UC Berkeley
Department of Electrical Engineering and Computer Science

EECS 227A
NONLINEAR AND CONVEX OPTIMIZATION

Problem Set 5
Fall 2009

Issued: Thursday, October 22 **Due:** Thursday, November 5, 2009

Reading: Boyd and Vandenberghe, Chapter 5

Problem 5.1
Use a Lagrange multiplier approach to solve the problem

$$\min_{x \in \mathbb{R}^n} \sum_{i=1}^{n} x_i \quad \text{such that } h(x) = \|x\|_2^2 - 1 = 0.$$

Give a geometric interpretation of the Lagrange multiplier conditions.

Problem 5.2
Given an $n \times n$ symmetric matrix Q, define

$$e_1 = \arg \min_{\|x\|_2=1} x^T Q x \quad \text{and} \quad \lambda_1 = \min_{\|x\|_2=1} x^T Q x,$$

and for $k = 1, 2, \ldots, n-1$,

$$e_{k+1} = \arg \min_{\|x\|_2=1} x^T Q x \quad \text{such that } e_i^T x = 0,\ i = 1, \ldots, k, \text{ and}$$
$$\lambda_{k+1} = \min_{\|x\|_2=1} x^T Q x \quad \text{such that } e_i^T x = 0,\ i = 1, \ldots, k.$$

Using optimization principles and theory:

(a) Show that $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$.

(b) Show that the vectors $e_1, \ldots, e_n$ are linearly independent.

(c) Show how $\lambda_1, \ldots, \lambda_n$ can be interpreted as Lagrange mulipliers.

Problem 5.3
This exercise develops the fact for problems with only linear constraints, the existence of Lagrange multipliers is guaranteed even when a local minimum is not regular.

Consider the problem $\min_{x \in \mathbb{R}^n} f(x)$ subject to $Ax = b$, where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

(a) Show that of $\text{rank}(A) = k < m$, then no local minimum can be regular.

(b) Show that *whether or not* x^* is a regular local minimum, there exists some $\lambda^* \in \mathbb{R}^m$ such that $\nabla f(x^*) + A^T \lambda^* = 0$.

(c) Show that λ^* need not be unique.

Problem 5.4
Given a vector $y \in \mathbb{R}^n$, consider the optimization problem $\max_{x \in \mathbb{R}^n} y^T x$ subject to $x^T Q x \leq 1$, where $Q \succ 0$ is a symmetric PD matrix. Show that the optimal value of this optimization problem is $\sqrt{y^T Q^{-1} y}$ and conclude that $(y^T x)^2 \leq (x^T Q x)\,(y^T Q^{-1} y)$. (*Note:* This is a generalization of the usual Cauchy-Schwarz inequality, which corresponds to the special case $Q = I$.)

Problem 5.5
B & V, Problem 5.6

Problem 5.6
B & V, Problem 5.11

Problem 5.7
B & V, Problem 5.14