Notes on 2-Dimensional Spaces

R^2 versus Euclidean 2-space

What's the difference? Why distinguish the vector space R^2 of pairs of real numbers from the familiar Euclidean plane? Or the Euclidean plane from the complex plane?

There really are differences. To visualize some of them imagine a vertical pane of glass upon which are drawn many of the familiar figures -- circles, squares, triangles, ... -- that you have seen in the Euclidean plane. Then look at the shadows cast on the floor at night by one star's light passing through the pane as through a window. Choose a star that lies in neither of the planes of the glass nor floor lest all shadows collapse into a line. What properties do shadows inherit from the figures that cast them? The *Affine* geometry of R^2 addresses this question.

Evidently straight lines in the pane cast straight line shadows. And triangles cast triangular shadows and parallelograms cast parallelogram shadows. But, unless the star is situated in a very special direction, certain figure's shadows will be distorted like the shadows cast by the sun at dawn or dusk; circles will cast elliptical shadows, and most rectangles will cast non-rectangular parallelograms as shadows.

How do you know that circles are really circular, not slightly elliptical? If you suffer from astigmatism, you cannot tell at a glance. (Astigmatism was alleged to have caused the Spanish painter *El Greco* (Domenico Theotocupuli, 1542? - 1614) to distort the faces of his subjects, but he probably did it for dramatic effect.) Many a programmer has been annoyed by ellipses he got on his computer screen when he thought he was plotting circles but overlooked the screen's aspect ratio. A TV set with vertical or horizontal sweep of improperly adjusted magnitude will either overflow the screen's boundaries or show a black margin; either way, it usually displays circles as ellipses. And, during the beginning or ending of many a movie shown on TV though intended originally to be projected onto a wide screen, horizontal compression of the picture causes characters to look as if El Greco had painted them.

The Affine geometry of R^2 concerns those shadows and screens. Lacking a way to compare distance in different directions, and lacking a measure of angle, the Affine geometry of R^2 concerns *only* those properties of figures that can be inferred from their shadows cast by the light of an unknown star. How are figures and their shadows related?

Linear Operators upon R^2

Casting a shadow by starlight is a Linear Operator in the sense studied in a Linear Algebra course. This is so because parallelograms cast parallelogram shadows, so a family of line segments parallel to each other and all of the same length, as might represent different pictures of the same vector in the pane, casts a shadow that is a similar family on the floor. In short, shadowing by starlight maps vectors in a plane to vectors in another plane. And a linear combination of two vectors maps to the same linear combination of the two mapped vectors because, by mapping *every* parallelogram to a parallelogram, the shadow map preserves the ratios of lengths of parallel line segments.

Why starlight? Why not take shadows by the light of a street lamp? The reason is that a star can be regarded as practically infinitely far away, so rays of light from it are all parallel. But rays of light from a street lamp diverge, causing a parallelogram on the window pane to cast a quadrilateral shadow that might not be a parallelogram. A similar effect causes "keystoning" when a slide projector is aimed askew at the screen, not perpendicular to it. Projection from a finite source through non-parallel planes, like a perspective view, maps lines to lines but does not necessarily preserve parallelism, so it is not a Linear Operator in the sense of a Linear Algebra course. (That kind of projection is the subject of *Projective Geometry*.)

Let $\mathbf{L}$ denote the linear operator that maps vectors in the pane to vectors on the floor; "linear" means $\mathbf{L}(ß\mathbf{x} + \mu\mathbf{y}) = ß\mathbf{L}\mathbf{x} + \mu\mathbf{L}\mathbf{y}$. Given two arbitrary regions F and G in the pane, what can we say about their shadows $\mathbf{L}F$ and $\mathbf{L}G$ on the floor without knowing exactly what $\mathbf{L}$ is? One thing we can say comes from cross-hatching the pane by two families of uniformly closely spaced parallel lines, thus covering the pane by a tiling of tiny parallelograms. If each of them is numbered, then their shadows on the floor will be numbered the same way, and the numbers of parallelograms intersected by F and G will be the same as for their respective shadows, so

(Oriented Area of $\mathbf{L}F$)/(Oriented Area of F) = (Oriented Area of $\mathbf{L}G$)/(Oriented Area of G) .

We have to say "Oriented Area" here instead of just "Area" to account for the possibility that an area is assigned a sign that depends upon the direction, clockwise or counter-clockwise, in which its boundary is traversed. The orientation of an area F reverses when you change your point of view from one side of the pane to the other.

The ratio of oriented areas in the last equation depends upon $\mathbf{L}$, not the regions. To compute it we must first choose a *Basis* in each of the two vector spaces, then compute a matrix L for $\mathbf{L}$, and then compute the determinant det(L) . Provided the basis vectors in each space span parallelograms of equal oriented area this computation defines "Determinant" for operators $\mathbf{L}$:

det($\mathbf{L}$) := (Oriented Area of $\mathbf{L}F$)/(Oriented Area of F) = det(L) .

For example, if the sides of the parallelograms in the cross-hatched pane serve as basis vectors, and their shadows serve as basis vectors for the floor, then the matrix of $\mathbf{L}$ will be the identity matrix I and det($\mathbf{L}$) = det(I) = 1 . But if the basis on the floor is chosen independently of the basis in the pane then det($\mathbf{L}$) will have some other nonzero value, namely the ratio of area of a parallelogram's shadow on the floor over the area of that same parallelogram in the pane. Like bases, units of area can be chosen so arbitrarily in R^2 that, without knowing them in advance, we can predict only that det($\mathbf{L}$) $\neq 0$. To define det($\mathbf{L}$) independently of different bases and different units of area, we must confine det(…) to linear operators that map a space to itself.

(Some authors take for granted that both spaces are Euclidean with orthonormal bases, thus defining det($\mathbf{L}$) for what appears to be a map $\mathbf{L}$ from one space to another; but since two Euclidean planes are as indistinguishable as are identical twins, little is lost by thinking of the two as the same plane. The English language is ill-equipped for discourse about different yet identical things. Try to find "a distinction without a difference" in a dictionary.)

Suppose another star, of a different color to avoid confusion, casts shadows from the pane to the floor. The mapping $\mathbf{C}$ that takes shadows of one color to those of the other is a linear map from an instance of R^2 to itself. As it did for $\mathbf{L}$, the ratio of oriented areas defines det($\mathbf{C}$) , but now it is independent of the basis chosen for the floor. This determinant turns out to be the same as the determinant of *every* matrix that represents $\mathbf{C}$ regardless of basis. (Exercise: The sign of det($\mathbf{C}$) tells whether the stars are on the same side of the pane or not. How?)

Euclidean 2-space versus R^2

What does Euclidean 2-space — we call it E^2 — have that R^2 hasn't? Every vector **z** in E^2 has a *Length* $\|\mathbf{z}\|$ that behaves as follows:

Four properties of Euclidean Length:

Positivity:	$\|\mathbf{z}\| > 0$ except that $\|\mathbf{o}\| = 0$.
Homogeneity:	$\|ß\cdot\mathbf{z}\| = \|ß\|\cdot\|\mathbf{z}\|$ for every scalar ß .
Triangle Inequality:	$\|\mathbf{y}+\mathbf{z}\| \le \|\mathbf{y}\| + \|\mathbf{z}\|$.
Parallelogram law:	$\|\mathbf{y}+\mathbf{z}\|^2 + \|\mathbf{y}-\mathbf{z}\|^2 = 2\cdot\|\mathbf{y}\|^2 + 2\cdot\|\mathbf{z}\|^2$.

The *Distance* between two points is the length of the vector that moves one to the other. Euclidean geometry concerns just those properties that figures retain after any transformation that preserves the distances between their points; such transformations turn out to include only

translations, rotations and reflections.

The geometry of R^2 , on the other hand, concerns a smaller set of properties retained after a far richer set of transformations that can map any parallelogram, even a square, onto any other parallelogram. Therefore, Euclidean notions of *Angle* and of *Congruent Triangles* (those whose respective sides have the same lengths) have no counterparts in R^2 . Two triangles are *Similar* in R^2 only if their respective sides are parallel, whereas similar triangles in E^2 need only have respective angles equal (and therefore respective sides proportional) regardless of orientation. Another difference between the geometries of R^2 and E^2 will be mentioned later.

Different Definitions of Length

The formula for computing distance in E^2 is usually derived from the *Pythagorean Formula* :

column vector $z = \begin{bmatrix} \xi \\ \eta \end{bmatrix}$ in E^2 has length $\|z\|_2 := \sqrt{(z^T z)} = \sqrt{(\xi^2 + \eta^2)}$.

This is not the only reasonable way to define length. Another way is $\|z\|_1 := |\xi| + |\eta|$; this is the "Taxi-cab" distance because $\|z–w\|_1$ is proportional to the mileage for a trip from w to z when streets are all laid out parallel to coordinate axes. Another way is $\|z\|_\infty = \max\{|\xi|, |\eta|\}$; this length could be the one that matters to a courier service that wishes to put all envelopes destined for the same city into the same square box.

Each of the three length functions $\|\ldots\|_2$, $\|\ldots\|_1$, $\|\ldots\|_\infty$ has the first three properties listed above, but only $\|\ldots\|_2$ has the fourth. (Confirm this yourself!) Geometry in a space with one of those lengths must vaguely resemble geometry with any other in so far as a vector "big" in one length is "big" in all three, and similarly for "tiny", since, for every column 2-vector z ,

$$\|z\|_2/\sqrt{2} \le \|z\|_\infty \le \|z\|_2 \le \|z\|_1 \le \|z\|_2\sqrt{2} .$$

(Can you prove this?) But, lacking the parallelogram law, $\|\ldots\|_1$ and $\|\ldots\|_\infty$ are preserved by a small group of rotations and reflections, so their geometry is less interesting than E^2 's.

The parallelogram law distinguishes E^2 from every other 2-dimensional *Normed Space*, but it does not determine uniquely the formula for length. Other formulas exist that work as well; in general $\|z\| := \sqrt{((L^{-1}z)^T(L^{-1}z))}$ for any invertible 2-by-2 matrix L . (You should confirm that this $||z||$ has all four properties of Euclidean length.) Must L be invertible? No; but if L^{-1} were replaced by a matrix S with a nonzero nullspace, the possibility that $Sz = o$ for a nonzero z would violate the *Positivity* of the last formula for $||z||$.

What does the last formula for $||z||$ mean geometrically? Recall again the linear operator **L** that mapped figures drawn in a pane of glass to their shadows cast upon the floor by starlight. Draw Cartesian axes on the pane to determine coordinates there, and independently draw axes on the floor to determine coordinates there too. (It is convenient, but not necessary, to place the origin **o** on the floor at the shadow of the pane's origin **o** .) Every vector **x** in the pane has for coordinates a column 2-vector x , and its shadow $\mathbf{z} = \mathbf{Lx}$ on the floor has coordinates $z = Lx$ where L is the 2-by-2 matrix that represents **L** in the chosen coordinate systems. Conversely, if z represents a vector on the floor, it is the shadow of $L^{-1}z$ in the pane. Once we know L we can determine the length $||\mathbf{x}|| = \sqrt{(x^Tx)}$ of a vector **x** in the pane by measuring the components $z = Lx$ of that vector's shadow on the floor; then $||\mathbf{x}|| = \sqrt{((L^{-1}z)^T(L^{-1}z))}$. When we use this formula to define $||\mathbf{z}||$, we assign to **z** the Euclidean length of the vector **x** of which **z** is merely the shadow, just as we assign to lines on a road map the lengths of the roads the lines represent.

But we live in a world where shadow can be hard to distinguish from substance. How do we know which space, the pane or the floor, deserves to be treated as the original? How can we tell when a picture of a painting distorts the painting? How can someone who suffers seriously from astigmatism tell which ellipses are really circles, which parallelograms really squares? Without a sample of a circle or square to use as a reference, the only way to tell is to perform appropriate experiments with rigid bodies. In a Euclidean space these experiments determine the elements of a real symmetric matrix A such that $||\mathbf{z}|| = \sqrt{(z^TA^{-1}z)}$ and from which L can be computed to satisfy $LL^T = A$, as we shall see later. Then $x := L^{-1}z$ is a transformation to orthonormal coordinates from which $||\mathbf{z}|| = \sqrt{(x^Tx)}$ may be computed. Thus the Pythagorean Formula can be taken for granted as the definition of length in E^2 or its shadows since we can always find (infinitely many) orthonormal coordinate systems for that purpose.

A crucial experiment that distinguishes E^2 from other normed spaces or from R^2 is to find for *each* line £ a family of length-preserving reflections **W** that merely reverse £ ; any line left unchanged by such a **W** is called *perpendicular* or *orthogonal* to £ . (Analogous length-preserving reflections preserve *planes* perpendicular to a line in E^3 .) R^2 has no notion of perpendicularity because length is undefined so reflections cannot preserve it, though they do preserve magnitudes of areas while reversing orientation. In other normed spaces there are lines that no length-preserving reflection can reverse.

Exercises:

Reflections W in R^n are transformations of the form $x \longrightarrow W(x) := \mu c + (I - (2/r^Tc)cr^T)(x-\mu c)$ where μ is any scalar constant, and c is any constant column n-vector and r^T any constant n-row such that $r^Tc \neq 0$. Which line (if $n = 2$) or plane (if $n = 3$) or hyperplane (if $n > 3$) does $W(x)$ preserve? Which family of lines £ does it reverse? What is $W(W(x))$?

Reflections in E^n have the same form but, if length $||z|| = \sqrt{(z^Tz)}$, how must c and r^T be correlated to ensure that $||W(x) - W(y)|| = ||x-y||$ for every x and y ?

A "Rigid Body Motion" of E^2 maps it to itself in such a way as preserves distances between points; such motions can be proved to consist of translations, rotations and reflections. Show how each translation can be composed out of two reflections, and so can each rotation.

The Spaces Dual to R^2 and E^2

The space dual to R^2 is the space of linear functionals that map vectors in R^2 to scalars and do so linearly. If vectors in R^2 are represented as column 2-vectors x, all linear functionals are represented by row 2-vectors w^T. Geometrically each nonzero row vector w^T defines a family of parallel lines drawn in R^2; each such line has " $w^Tx = \text{constant}$ " as its equation. In R^2 there is no geometrical relationship between a row w^T and a column w that happens to have the same components; a change of basis can change w to Cw and w^T to w^TC^{-1} without keeping Cw and w^TC^{-1} transposes of each other. Geometrically this means that, since a linear map of R^2 to itself can map any two non-parallel straight lines to any other two, pairs of straight lines can have no special relationship other than parallelism.

E^2 is its own dual space because the connection between column vectors x and row vectors x^T with the same components is preserved by changes of coordinates that preserve length as determined by the Pythagorean formula; such coordinate systems are called " orthonormal." These coordinate changes are linear maps of E^2 to itself that also preserve the angles at which lines intersect. Angle and perpendicularity (the relation between two lines confirmed when a length-preserving reflection preserves one line while reversing the other) are not relationships unique to Euclidean geometry but relatively few maps of a non-Euclidean normed space to itself preserves them.

E^2 versus the Complex Plane C

They certainly look alike to casual observers, but they differ because certain operations natural for C make little sense in E^2. Complex numbers can be multiplied or divided to produce other complex numbers; when reinterpreted as operations upon vectors in E^2 these operations lose their significance. The significant maps of C to itself, called *conformal maps*, preserve the angles at which curves intersect; such a map can wrap C by *stereographic projection* 1-to-1 onto a sphere, called the *Riemann* sphere, with a single point at ∞. Identifying C with this sphere makes conformal maps, each an *Analytic* function of a complex variable, easier to understand. For instance the conformal map that takes z to $2 + 1/(z-3)$ maps the sphere C to itself so smoothly that it is infinitely differentiable despite that it takes 3 to ∞. Maps of E^2 to itself that preserve distances between points cannot map a finite point to ∞; this plane cannot have a single point at ∞ but must instead be bounded by a remote "circle at infinity" or "line at infinity" on which each "point" represents a different direction of departure to infinity. In short, there is no continuous 1-to-1 map between the plane E^2 and the sphere C.

Just as a balloon may be punctured, opened out and spread flat onto a plane, so the punctured sphere C may be spread onto E^2, but doing so does not make them the same thing.