UC Berkeley
Department of Electrical Engineering and Computer Science

EECS 227A
NONLINEAR AND CONVEX OPTIMIZATION

Solutions 3
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Solution 3.1

(a) We have $\nabla f(x) = \beta\|x\|^{\beta-2}x$ and $\nabla^2 f(x) = \beta(\beta-2)\|x\|^{\beta-4}xx^T + \beta\|x\|^{\beta-2}I$, so

$$\begin{aligned}\nabla^2 f(x)x &= \beta(\beta-2)\|x\|^{\beta-4}xx^Tx + \beta\|x\|^{\beta-2}x \\ &= \beta(\beta-2)\|x\|^{\beta-2}x + \beta\|x\|^{\beta-2}x = (\beta-1)\beta\|x\|^{\beta-2}x = (\beta-1)\nabla f(x)\end{aligned}$$

Thus $(\nabla^2 f(x))^{-1}\nabla f(x) = x/(\beta-1)$. Alternatively, we could use Sherman-Morrison formula to find the inverse of the Hessian

$$(A + CBC^T)^{-1} = A^{-1} - A^{-1}C(B^{-1} + C^TA^{-1}C)^{-1}C^TA^{-1}$$

The pure Newton's method has the form

$$x^{k+1} = x^k - \frac{1}{\beta-1}x^k = \frac{\beta-2}{\beta-1}x^k \Rightarrow \|x^{k+1}\| = \left|\frac{\beta-2}{\beta-1}\right| \|x^k\|$$

Hence

- If $\beta > \frac{3}{2}$ then $\left|\frac{\beta-2}{\beta-1}\right| < 1$ and the method converges to 0 for any initial point.
- If $\beta = \frac{3}{2}$, we have $x^{k+1} = -x^k, \forall k$ so it does not converges for any initial point $x^0 \neq 0$.
- For $1 < \beta < \frac{3}{2}$, we have $\left|\frac{\beta-2}{\beta-1}\right| > 1$ and the diverges for all $x^0 \neq 0$.
- When $\beta \leq 1$, we have

$$(\nabla^2 f(x))^{-1} = \frac{1}{\beta\|x\|^{\beta-2}}\left(I - \frac{\beta-2}{\beta-1}\frac{xx^T}{\|x\|^2}\right)$$

  so $(\nabla^2 f(x))^{-1}$ does not exist
- If $\beta < 1$ then $\left|\frac{\beta-2}{\beta-1}\right| > 1$ and the method diverges for any initial point $x^0 \neq 0$

(b) With the Armijo rule we have

$$\|x^{k+1}\| = \left|1 - \frac{\alpha^k}{\beta-1}\right| \|x^k\|$$

At each step, Armijo rule sets the stepsize $\alpha^k = a^{m_k}s$ where s is the initial stepsize, a is the reduction factor, and m_k is the smallest nonnegative integer such that

$$f(x^{k+1}) - f(x^k) \leq \sigma a^{m_k} s \nabla f(x^k)^T d^k$$

which is equivalent to

$$\|x^k\|\left(1-\left|1-\frac{\alpha^k}{\beta-1}\right|^\beta\right)\geq\sigma\alpha^k\frac{\beta\|x^k\|^\beta}{\beta-1}\Leftrightarrow 1-\left|1-\frac{\alpha^k}{\beta-1}\right|^\beta\geq\sigma\alpha^k\frac{\beta}{\beta-1}>0$$

Therefore, the stepsize is the same at each iteration $\alpha^k=\alpha,\forall k$, we then have

$$\|x^{k+1}\|=\left|1-\frac{\alpha}{\beta-1}\right|\|x^k\|$$

and also $\left|1-\frac{\alpha}{\beta-1}\right|<1$ (due to Armijo rule), so the method converges for any starting point when $\beta>1$

Solution 3.2

(a) We need to verify the properties of a norm (see Appendix A.1.2 in Boyd and Vandenberghe. Since $Q\succ 0$, we have $x^TQx\geq 0$, with equality if and only if $x=0$. For any $t\in\mathbb{R}$, we have

$$\|tx\|_Q=\sqrt{(tx)^TQ(tx)}=t\sqrt{x^TQx}=t\|x\|_Q,$$

which shows that $\|\cdot\|_Q$ is homogeneous. Finally, we need to verify the triangle inequality. Let $\sqrt{Q}$ denote the symmetric matrix square root of Q, which exists since Q is symmetric and positive definite. For any pair $x,y\in\mathbb{R}^n$, we have

$$\begin{aligned}\|(x+y)\|_Q&=\|\sqrt{Q}x+\sqrt{Q}y\|_2\\&\leq\|\sqrt{Q}x\|_2+\|\sqrt{Q}y\|_2\\&=\|x\|_Q+\|y\|_Q,\end{aligned}$$

where we have used triangle inequality for the Euclidean norm $\|\cdot\|_2$.

(b) A suitable generalization is the following: Let C be a non-empty closed and convex set in $\mathbb{R}^n$. Then for any point $z\in\mathbb{R}^n$:

(i) the optimization problem $\min_{x\in C}\frac{1}{2}\|z-x\|_Q^2$ has a unique solution $\mathcal{P}_C(z;Q)$, and

(ii) the projection $\mathcal{P}_C(z;Q)$ satisfies the condition

$$\langle z-\mathcal{P}_C(z;Q),\,x-\mathcal{P}_C(z;Q)\rangle_Q\leq 0\quad\text{for all } x\in C,\tag{1}$$

where $\langle a,\,b\rangle_Q=a^TQb$ is the inner product defined by Q.

Part (i) follows from the same argument that we used for the Euclidean norm. If C is bounded, then existence follows from the Weierstrass theorem. If C is unbounded, then we can always choose an arbitrary point $y\in C$, and add the extra constraint $\|x-z\|_Q\leq\|y-z\|_Q$, which converts the problem into one with a bounded and closed constraint set. Given existence, uniqueness follow because the objective function $\frac{1}{2}\|z-x\|_Q^2$ is strictly convex.

As for part (ii), the gradient of the objective function is $\nabla f(x)=Q(x-z)$. The optimum $z^*=\mathcal{P}_C(z;Q)$ must satisfy the condition $\nabla^Tf(z^*)(z-x)\geq 0$ for all $x\in C$. Some algebra yields the condition (1).

(c) We have with $z = x^k - s^k(H^k)^{-1}\nabla f(x^k)$,

$$\begin{aligned}\mathcal{P}_C(z;Q) &= \arg\min_{x\in C} \left(x^k - s^k(H^k)^{-1}\nabla f(x^k) - x\right)^T H^k \left(x^k - s^k(H^k)^{-1}\nabla f(x^k)\right) \\ &= \arg\min_{x\in C} \left\{(x^k - x)^T H^k (x^k - x) - 2s^k \nabla f(x^k)^T (H^k)^{-1} H^k (x^k - x)\right\} \\ &= \arg\min_{x\in C} \left\{\nabla f(x^k)^T (x - x^k) + \frac{1}{2s^k}(x - x^k)^T H^k (x - x^k)\right\} \\ &= \bar{x}^k\end{aligned}$$

Solution 3.3

Let C be a circle with radius r. Let d be the length of the rectangle's diagonals and θ be the angle between them. Clearly, if the rectangle is within C, then we must have $0 < d \le 2r$ and $0 \le \theta \le \pi$. On the other hand, if $0 < d \le 2r$ and $0 \le \theta \le \pi$, then we can always contruct a rectangle inside the circle C for with a diagonal length equal to d and the angle between them given by θ. (Note that the middle of such a diagonal is the center of the circle.) In terms of d and θ, the area of the rectangle is $f(d,\theta) = d^2 \sin\theta$, therefore we can find the rectangle of maximal area by solving the following optimization problem

$$\max_{d,\theta} f(d,\theta) = d^2 \sin\theta, \text{ subject to } 0 \le d \le 2r, 0 \le \theta \le \pi$$

Note that the set $[0, 2r] \times [0, \pi]$ is convex. Now,

$$f(d,\theta) = d^2 \sin\theta \le d^2 \le (2r)^2 = 4r^2.$$

Equality occurs when $\theta = \pi/2$ and $d = 2r$, thus the rectangle of maximal area is a square.

Solution 3.4

Let us re-write the objective. By expanding out the quadratic terms, we have

$$f(x) = m\{\|x\|_2^2 - 2x^T \frac{\sum_{j=1}^m b_j}{m} + \sum_{j=1}^m \|b\|_j^2\}.$$

. Minimizing this quantity is the same as minimizing the objective

$$\|x - \frac{\sum_{j=1}^m b_j}{m}\|_2^2,$$

for $x \in C$, and so the problem is equivalent to the stated projection.

Solution 3.5

(a) Suppose that there exists some $x \in C$ with $\nabla f(x^*)^T(x - x^*) = 0$ and

$$(x - x^*)^T \nabla^2 f(x^*)(x - x^*) < 0.$$

Then we also have $\nabla f(x^*)(x_\alpha - x^*) = 0$ and $(x_\alpha - x^*)^T \nabla^2 f(x^*)(x_\alpha - x^*) < 0$ for all $x_\alpha = (1-\alpha)x^* + \alpha x$ on the line joining x^* and x.

By a second order Taylor series expansion, we have

$$\begin{aligned} f(x_\alpha) - f(x^*) &= \nabla^T f(x^*)(x_\alpha - x^*) + \frac{1}{2}(x_\alpha - x^*)^T \nabla^2 f(\widetilde{x}_\alpha)(x_\alpha - x^*) \\ &= \frac{1}{2}(x_\alpha - x^*)^T \nabla^2 f(\widetilde{x}_\alpha)(x_\alpha - x^*), \end{aligned}$$

where $\widetilde{x}_\alpha$ lies on the line joining x_α and x^*.

Since $\nabla^2 f$ is continuous, by taking $\alpha > 0$ small enough, we are guaranteed that $\frac{1}{2}(x_\alpha - x^*)^T \nabla^2 f(\widetilde{x}_\alpha)(x_\alpha - x^*) < 0$, which shows that x^* is not a local minimum.

(b) Let $C = [-1, +1]$ and consider $f(x) = x^3$. At the point $x^* = 0$, we have $f'(0) = 0$ and $f''(0) \geq 0$, but $x^* = 0$ is not a local minimum.

(c) By the Taylor series expansion, for all $x \in C$ such that $\nabla^T f(x^*)(x - x^*) = 0$, we have

$$f(x^*) - f(x) = \frac{1}{2}(x - x^*)^T \nabla f(\widetilde{x})(x - x^*)$$

for some $\widetilde{x}$ on the line joining x and x^*. If $\nabla^2 f(x^*) \succ 0$, then by continuity, it remains strictly positive definite in a neighborhood of x^*, so that $(x - x^*)^T \nabla f^2(\widetilde{x})(x - x^*) > 0$ as long as x is close enough to x^*.

Solution 3.6

Let $a = (a_1, \ldots, a_n)$ where $a_i > 0$ and $Q = \operatorname{diag}(a)$, and let us find the projection of z onto the simplex $S = \{x \in \mathbb{R}^n | x \geq 0, a^T x \leq 1\}$ under the $\|\cdot\|_Q$ norm

$$x^* = \arg\min_{x \in S} \quad \|z - x\|_Q^2$$

We claim that the $x_i^* = \max(0, z_i - \mu^*)$ where μ^* is the solution of

$$\min \mu \quad \text{such that } \mu \geq 0 \text{ and } \sum_{i=1}^n a_i \max(0, z_i - \mu) \leq 1. \tag{2}$$

We do so using the "modified" projection theorem from the earlier problem 3.2(b). Indeed, for any $x \in S$, we have

$$\begin{aligned} (z - x^*)^T Q(x - x^*) &= \sum_{i=1}^n (z_i - x_i^*) a_i (x_i - x_i^*) = \sum_{i=1}^n (z_i - x_i^*) a_i x_i - \sum_{i=1}^n (z_i - x_i^*) a_i x_i^* \\ &\leq \max_{1 \leq i \leq n} \{z_i - x_i^*\} - \sum_{i=1}^n (z_i - x_i^*) a_i x_i^* \end{aligned}$$

where the last step follows since $0 < x_i$ and $\sum_{i=1}^n a_i x_i = 1$. We now observe that

$$z_i - x_i^* = -\max(-z_i, -\mu^*) = \min(z_i, \mu^*),$$

and hence

$$
\begin{aligned}
(z-x^*)^T Q(x-x^*) &\leq \max_{1\leq i\leq n}\{\min(z_i,\mu^*)\} - \sum_{i=1}^{n}\min(z_i,\mu^*)a_i x_i^* \leq \mu^* - \sum_{i=1}^{n}\min(z_i,\mu^*)a_i x_i^* \\
&= \mu^* - \sum_{i=1}^{n}\min(z_i,\mu^*)a_i\max(0,z_i-\mu^*) \\
&= \mu^* - \sum_{i|z_i>\mu^*}\mu^* a_i\max(0,z_i-\mu^*) = \mu^* - \mu^*\sum_{i=1}^{n}a_i\max(0,z_i-\mu^*)
\end{aligned}
$$

Since μ^* is the solution of (2), either $\mu^*=0$ or if $\mu^*>0$ then $\sum_{i=1}^{n}a_i\max(0,z_i-\mu^*)=1$. (Otherwise, by decreasing μ^* a bit, we could get better solution, which would be a contradiction). Therefore, we have shown that

$$(z-x^*)^T Q(x-x^*) \leq 0$$

so $x^*=\mathcal{P}_S(x;Q)$.

We design the following algorithm for finding $\mathcal{P}_S(x;Q)$, based on finding μ^*:

(1) If $\sum_{i=1}^{n}a_i\max(0,z_i)\leq 1$ then $\mu^*=0$

(2) Else *(we need to find* $\min\mu:\mu\geq 0$ *and* $\sum_{i=1}^{n}a_i\max(0,z_i-\mu)=1$*)*

  (2.1) Reorder $\{z_i\}_{i=1}^{n}$ in decreasing order, assuming $\{\tilde{a}_i\}$ is $\{a_i\}$ rearranged using this order, set $k=n$

  (2.2) If $z_k \geq \dfrac{\sum_{i=1}^{k}\tilde{a}_i z_i - 1}{\sum_{i=1}^{k}\tilde{a}_i}$ then go to (2.3). Otherwise set $k=k-1$, go to (2.2)

  (2.3) Set $\mu^* = \dfrac{\sum_{i=1}^{k}\tilde{a}_i z_i - 1}{\sum_{i=1}^{k}\tilde{a}_i}$ and obtain $x_i=\max(0,z_i-\mu^*)$, undo the order in (2.1) and finish.

(a) To solve this part, we apply the above algorithm for $a=\vec{1}$ and $Q=I$

(b) We have

$$
\begin{aligned}
f(x) &= \sum_{i=1}^{n}(\alpha_i x_i + \frac{1}{2}\beta_i x_i^2) \\
&= \frac{1}{2}\sum_{i=1}^{n}\frac{1}{\beta_i}(\beta_i x_i+\alpha_i)^2 - \frac{1}{2}\sum_{i=1}^{n}\frac{\alpha_i^2}{\beta_i}
\end{aligned}
$$

Let $a=\left(\dfrac{1}{\beta_i}\right)_{i=1}^{n}$, $y_i=\beta_i x_i$ and $z=-\alpha$, then minimizing $f(x)$ over the set

$$S=\{x\in\mathbb{R}^n | x\geq 0, \sum_{i=1}^{n}x_i\leq 1\}$$

is equivalent to minimizing

$$\sum_{i=1}^{n} a_i (y_i - z_i)^2 = \|z - y\|_Q^2$$

subject to $\sum_{i=1}^{n} y_i a_i \leq 1$ and $y \geq 0$. Consequently, the algorithm presented above can be used to solve this problem.

Solution 3.7

Let us consider the equivalent problem of minimizing

$$g(x) = -\sum_{i=1}^{n} \frac{\alpha_i}{\sum_{j=1}^{m} \alpha_j} \log(x_i) \;=\; -\sum_{j=1}^{n} \beta_i \log(x_i)$$

over the simplex. Here we have defined the vector $\beta \in S$ via $\beta_i = \frac{\alpha_i}{\sum_{j=1}^{n} \alpha_j}$.

The minimum exists since g is continuous and S is compact. We have

$$\nabla g(x) = (-\frac{\beta_1}{x_1}, \ldots, -\frac{\beta_n}{x_n}).$$

Noting that $\nabla^2 g(x)$ is strictly positive definite for all $x > 0$, we conclude that the optimum is unique.

We claim that $x^* = \beta$ is the optimum. Note that $x^* \in S$ since $\beta > 0$ and $\sum_{j=1}^{n} \beta_j = 1$ by construction. Since g is strictly convex and hence $\nabla^2 g(x^*) \succ 0$, we simply need to verify the first order condition $\nabla^T g(x^*)(x - \beta) \geq 0$ for all $x \in S$. We have

$$\nabla^T g(x^*)(x - \beta) = -\sum_{i=1}^{n} (x_i - \beta_i) \;= 0$$

for any $x \in S$.