Problem 1: Calculus is bunk! Any silliness can be "proved" by calculus. Here is an example: Choose any positive unteger K . Then $K^2 = K + K + K + \dots + K + K + K$ with "K" repeated K times on the right-hand side. The derivative of K^2 is 2K . The derivative of K is 1 repeated K times on the right-hand side. Equating derivatives yields $2K = 1 + 1 + 1 + \dots + 1 + 1 + 1 = K$, and then dividing out the chosen positive integer K yields 2 = 1 . Agreed? If not, why not?

Solution 1: The definition $f'(x) := \lim_{\Delta x \to 0} \big(f(x+\Delta x) - f(x) \big)/\Delta x$ of the derivative $f'(x)$ makes sense only at points x inside or on the boundary of an open neighborhood in f's domain. This becomes relevant when "$K+\Delta K$" has to be repeated $K+\Delta K$ times on the right-hand side. Neither is "…" an operation allowed in algebraic expressions, so purely algebraic derivatives are inapplicable too.

Problem 2: This problem responds to a classroom request for mathematical coincidences. In each of four instances below, two numbers x and y are defined that agree to at least ten sig. dec., as you may confirm with the aid of a calculator or computer. Does x = y ? If you think so, prove it. If you think not, estimate roughly a positive lower bound for $|x - y|$. You may use a computer provided you document your work as a scientist should.

Problem 2.1: $x := 1 + \sqrt{3}$ and $y := \sqrt{(3 + \sqrt{(13 + 4\sqrt{3})})}$.

Solution 2.1: x = y because x > 0 , y > 0 and $x^2 - y^2 = 1 + 2\sqrt{3} - \sqrt{(13 + 4\sqrt{3})} = 0$.

Many computerized algebra systems simplify expression y to x ; *Maple V r5* converts y to x upon input. This version of *Maple* is obsolete but is the latest one that runs on my old 33 MHz. 68040-based *Apple Quadra 950*.

Problem 2.2: $x := \sqrt{5} + \sqrt{(22 + 2\sqrt{5})}$ and $y := \sqrt{(11 + 2\sqrt{29})} + \sqrt{\big(16 - 2\sqrt{29} + 2\sqrt{(55 - 10\sqrt{29})}\big)}$.

Solution 2.2: x = y : First $\sqrt{\big(16 - 2\sqrt{29} + 2\sqrt{(55 - 10\sqrt{29})}\big)} = \sqrt{5} + \sqrt{(11 - 2\sqrt{29})}$, simplifying y to $y = \sqrt{5} + \sqrt{(11 + 2\sqrt{29})} + \sqrt{(11 - 2\sqrt{29})}$; then $\big(\sqrt{(11 + 2\sqrt{29})} + \sqrt{(11 - 2\sqrt{29})} \big)^2 = 22 + 2\sqrt{5}$, so x = y .

Multiplying (z–x) by its three *Conjugates* obtained by reversing signs of $\sqrt{\dots}$ in x produces a polynomial $f(z) := z^4 - 54z^2 - 40z + 269$; now *Maple V r5* simplifies $f(x)$ to 0 but not $f(y)$, though it does simplify expand($f(y)*f(-y)$) to 0 . *Maple 7* on my *Power Mac* simplifies x-y , $f(x)$ and $f(y)$ to 0 promptly.

Problem 2.3: $x := \sqrt{75025} + \sqrt{121393} + \sqrt{196418} + \sqrt{317811}$ and $y := \sqrt{514229} + \sqrt{832040}$.

Solution 2.3: About 13 leading sig. dec. cancel when $x - y = 2.953\ldots/10^9$ is computed carrying at least about 15 sig. dec. as provided by, say, *Matlab.*

See "When Close Enough is Close Enough" by E.R. Scheinerman, pp. 489-499 in *Amer. Math Monthly* **107** #6 (June-July 2000), for a strategy that systematically (though onerously) solves problems like the foregoing three

Problem 2.4: $x := \sum_{n\ge1} \exp(-(n/10^5)^2)$ and $y := 50000\cdot\sqrt{\pi} - 1$.

Solution 2.4: $x \ne y$ though they agree to at least about 42 billion sig. dec.

See "Strange Series and High Precision Fraud" by J.M. and P.B. Borwein, pp. 622-640 in *Amer. Math. Monthly* **99** #7 Aug.-Sept. 1992.

Problem 3: Explain why any matrix B of rank $r > 0$, regardless of its other dimensions, can be factored into a product $B = C\cdot E^T$ in which each matrix C and E has r columns and rank r ; and then explain why the *Kernel* (or *Nullspace*) of E^T is the same as the Kernel of B .

Solution 3: The column-rank r of B is the dimension of its *Range*, the (sub)space *Spanned* by its columns. Choose *any* matrix C whose r columns constitute a *Basis* for Range(B) . The r columns of C must be *Linearly Independent*, and each column **b** of B must be some linear combination $C\cdot\mathbf{e} = \mathbf{b}$ of the columns of C . All such columns **e** can be assembled to constitute the columns of a matrix E^T satisfying $B = C\cdot E^T$. Evidently $\text{rank}(E) \le r = \#(\text{columns of } E)$. Actually $\text{rank}(E) = r$; this can be proved in at least two ways. The simplest way uses this …

Fact: A matrix product's rank cannot exceed the rank of any matrix factor of the product. Therefore $r = \text{rank}(B) = \text{rank}(C\cdot E^T) \le \text{rank}(E^T) \le r$, whence $\text{rank}(E) = \text{rank}(E^T) = r$ as claimed.

If you don't know the Fact you must proceed differently: Suppose $\text{rank}(E) < r$ for the sake of an argument by contradiction. Then some one of the r columns of E , say the last, would have to be a linear combination of the others; this would make $E = \acute{E}\cdot[I, \mathbf{h}]$ in which $\acute{E}$ consists of the first $r-1$ columns of E and column **h** exhibits the $r-1$ coefficients of the aforementioned linear combination. Since $B = C\cdot([I, \mathbf{h}]^T\cdot\acute{E}^T) = (C\cdot[I, \mathbf{h}]^T)\cdot\acute{E}^T$ implies $\text{Range}(C\cdot[I, \mathbf{h}]^T) \supseteq \text{Range}(B)$, we would have to infer that $r-1 \ge \text{column-rank}(C\cdot[I, \mathbf{h}]^T) \ge \text{column-rank}(B) = r$. Impossible**!**

Next we see why $\text{Kernel}(E^T) = \text{Kernel}(B)$: Evidently $\text{Kernel}(B) = \text{Kernel}(C\cdot E^T) \supseteq \text{Kernel}(E^T)$, so suppose for the sake of an argument by contradiction that $B\cdot\mathbf{z} = \mathbf{o}$ but $E^T\cdot\mathbf{z} \ne \mathbf{o}$; then $C\cdot(E^T\cdot\mathbf{z}) = \mathbf{o}$ would imply that the r columns of C are linearly dependent and consequently $\text{column-rank}(C) < r$ contrary to C 's definition. Instead $\text{Kernel}(E^T) \supseteq \text{Kernel}(B) \supseteq \text{Kernel}(E^T)$.

This problem's solution exhumes memories of crucial facts about matrix rank taught in Math. 54 and Math. 110. Three definitions of $r = \text{rank}(B)$ are …

- column-rank(B) := the maximum number r of linearly independent columns of B
  = the dimension r of Range(B) .
- row-rank(B) := the maximum number r of linearly independent rows of B .
- determinantal-rank(B) := the maximum size r of an r-by-r submatrix of B with nonzero determinant.

That these definitions coincide is an important *Theorem*: There is just one $r = \text{rank}(B)$. It connects closely to …

- nullity(B) := the maximum number of linearly independent vectors **z** that satisfy $B\cdot\mathbf{z} = \mathbf{o}$
  = the dimension of Kernel(B) (also called Nullspace(B)) .
- *Theorem*: nullity(B) + rank(B) = the number of columns B has.

The foregoing problem provides a fourth definition of $r = \text{rank}(B)$:

- tensor-rank(B) := the minimum number r of *Dyads* $\mathbf{c}_n\cdot\mathbf{e}_n^T$ whose sum $\sum_{1\le n\le r} \mathbf{c}_n\cdot\mathbf{e}_n^T = C\cdot E^T = B$.

The proof that all four definitions of rank(B) coincide amounts to an algorithm that reduces B to an *Echelon* form at the cost of arithmetic operations and comparisons numbering at most a polynomial function of B's dimensions, at least if B's elements are all rational numbers. If there are transcendental numbers among B's elements, and if rank(B) is possibly smaller than the lesser of the numbers of rows and columns that B has, the computation of rank(B) becomes problematical because we have no algorithm to decide whether any arbitrary arithmetic expression involving transcendental numbers vanishes exactly. Also problematical is rank's estimation in rounded arithmetic.

Tensor-rank can be defined also for multi-dimensional arrays that represent multi-linear operators. Suppose array $M := [\mu_{i,j,k}]$ is a three-dimensional array whose every element $\mu_{i,j,k}$ is tabulated for three indices i, j and k each over some range. Then tensor-rank(M) is the least number r of *Triads* with elements $c_{i,n}\cdot e_{j,n}\cdot g_{k,n}$ whose sum $\sum_{1\le n\le r} c_{i,n}\cdot e_{j,n}\cdot g_{k,n} = \mu_{i,j,k}$ for all indices i, j and k in their allowed ranges. Nobody knows how to compute r .

Problem 4: Given that N and $M := 2 + 2\cdot\sqrt{(1 + 28\cdot N^2)}$ must be a pair of positive integers, show why M is a perfect square in each such pair, and show that some such pairs do exist.

Solution 4: The given equation simplifies to $M\cdot(M - 4) = 7\cdot(4\cdot N)^2$. Evidently M cannot be odd; in fact $M := 4\cdot k$ must be a multiple of 4 satisfying $k\cdot(k - 1) = 7\cdot N^2$. Now only two mutually exclusive possibilities require consideration:

- $k := 7\cdot j$ is a multiple of 7 satisfying $j\cdot(7\cdot j - 1) = N^2$. This equation would imply $j := x^2$ and $7\cdot j - 1 = 7\cdot x^2 - 1 = y^2$ for some integers x and y because $\mathrm{GCD}(j, 7\cdot j - 1) = 1$; but this possibility is impossible because, by trial, no integer y satisfies $y^2 \equiv -1 \ \mathbf{mod}\ 7$.
- $k - 1 := 7\cdot j$ is a multiple of 7 satisfying $(7\cdot j + 1)\cdot j = N^2$. This equation implies $j := y^2$ and $7\cdot j + 1 = 7\cdot y^2 + 1 = x^2$ for some positive integers x and y because $\mathrm{GCD}(j, 7\cdot j + 1) = 1$.

The equation $x^2 - 7\cdot y^2 = 1$ does have positive integer solutions $x = X_L$ and $y = Y_L$ obtained from $X_L \pm Y_L\cdot\sqrt{7} := (8 \pm 3\cdot\sqrt{7})^L$ for $L = 1, 2, 3, \dots$. These solutions provide infinitely many pairs $N_L := X_L\cdot Y_L$ and $M_L := (2\cdot X_L)^2$ satisfying the problem's requirements. For example,

$$X_1 = 8, \quad Y_1 = 3, \quad N_1 = 24, \quad M_1 = 16^2 = 2 + 2\cdot\sqrt{(1 + 28\cdot N_1^2)};$$
$$X_2 = 127, \quad Y_2 = 48, \quad N_2 = 6096, \quad M_2 = 254^2 = 2 + 2\cdot\sqrt{(1 + 28\cdot N_2^2)};$$
$$X_3 = 2024, \quad Y_3 = 765, \quad N_3 = 1548360, \quad M_3 = 4048^2 = 2 + 2\cdot\sqrt{(1 + 28\cdot N_3^2)}.$$

The *Diophantine* equation $x^2 - K\cdot y^2 = 1$ with a non-square integer $K > 0$ is called "Pell's Equation" and has a long history. See ch. 11.4 on *Pell's Equation* in K.H.Rosen's *Elementary Number theory and its Applications* (3rd. ed., 1993, Addison-Wesley). More advanced are Prof. H.W. Lenstra Jr.'s articles "Solving the Pell Equation" on pp. 182-192 of the *Notices of the AMS* **49** #2 (Feb. 2002), http://www.ams.org/notices/200202/fea-lenstra.pdf , and a later postscript version on his web page http://www.math.leidenuniv.nl/~hwl/pell/to.ps . Prof. R.J. Fateman supplied URLs for computerized treatments: Greg Fee's course on Elementary Number Theory, Math. 342 at Simon Fraser University (B.C., Canada) includes Lecture 24, http://www.math.sfu.ca/~gfee/Math324/L241.html, on the continued fraction for $\sqrt{\dots}$ and application to Pell's Equation worked out in *Maple*. Eric Weisstein's posting http://mathworld.wolfram.com/PellEquation.html includes an extensive bibliography about Pell's Equation.

Problem 5: For which positive integers n is $n! \ge n^n \cdot e^{1-n}$? Why? ($e := 2.718281828459\ldots$ is the base of natural logarithms.)

Solution 5: $n! \ge n^n \cdot e^{1-n}$ for every integer $n > 0$. To see why take the logarithm of both sides: $\log(n!) = \sum_{2\le k\le n} \log(k) \ge \sum_{2\le k\le n} \int_{k-1}^{k} \log(x)\,dx = \int_1^n \log(x)\,dx = n\cdot\log(n) - n + 1 = \log(n^n\cdot e^{1-n})$.

Better estimates of $n!$ appear in §6 of the *Handbook of Mathematical Functions* … ed. by M. Abramowitz & I.A. Stegun, reprinted by Dover.

Problem 6(a): Prove that no equilateral triangle in the Cartesian plane can have vertices all of whose coordinates are pairs of integers.

Proof 6(a): Suppose the contrary for the sake of an argument by contradiction. No generality is lost by translating one of the triangle's vertices to the origin $(0, 0)$. Let (x, y) and (p, q) be the other vertices' coordinates. We can presume that $\mathrm{GCD}(x, y, p, q) = 1$ since otherwise the GCD could be divided out of all the coordinates. If the triangle were equilateral we would find that $x^2 + y^2 = p^2 + q^2 = (x-p)^2 + (y-q)^2 > 0$. The equations imply $x^2 + y^2 = p^2 + q^2 = 2(x\cdot p + y\cdot q)$, whence follows that $x^2 + y^2 + p^2 + q^2 = 4(x\cdot p + y\cdot q) \equiv 0 \ \mathbf{mod}\ 4$. This congruence, combined with the presumption that x, y, p and q cannot all be even, would imply that all of them must be odd since every odd square is congruent to $1 \ \mathbf{mod}\ 4$. But this is contradicted by the equation $p^2 + q^2 = (x-p)^2 + (y-q)^2 \equiv 0 \ \mathbf{mod}\ 4$. Therefore no equilateral triangle can have vertices all of whose coordinates are pairs of integers.

The foregoing proof was found by E. Lucas in 1878. For a longer proof see problem 6(b). Roman Vaisberg's shorter proof follows.

The area of any triangle whose coordinates are pairs of integers, say $(0, 0)$, (x, y) and (p, q) , is $|\det(\begin{bmatrix} x & p \\ y & q \end{bmatrix})|/2$, which is an integer or half-integer. The area of an equilateral triangle is $s^2\cdot\sqrt{3}/4$ where its squared edge-length would be $s^2 = x^2 + y^2 = p^2 + q^2 = (x-p)^2 + (y-q)^2$, an integer, if all this triangle's vertices had the aforementioned pairs of integers as coordinates. This is not possible; $s^2\cdot\sqrt{3}/4$ would be irrational, neither an integer nor a half-integer.

Problem 6(b): A triangle in the Cartesian plane can have vertices all of whose coordinates are pairs of integers only if each angle's tangent is a rational number or ∞ . Conversely, if each angle's tangent is rational or ∞ then a *Similar* (a scalar multiple of a rotated and translated copy of the) triangle has vertices all of whose coordinates are pairs of integers. Prove these assertions.

Proof 6(b): The "only if" assertion is proved by supposing triangle ABC has each vertex at a point with integer coordinates. If neither edge emanating from A is parallel to, say, the vertical axis draw a horizontal line segment AP and treat angle $\angle A$ as the difference between two angles BAP and CAP each of which obviously has a finite rational tangent. In this case the formula $\tan(\alpha - ß) = \big(\tan(\alpha) - \tan(ß)\big)/\big(1 + \tan(\alpha)\cdot\tan(ß)\big)$ provides a rational tangent for $\angle A$. If both edges emanating from A are parallel to coordinate axes then $\angle A$ is a right angle with ∞ as its tangent, and the other two angles obviously have rational tangents. If just one edge emanating from A is either vertical or horizontal, the edge joins two angles with rational tangents, and the third angle has a rational tangent too because the three angles add up to π. End of "only if".

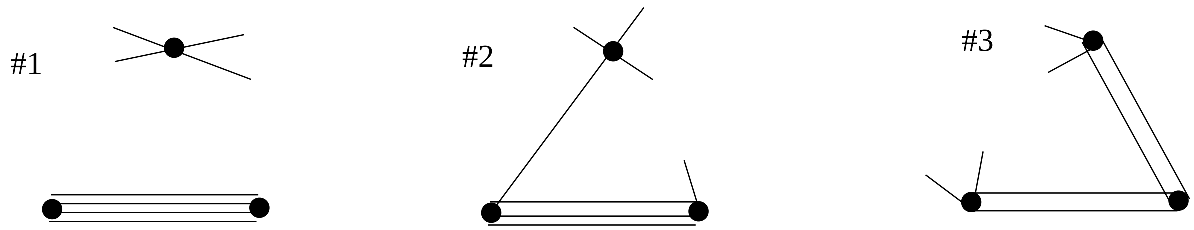
To prove the "if" assertion suppose every angle of ABC has a rational tangent; the exceptional case of a right-angled triangle is dispatched by placing the right angle at (0, 0) and aligning its adjacent edges along the coordinate axes. Thus we may presume henceforth that no angle of ABC is a right angle. Drop the altitude AP from vertex A perpendicular to edge BC or its extension so that P falls on line BC. Now both $\tan(\angle B) = AP/BP$ and $\tan(\angle C) = AP/PC$ are rational numbers that can be assigned a common denominator: $AP/BP = k/n$ and $AP/PC = m/n$, perhaps with different signs in case point P does not lie between vertices B and C. Thus ABC is Similar to triangle $\overline{A}\,\overline{B}\,\overline{C}$ with vertices $\overline{A} := (0, k\cdot m)$, $\overline{B} := (-m\cdot n, 0)$ and $\overline{C} := (k\cdot n, 0)$ because $\tan(\angle\overline{B}) = \tan(\angle B)$ and $\tan(\angle\overline{C}) = \tan(\angle C)$, so $\angle\overline{B} = \angle B$ and $\angle\overline{C} = \angle C$.

This problem was attributed in the 1980s to Stanford's John McCarthy, a pioneer in Artificial Intelligence. See "Triangles with Vertices on Lattice Points" by M.J. Beeson, pp. 243-252 in *Amer. Math. Monthly* **99** #3 (Mar. 1992).

Problem 7: For any collection X of students let F(X) be how many of them speak French, G(X) how many speak German, and R(X) how many speak Russian. Some students speak more than one of these languages. Given a collection S satisfying $F(S) = G(S) = R(S) = 24$, show how to choose a subset W of S satisfying $F(W) = G(W) = R(W) = 12$. (Lengthy)

Solution 7: The desired subset W will be constructed as the disjoint union of smaller subsets $V_1, V_2, V_3, \ldots$ each of which has either $F(V_j) = G(V_j) = R(V_j) = 1$ or $F(V_j) = G(V_j) = R(V_j) = 2$ as follows: Dismiss every student who speaks none of the three languages. Any student who speaks all three languages constitutes a *Singleton* subset U satisfying $F(U) = G(U) = R(U) = 1$. Any three students of whom one speaks only French, one only German and one only Russian constitute a *Trio* subset T that satisfies $F(T) = G(T) = R(T) = 1$. Any three students of whom one speaks just French and German, one speaks just German and Russian, and the third speaks just Russian and French constitute a *Triad* Δ that satisfies $F(\Delta) = G(\Delta) = R(\Delta) = 2$. Segregate every subset like U or T or Δ from S until, in the collection Z of students remaining, each student speaks just one or two of the three languages, and no three students in Z each speaks one of each language, and no three students include just two speakers of each of the three languages.
Note: $F(Z) = G(Z) = R(Z) = F(S) - \#(\text{singletons like } U) - \#(\text{trios like } T) - 2\cdot\#(\text{triads like } \Delta)$.

To better visualize the kinds of students remaining in Z , represent each language by a dot • , and represent each bilingual student by an *Edge*, a line segment joining the two dots of this student's languages. The edges cannot include a triangle because it would already have been segregated from Z as a triad like Δ . Represent each monolingual student by a *Stub*, a short line segment emanating from the dot of this student's language and connecting with nothing else. At least one of the dots is connected to no stubs; otherwise another trio like T would have been segregated from Z . Still, from each dot emanates the same number, namely F(Z) = G(Z) = R(Z) , of line segments, be they edges or stubs. When this number is 4 the following pictures are possible:



• If some dot is connected to no other by an edge, as happens in picture #1, then that dot has the same number of stubs as there are edges connecting the other two dots; neither of these can have any stubs lest both have them, contradicting the segregation of trios. That number of *Pairs* can now be segregated from Z : Each pair ∏ is a subset including one monolingual and one bilingual student, so F(∏) = G(∏) = R(∏) = 1 . These pairs exhaust Z .

• If one dot, say French, is connected by at least one edge to each of the other two dots, these two cannot be joined by an edge since Z has no more triads like Δ . The number of edges emanating from French must equal the sum of the numbers of edges emanating from the other two dots, each of which has as many stubs as the other has edges. French can have no stubs. Segregate now a *Quad* Q of four students from Z chosen thus: One student (stub) speaks only German, one (stub) speaks only Russian, one (edge) speaks French and German, and the fourth (edge) speaks French and Russian, so F(Q) = G(Q) = R(Q) = 2 . Keep on segregating quads from Z until either it is exhausted or one dot is connected to no other by an edge, and then revert to the previous paragraph to exhaust Z by segregating pairs.

Finally, 24 = F(S) = 2·#(quads Q) + 2·#(triadsΔ) + #(pairs ∏) + #(trios T) + #(singletons U) .

Now we can assemble the desired subset W satisfying F(W) = G(W) = R(W) = 12 by uniting first some quads, and then (if needed) some triads, and then (if needed) some pairs, and then (only if needed) some trios and/or singletons until
12 = F(W) = G(W) = R(W) = 2·#(quads) + 2·#(triads) + #(pairs) + #(trios) + #(singletons) ,
counting only the quads, triads, pairs, trios and/or singletons united into W .