arc\,sinh} x - \operatorname{arc\,sinh} x_0}{2\sqrt{t}},$$

$$f_n(x_0, x, t) = \frac{1}{\sqrt{\pi}}\int_{u-(K-n)\sqrt{t}}^{u+(K-n)\sqrt{t}} e^{-z^2}\, dz,$$

and θ_n are polynomials defined by (35).

F. Corresponding to the situation where "$B(x) = 0$" has a double real root, we consider the first-order density function

$$W(x) = \frac{1}{\Gamma(2\alpha)} x^{-(2\alpha+1)} e^{-1/x}, \qquad \alpha > 0,\ 0 \leqq x < \infty, \tag{37}$$

with corresponding Pearson equation

$$\frac{dW(x)}{dx} = \frac{1 - (2\alpha + 1)x}{x^2} W(x).$$

The resulting Fokker–Planck equation is given by

$$\frac{\partial^2}{\partial x^2}[x^2 p] - \frac{\partial}{\partial x}\{[1 - (2\alpha - 1)x]p\} = \frac{\partial p}{\partial t}.$$

The solution of the Sturm–Liouville equation for this case consists of $N + 1$ discrete eigenvalues ($\alpha - 1 \leqq N < \alpha$), and a continuous range of eigenvalues:

$$\lambda_n = n(2\alpha - n), \qquad n = 0, 1, \cdots, N,$$
$$\lambda = \alpha^2 + \mu^2, \qquad \mu \geqq 0.$$

The transitional probability density function $p(x|x_0, t)$ is found to be

$$p(x|x_0, t) = x^{-(2\alpha+1)} e^{-1/x} \left\{ \sum_{n=0}^{N} \frac{2(\alpha - n)}{\Gamma(2\alpha + 1 - n)} \frac{1}{n!} e^{-n(2\alpha-n)t} \theta_n(x_0)\theta_n(x) + \frac{1}{2\pi} \int_0^\infty e^{-(\alpha^2+\mu^2)t} A(\mu)\psi(\mu, x_0)\psi(\mu, x)\, d\mu \right\}. \tag{38}$$

Here, $\theta_n(x)$ are orthogonal polynomials of degree n,

$$\theta_n(x) = (-1)^n x^{2\alpha+1} e^{1/x} \frac{d^n}{dx^n}(x^{2n-2\alpha-1} e^{-1/x}),$$

and

$$\psi(\mu, x) = {}_2F_0(-\alpha - i\mu, -\alpha + i\mu, -x),$$

${}_2F_0$ being the generalized hypergeometric series [7]. Alternative representations of $\psi(\mu, x)$ in terms of confluent hypergeometric functions follow from properties of ${}_2F_0$, e.g.,

$${}_2F_0(-\alpha - i\mu, -\alpha + i\mu, -x) = x^{(\alpha+i\mu)}\Psi\left(-\frac{\alpha}{2} - i\mu, 1 - 2i\mu, \frac{1}{x}\right),$$

where Ψ is the hypergeometric Ψ-function [7]. The quantity $A(\mu)$ is a normalization factor given by

$$A(\mu) = \frac{\Gamma(-\alpha + i\mu)\Gamma(-\alpha - i\mu)}{\Gamma(i\mu)\Gamma(-i\mu)}.$$

3. Physical interpretation and properties. The processes outlined in (A) through (D) of §2 are familiar processes with some well-known interpretations. The process of (2 A) represents Brownian motion in a constant force-field (e.g., a gravitational field) with a reflecting barrier [3; 11]. The process of (2 B) is the Ornstein–Uhlenbeck process.

The process of (2 C) has been studied in connection with population growth [8]. For $\alpha = N/2 - 1$, $N = 1, 2, \cdots$, it can also be considered as the sum of the squares of N independent and identical Ornstein–Uhlenbeck processes, or alternatively as the square of the radial component of a Brownian motion in N dimensions with radial restoring force proportional to the distance from the origin [12].

The process of (2 D), whose transitional density function involves Jacobi polynomials, has been studied by Bochner, and Karlin and McGregor [2; 12]. For special values of the parameters α and γ, $\alpha = \gamma = (N-3)/2$, $N = 2, 3, \cdots$, a geometric interpretation of the Jacobi diffusion process in terms of Brownian motion on a unit sphere in N dimensions has been given by Karlin and McGregor. It has also been used in connection with biological applications (see Karlin and McGregor).

The process of (2 E) and (2 F) appear to be new. It is of some interest to note that the density function $W(x)$ in (2 E) and (2 F), for $\alpha = \frac{1}{2}$ and $\alpha = \frac{1}{4}$ respectively, become $1/\pi(1 + x^2)$ and $(1/\sqrt{\pi})x^{-3/2}\, e^{-1/x}$. Both of these are known to be density functions for stable distributions [10].

The process $X(t)$ of (2 E) can be interpreted in terms of a Langevin type stochastic differential equation. Specifically, if we make the transformation $Y(t) = \sinh X(t)$, then $Y(t)$ is the solution of a differential equation [6]

$$dY(t) + 2\alpha \tanh Y(t)\, dt = dU(t),$$

where $U(t)$ is a Brownian motion process. The term $2\alpha \tanh Y(t)$ is interesting from the point of view of application, since it approximates a saturating linear element (e.g., a saturating linear amplifier).

For $\alpha = K$ an integer, the form of $p(x|x_0, t)$ given by (36) suggests that a simple interpretation of the process (2 E) in terms of Brownian motion should be possible. However, no such interpretation has yet been found.

The process $X(t)$ of (2 F) can also be interpreted in terms of a differential equation

$$dY + [2\alpha - \exp(-Y)]\, dt = dU(t),$$

where $Y(t) = \ln X(t)$ and $U(t)$ is a Brownian motion process.

A feature of processes of this class is that the transitional probability density function can always be represented in terms of relatively simple classical orthogonal functions. This is due to the fact that $B(x)$ and $A(x)$ are chosen to be polynomials of second and first degrees, thus simplifying the resulting Sturm–Liouville equation. In particular, we note the presence of classical orthogonal polynomials. The transitional probability density function $p(x|x_0, t)$ given by (26), (29) and (32) have the following common form of representation:

$$p(x|x_0, t) = W(x)\sum_{n=0}^{\infty} e^{-\lambda_n t}\theta_n(x_0)\theta_n(x), \tag{39}$$

where $\theta_n(x)$ are normalized orthogonal polynomials. Representations of second-order probability density functions as a single sum of orthogonal polynomials were studied by Barrett and Lampard [**1**], where they discussed the properties and application of such density functions. It has been shown [**16**] that if $p(x|x_0, t)$ satisfies a Fokker–Planck equation of the form given by (1), and if

$$\lim_{t\to\infty} p(x|x_0, t) = W(x),$$

then $p(x|x_0, t)$ has the representation given by (39) if and only if

(1) $B(x) = ax^2 + bx + c$, $A(x) = dx + e$,
(2) $B(x_1)W(x_1) = B(x_2)W(x_2) = 0$,
(3) $\int_{x_1}^{x_2} W(x)x^n\,dx < \infty$, $x = 0, 1, 2, \cdots$.

In effect, (2 B), (2 C) and (2 D) exhaust all such possibilities.

A property, satisfied by most of the processes discussed in §2, is that the normalized covariance function is an exponential function of t, i.e.,

$$E\left(\frac{X(t+\tau)-m}{\sigma}\right)\left(\frac{X(\tau)-m}{\sigma}\right) = e^{-\beta t}; \qquad \beta > 0,\ t \geqq 0, \tag{40}$$

where m and σ^2 are the mean and variance respectively. This property has long been known for Ornstein–Uhlenbeck process, and is closely connected with the question of representing $p(x|x_0, t)$ in terms of orthogonal polynomials. It can be shown that if the transitional probability density function $p(x|x_0, t)$ of a Markoff process $X(t)$ satisfies (1) and if $\lim_{t\to\infty} p(x|x_0, t) = W(x)$, then (40) is satisfied if and only if

(1) $B(x) = ax^2 + bx + c$, $A(x) = dx + e$,
(2) $B(x_1)W(x_1) = B(x_2)W(x_2) = 0$,
(3) $\int_{x_1}^{x_2} W(x)x^n\,dx < \infty$, $n = 0, 1, 2$.

In addition to (2 B), (2 C) and (2 D), the process of (2 E) for $\alpha > 1$, and the process of (2 F) for $\alpha > 1$, also satisfy (40). In each of these two cases the representation of $p(x|x_0, t)$ is in part in terms of a sum of orthogonal polynomials.

4. The distribution of functionals of Markoff processes. We consider a functional of the form

$$Y(t) = \int_0^t f[X(\tau)]\,d\tau, \tag{41}$$

where $X(\tau)$ is a stationary Markoff process of the class being considered in this paper. Darling and Siegert [**5**] have defined a function

$$r(x|x_0, t, \eta) \equiv E\{e^{-\eta Y(t)}|X(0) = x_0, X(t) = x\}\cdot p(x|x_0, t),$$

which in our case is the principal solution of

$$\frac{\partial^2}{\partial x^2}[B(x)r] - \frac{\partial}{\partial x}[A(x)r] - \eta f(x)r = \frac{\partial r}{\partial t}. \tag{42}$$

The corresponding Sturm–Liouville equation becomes

$$\frac{d}{dx}\left[B(x)W(x)\frac{d\varphi(x)}{dx}\right] + W(x)[\lambda - \eta f(x)]\varphi(x) = 0. \tag{43}$$

A comparison of (43) and (12) shows that the only difference is the presence of a term $\eta f(x)$ in (43). If the addition of the term $\eta f(x)$ does not significantly complicate the Sturm–Liouville equation, then it is to be expected that the distribution of the corresponding functional is not too difficult to find.

With a standard transformation, (43) can be rewritten in the form

$$\frac{d^2\psi}{dz^2} + [\lambda - V(z) - \eta f]\psi = 0, \tag{44}$$

with

$$z = \int \frac{1}{\sqrt{B(x)}}\,dx,$$

$$q = [B(x)W^2(x)]^{1/4},$$

$$\psi = q\varphi,$$

$$V(z) = q^{-1}(z)\frac{d^2}{dz^2}q(z)$$

and

$$f = f(x(z)).$$

If the kernel f of the functional is such that

$$V(z) + \eta f(x(z)) = a^2V(az + b) + c, \tag{45}$$

where a, b, c are constants (i.e., independent of z), then (44) can be rewritten as

$$\frac{d^2\psi}{d\zeta^2} + \left[\left(\frac{\lambda - c}{a^2}\right) - V(\zeta)\right]\psi = 0, \tag{46}$$

with

$$\zeta = az + b.$$

In that case the solutions of (43) follow immediately from the solution of (12).

As an example, consider the functional

$$Y(t) = \int_0^t X(\tau)\,d\tau, \tag{47}$$

where $X(\tau)$ is the process of (2 C) with first-order probability density function

$$W(x) = \frac{1}{\Gamma(\alpha + 1)}x^\alpha e^{-x}.$$

For this case (43) becomes

$$\frac{d}{dx}\left[x^{\alpha+1} e^{-x} \frac{d\varphi}{dx}\right] + x^{\alpha} e^{-x}[\lambda - \eta x]\varphi(x) = 0. \tag{48}$$

It can be verified that $f(x) = x$ for this case satisfies (45) with

$$\begin{aligned} a &= (1 + 4\eta)^{1/4}, \\ b &= c = 0. \end{aligned} \tag{49}$$

The solutions of (48) are

$$\lambda_n = a^2(n + \alpha + 1) - (\alpha + 1) \tag{50}$$

and

$$\varphi_n(x) = e^{(1-a^2)x/2} L_n^{\alpha}(a^2 x), \tag{51}$$

with

$$L_n^{\alpha}(x) = \frac{1}{n!} x^{-\alpha} e^{x} \frac{d^n}{dx^n}(x^{n+\alpha} e^{-x}).$$

The function $r(x|x_0, t; \eta)$ can be written as

$$r(x|x_0, t; \eta) = a^2(a^2 x)^{\alpha} e^{-x} \sum_{n=0}^{\infty} \frac{n!}{\Gamma(n + \alpha + 1)} \varphi_n(x_0)\varphi_n(x). \tag{52}$$

From (52) we find

$$\begin{aligned} F(\eta, t) = E(e^{-\eta Y(t)}) &= \int_0^{\infty}\int_0^{\infty} W(x_0) r(x|x_0, t, \eta)\, dx_0\, dx \\ &= \left[\frac{4a^2 e^{-(a^2-1)t/2}}{(a^2+1)^2 - (a^2-1)\, e^{-a^2 t}}\right]^{\alpha+1}. \end{aligned} \tag{53}$$

Finally, the probability density function for $Y(t)$ can be found from (53) using the inversion integral of Laplace transform. Specifically, let $W(y, t)$ be the density function. Then

$$W(y, t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\eta y} F(\eta, t)\, d\eta, \qquad c > -\tfrac{1}{4}. \tag{54}$$

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