UC Berkeley
Department of Electrical Engineering and Computer Science

EECS 227A
NONLINEAR AND CONVEX OPTIMIZATION

Problem Set 4

Fall 2009

Issued: Tuesday, October 6 **Due:** Thursday, October 15, 2009

Reading: Boyd and Vandenberghe: Chapter 2, §4.3, 4.4

Problem 4.1

(d) This set is convex because it can be expressed as

$$\bigcap_{y \in S} \{x | \|x - x_0\|_2 \leq \|x - y\|_2\},$$

i.e., an intersection of half-spaces.
(Note that

$$\|x-x_0\|_2 \leq \|x-y\|_2 \Leftrightarrow x^T x-2x^T x_0+x_0^T x_0 \leq x^T x-2x^T y+y^T y \Leftrightarrow 2(y-x_0)^T x+x_0^T x_0-y^T y \leq 0$$

so the set $y \in S\{x | \|x - x_0\|_2 \leq \|x - y\|_2\}$ is a half-space)

(e) In general this set is not convex, for example with $S = \{-1, 1\}$ and $T = \{0\}$, we have

$$\{x | dist(x, S) \leq dist(x, T)\} = \{x \in \mathbb{R} | x \leq -1/2 \text{ or } x \geq 1/2\}$$

which clearly is not convex.

(g) This set is convex, in fact a ball

$$\begin{aligned} \{x | \|x - a\|_2 \leq \theta \|x - b\|_2\} &= \{x | \|x - a\|_2^2 \leq \theta \|x - b\|_2^2\} \\ &= \{x | (1 - \theta^2)x^T x - 2(a - \theta^2 b)^T x + (a^T a - \theta^2 b^T b) \leq 0\} \end{aligned}$$

If $\theta = 1$, this is a half-space. If $\theta < 1$, it is a ball

$$\{x | (x - x_0)^T (x - x_0) \leq R^2\}$$

where the center x_0 and radius R given by

$$x_0 = \frac{a - \theta^2 b}{1 - \theta^2}, R = \left(\frac{\theta^2 \|b\|_2^2 - \|a\|_2^2}{1 - \theta^2} - \|x_0\|_2^2 \right)^{1/2}$$

Problem 4.2

We have $XX^T \succeq 0$ and $rank(XX^T) = k$ (using SVD decomposition of X). A positive linear combination of such matrices can have rank up to n, but never less than k. Indeed, let A and B be positive semidefinite matrices of rank k, with $rank(A+B) = r < k$. Ket $V \in \mathbb{R}^{n\times(n-r)}$ be a matrix such that $\mathcal{R}(V) = \mathcal{N}(A+B)$, i.e.

$$V^T(A+B)V = V^TAV + V^TBV = 0$$

Since $A, B \succeq 0$, this means $V^TAV = V^TBV = 0$, which implies that $rank(A) \leq r$ and $rank(B) \leq r$, which is contradiction. We conclude that $rank(A+B) \geq k$ for any matrix A, B such that $rank(A) = rank(B) = k$ and $A, B \succeq 0$.

Conversely, any non-zero matrix of rank at least k can be written as the sum of several matrices of rank k (using SVD decomposition for of $A = XX^T = U\Sigma V^T$ and note that the diagonal of Σ has at least k positive entries).

It follows that the conic hull of the set of rank-k outer products is the set of positive semidefinite matrices of rank greater than or equal to k, along with the zero matrix.

Problem 4.3

Obviously if $C = D$ the support functions are equal. We show that if the support functions are equal, then $C = D$, by showing that $D \subseteq C$ and $C \subseteq D$.

We first show that $D \subseteq C$. Suppose there exists a point $x_0 \in D, x_0 \notin C$. Since C is closed, x_0 can be strictly separated from C, i.e., there exists an $a \neq 0$ with $a^Tx_0 > b$ and $a^Tx < b, \forall x \in C$. This means that

$$\sup_{x\in C} a^Tx \leq b < a^Tx_0 \leq \sup_{x\in D} a^Tx,$$

which implies that $S_C(a) \neq S_D(a)$. By repeating the argument with the roles of C and D reversed, we can show that $C \subseteq D$.

Problem 4.4

(a) The set K_{m+} is defined by n homogenous linear inequalities, hence it is a closed (polyhedral) cone.

The interior of K_{m+} is non-empty, because there are points that satisfy the inequality with strict inequality, for example $x = (n, n-1, \ldots, 1)$.

To show that K_{m+} is pointed, we note that if $x \in K_{m+}$ then $-x \in K_{m+}$ only if $x = 0$. This implies that the cone does not contain an entire line.

(b) Using the hint, we see that $y^Tx \geq 0$ for all $x \in K_{m+}$ if and only if

$$y_1 \geq 0, y_1 + y_2 \geq 0, \ldots, y_1 + \ldots + y_n \geq 0$$

Therefore,

$$K^*_{m+} = \{y | \sum_{i=1}^{k} y_i \geq 0, k = 1, \ldots, n\}$$

Problem 4.5

The problem can be formulated as an LP

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & -S \preceq_K A_0 + x_1 A_1 + \ldots + x_k A_k \preceq_K S \\ & S\mathbf{1} \leq t\mathbf{1} \end{array}$$

with variables $S \in \mathbb{R}^{m \times n}, t \in \mathbb{R}$ and $x \in \mathbb{R}^k$. The inequality $\preceq_K$ denote component-wise inequality between matrices, i.e. with respect to the cone

$$K = \{X \in \mathbb{R}^{m \times n} | X_{ij} \geq 0, \forall i = 1, \ldots, m, j = 1, \ldots, n\}$$

To see the equivalence, suppose x and S are feasible in the LP. The last constraint means that

$$t \geq \sum_{j=1}^{n} s_{ij}, i = 1, \ldots, m$$

So the optimal choice of t is

$$t = \max_i \sum_{j=1}^{n} S_{ij}$$

This shows that the LP is equivalent to

$$\begin{array}{ll} \text{minimize} & \max_i \sum_{j=1}^{n} S_{ij} \\ \text{subject to} & -S \preceq_K A_0 + x_1 A_1 + \ldots + x_k A_k \preceq_K S \end{array}$$

Suppose x is given in the problem, and we optimize over S. The constraint in the LP state that

$$-S_{ij} \leq A(x)_{ij} \leq S_{ij}$$

(where $A(x) = A_0 + x_1 A_1 + \ldots + x_k A_k$), and since the objective is monotone increasing in S_{ij}, the optimal choice of S_{ij} is

$$S_{ij} = |A(x)_{ij}||$$

The problem is now reduced to the original problem

$$\text{minimize } \max_{1 \leq i \leq m} \sum_{j=1}^{n} |A(x)_{ij}|$$