# On the Convergence of No-regret Learning in Selfish Routing

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# Introduction

Routing game: players choose routes.
Population distributions: $\mu^{(t)} \in \Delta^{\mathcal{P}_1} \times \cdots \times \Delta^{\mathcal{P}_K}$
Nash equilibria: $\mathcal{N}$

- Under no-regret dynamics, $\bar{\mu}^{(t)} = \frac{1}{t}\sum_{\tau \leq t} \mu^{(\tau)} \to \mathcal{N}$.
- Does $\mu^{(t)} \to \mathcal{N}$?

# Outline

# Routing game

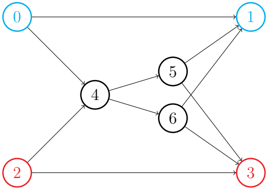
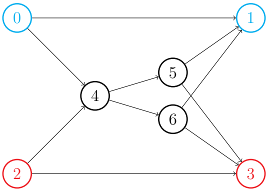



Figure : Example network

- Directed graph $(V, E)$
- Population $\mathcal{X}_k$: paths $\mathcal{P}_k$

# Routing game

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- Population $\mathcal{X}_k$: paths $\mathcal{P}_k$
- Player $x \in \mathcal{X}_k$: distribution over paths $\pi(x) \in \Delta^{\mathcal{P}_k}$,
- Population distribution over paths $\mu^k \in \Delta^{\mathcal{P}_k}$, $\mu^k = \int_{\mathcal{X}_k} \pi(x) dm(x)$
- Loss on path $p$: $\ell_p^k(\mu)$

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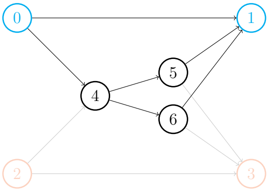


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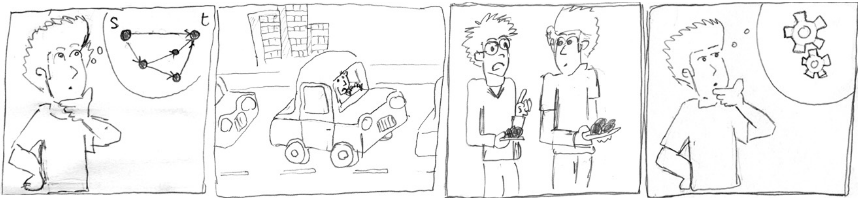
# Online learning model

$\pi^{(t)} \in \Delta^{\mathcal{P}_1}$ | Sample $p \sim \pi^{(t)}$ | Discover $\ell^{(t)} \in [0,1]^{\mathcal{P}_1}$ | Update $\pi^{(t+1)}$



# The Hedge algorithm

## Hedge algorithm

- Update the distribution according to observed loss

$$\pi_p^{(t+1)} \propto \pi_p^{(t)} e^{-\eta_t \ell_p^{k(t)}}$$

# Nash equilibria

**Nash equilibrium**

$\mu \in \mathcal{N}$ if $\forall k$, $\forall p \in \mathcal{P}_k$ with positive mass,

$$\ell_p^k(\mu) \leq \ell_{p'}^k(\mu) \ \forall p' \in \mathcal{P}_k$$

- How to compute Nash equilibria?

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- How to compute Nash equilibria? Convex formulation

# Nash equilibria

## Convex potential function

$$V(\mu) = \sum_e \int_0^{(M\mu)_e} c_e(u)du$$

- $V$ is convex.
- $\nabla_{\mu^k} V(\mu) = \ell^k(\mu)$.
- Minimizer not unique.

- How do players find a Nash equilibrium?
- Iterative play.

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- How do players find a Nash equilibrium?
- Iterative play.
- Ideally: distributed, and has reasonable information requirements.

# Assume sublinear regret dynamics

- Losses are in $[0, 1]$.
- Expected loss is $\left\langle \pi^{(t)}(x), \ell^k(\mu^{(t)}) \right\rangle$
- Discounted regret

$$\bar{r}^{(T)}(x) = \frac{\sum_{t \leq T} \gamma_t \left\langle \pi^{(t)}(x), \ell^k(\mu^{(t)}) \right\rangle - \min_p \sum_{t \leq T} \gamma_t \ell_p^{k(t)}}{\sum_{t \leq T} \gamma_t}$$

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## Assumptions

- $\gamma(t) > 0$
- $\gamma(t) \downarrow 0$
- $\sum_t \gamma(t) = \infty$

# Convergence to Nash equilibria

## Population regret

$$\bar{r}^{k(T)} = \frac{1}{m(\mathcal{X}_k)} \int_{\mathcal{X}_k} \bar{r}^{(T)}(x) dm(x)$$

## Convergence of averages to Nash equilibria

If an update has sublinear population regret, then $\bar{\mu}^{(T)} = \sum_{t \leq T} \gamma_t \mu^{(t)} / \sum_{t \leq T} \gamma_t$ converges

$$\lim_{T \to \infty} d\left(\bar{\mu}^{(T)}, \mathcal{N}\right) = 0$$

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Proof: show

$$V(\bar{\mu}^{(T)}) - V(\mu^*) \le \sum_k \bar{r}^{k(T)}$$

Similar result in Blum et al. (2006)

# Convergence of a dense subsequence

**Proposition**

Under any algorithm with sublinear discounted regret, a dense subsequence of $(\mu^{(t)})_t$ converges to $\mathcal{N}$

- Subsequence $(\mu^{(t)})_{t \in \mathcal{T}}$ converges
- $\lim_{T \to \infty} \frac{\sum_{t \in \mathcal{T} : t \leq T} \gamma_t}{\sum_{t \leq T} \gamma_t} = 1$

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*Proof.*

- Absolute Cesàro convergence implies convergence of a dense subsequence.

# Example: Hedge with learning rates $\gamma_\tau$

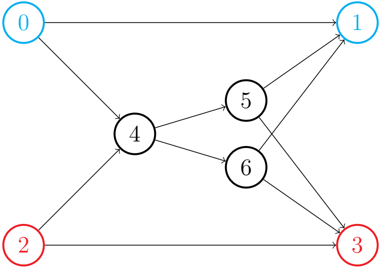
$$\pi_p^{(t+1)} \propto \pi_p^{(t)} e^{-\eta_t \ell_p^{k(t)}}$$

**Regret bound**

Under Hedge with $\eta_t = \gamma_t$,

$$\bar{r}^{(T)}(x) \leq \rho \frac{\ln \pi_{\min}^{(0)}(x) + c \sum_{t \leq T} \gamma_t^2}{\sum_{t \leq T} \gamma_t}$$

# Simulations

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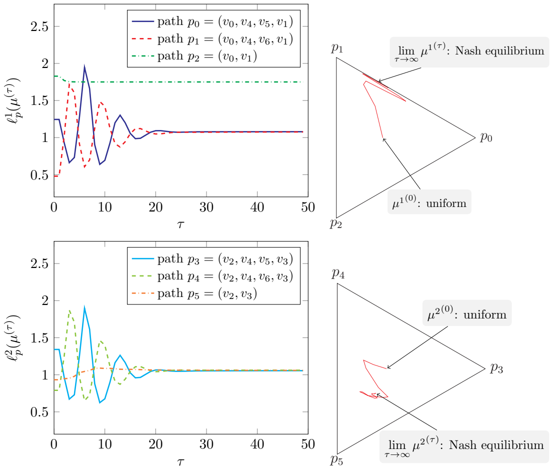


Figure : Path losses and strategies for the Hedge algorithm with $\gamma_\tau = 1/(10 + \tau)$

# Sufficient conditions for convergence of $(\mu^{(t)})_t$

- Have $\bar{\mu}^{(t)} \to \mathcal{N}$.

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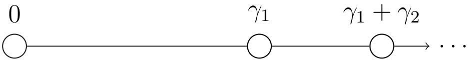
**Sufficient condition**

If $V(\mu^{(t)})$ converges ($\mu^{(t)}$ need not converge), then

- $V(\mu^{(t)}) \to V_*$
- $\mu^{(t)} \to \mathcal{N}$ ($V$ is continuous, $\mu^{(t)} \in \Delta$ compact)

# Replicator dynamics

Imagine an underlying continuous time. Updates happen at $\gamma_1, \gamma_1 + \gamma_2, \dots$

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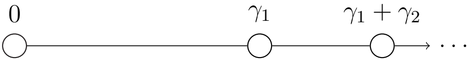


Figure : Underlying continuous time

In the update equation $\mu_p^{(t+1)} \propto \mu_p^{(t)} e^{-\gamma_t \ell_p(t)}$, take $\gamma_t \to 0$
We obtain the autonomous ODE:

**Replicator equation**

$$\forall p \in \mathcal{P}_k, \frac{d\mu_p^k}{dt} = \mu_p^k \left( \langle \ell^k(\mu), \mu^k \rangle - \ell_p^k(\mu) \right) \tag{1}$$

Also in evolutionary game theory.

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**Theorem (Fischer and Vöcking (2004))**

Every solution of the ODE (1) converges to the set of its stationary points.

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Proof: $V$ is a Lyapunov function.

# AREP update

## Discretization of the continuous-time replicator dynamics

$$\pi_p^{(t+1)} - \pi_p^{(t)} = \eta_t \pi_p^{(t)} \left( \left\langle \ell^k(\mu^{(t)}), \pi^{(t)} \right\rangle - \ell_p^k(\mu^{(t)}) \right) + \eta_t U_p^{k(t+1)}$$

$(U^{(t)})_{t\geq 1}$ perturbations that satisfy for all $T > 0$,

$$\lim_{\tau_1 \to \infty} \max_{\tau_2 : \sum_{t=\tau_1}^{\tau_2} \eta_t < T} \left\| \sum_{t=\tau_1}^{\tau_2} \eta_t U^{(t+1)} \right\| = 0$$

Benaïm (1999)

# Convergence to Nash equilibria

**Theorem**

Under any no-regret algorithm which is approximate REP, $\mu^{(t)} \to \mathcal{N}$.

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**Theorem**

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Proof uses two facts

- Affine interpolation of $\mu^{(t)}$ is an asymptotic pseudo trajectory for the ODE.
- $V$ is a Lyapunov function for Nash equilibria.

# REP update

In particular

- REP update: take $U = 0$

$$\pi_p^{(t+1)} - \pi_p^{(t)} = \eta_t \pi_p^{(t)} \left( \left\langle \ell^k(\mu^{(t)}), \pi^{(t)} \right\rangle - \ell_p^k(\mu^{(t)}) \right)$$

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- Hedge

$$\pi_p^{(t+1)} - \pi_p^{(t)} = \eta_t \pi_p^{(t)} \frac{e^{-\eta_t \ell_p^k(\mu^{(t)})} - 1}{\eta_t \sum_{p'} e^{-\eta_t \ell_{p'}^k(\mu^{(t)})}}$$

# Mirror Descent

Consider the convex problem

$$\text{minimize}_{\mu \in \Delta} V(\mu)$$

**Algorithm 1** Mirror Descent Method

1: **for** $t \in \mathbb{N}$ **do**

2: $\mu^{(t+1)} = \arg\min_{\mu \in \Delta} \langle \nabla V(\mu^{(t)}), \mu \rangle + \frac{1}{\eta_t} D_\psi(\mu, \mu^{(t)})$

3: **end for**

where $D_\psi$ is a Bregman divergence

$$D_\psi(\mu, \nu) = \psi(\mu) - \psi(\nu) - \langle \nabla \psi(\nu), \mu - \nu \rangle$$

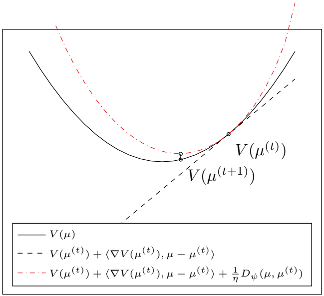


Figure : Mirror Descent iteration.

# Mirror Descent

**Hedge = Mirror descent on $V$**

- Take $D_\psi(\mu, \nu) = \sum_k D_{KL}(\mu^k, \nu^k)$
- Update:

$$\mu^{(t+1)} = \arg \min_{\mu \in \Delta^1 \times \cdots \times \Delta^K} \sum_k \left( \left\langle \ell^k(\mu^{(t)}), \mu^k \right\rangle + \frac{1}{\eta_t} D_{KL}(\mu^k, \mu^{k(t)}) \right)$$

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- Solution: Hedge algorithm with learning rate $\eta$

$$\mu_p^{k(t+1)} \propto \mu_p^{k(t)} e^{-\eta \ell_p^{k(t)}}$$

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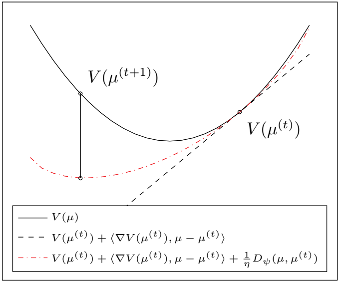
## General result

Convergence of $\bar{\mu}^{(T)} = \frac{\sum_{t \leq T} \eta_t \mu^{(t)}}{\sum_{t \leq T} \eta_t}$ to $\mathcal{N}$ for Any Mirror Descent method

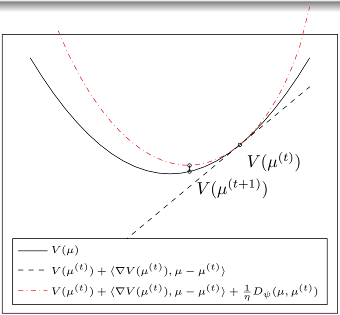
# Strong convergence of Mirror Descent

## Convex $V$ with $L$-Lipschitz gradient

If $\eta_t$ small enough, MD update guarantees $V(\mu^{(t+1)}) \leq V(\mu^{(k)})$.




(a) Large $\eta$




(b) Small $\eta$

Figure : Mirror Descent iteration for a function with $L$-Lipschitz gradient.

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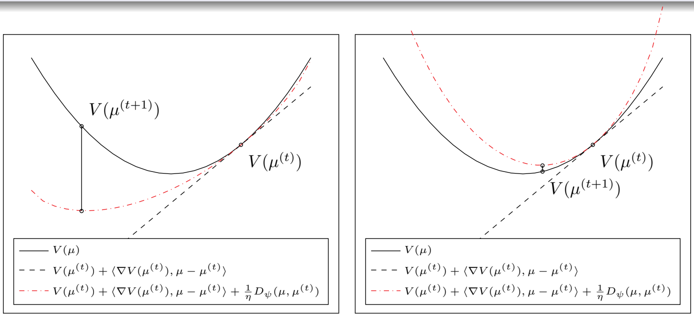


(a) Large $\eta$ (b) Small $\eta$

Figure : Mirror Descent iteration for a function with $L$-Lipschitz gradient.

$V(\mu^{(t)})$ is monotone, converges, so $\mu^{(t)} \to \mathcal{N}$.

# Summary

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- Convergence of $\bar{\mu}^{(t)}$ under no-regret updates.
- Convergence of a dense subsequence $(\mu^{(t)})_{t \in \mathcal{T}}$.
- Convergence of $\mu^{(t)}$ for no-regret AREP updates.
  - Hedge, REP
- Convergence of $\mu^{(t)}$ for MD updates (+ convergence rate)
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Future work

- Bandit setting.
- Stochastic perturbations on the losses.

Thank you.

Poster M43

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