UC Berkeley
Department of Electrical Engineering and Computer Science

EECS 227A
NONLINEAR AND CONVEX OPTIMIZATION

Problem Set 3
Fall 2009

Issued: Tuesday, September 22 **Due:** Tuesday, October 6, 2009

Problem 3.1
Consider applying Newton's method to the cost function $\|x\|^\beta$, where $\beta > 1$.

(a) Suppose that we use the pure form of Newton's method (i.e., stepsize $\alpha^k = 1$). For what starting points and values of β does the method converge to the optimal solution? What happens when $\beta \leq 1$?

(b) Repeat part (a) for Newton's method using Armijo rule to choose step sizes.

Problem 3.2
Let $Q \in \mathbb{R}^{n\times n}$ be a strictly positive definite symmetric matrix.

(a) Show that $\|x\|_Q = \sqrt{x^T Q x}$ defines a valid norm on $\mathbb{R}^n$.

(b) State and prove a generalization of the projection theorem from class that involves $\|z - x\|_Q$.

(c) Let $H^k \in \mathbb{R}^{n\times n}$ be a positive definite matrix, and let C be a convex set. For a current iterate x^k and parameter $s^k > 0$, define $\bar{x}^k$ via

$$\bar{x}^k = \arg\min_{x\in C} \left\{\nabla f(x^k)^T (x - x^k) + \frac{1}{2s^k}(x - x^k)^T H^k (x - x^k)\right\}.$$

Show that $\bar{x}^k$ is equal to the projection of the point $z = x^k - s^k (H^k)^{-1}\nabla f(x^k)$ onto the set C under the norm $\|\cdot\|_{H^k}$.

Problem 3.3
Given a circle, consider the set of all possible rectangles that are contained within in it. Formulating this problem as optimization over a convex set, show that the rectangle of maximal area is a square.

Problem 3.4
Let $b_1, \ldots, b_m$ be given vectors in $\mathbb{R}^m$, and consider the problem of minimizing $f(x) = \sum_{j=1}^m \|x - b_j\|^2$ subject to the constraint $x \in C$, where C is some convex set. Show that this problem is equivalent to projecting the center of gravity $\frac{1}{m}\sum_{j=1}^m b_j$ onto C.

Problem 3.5
Consider a twice differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ over a convex set C.

(a) If x^* is a constrained local minimum (subject to $x \in C$), show that

$$(x - x^*)^T \nabla^2 f(x^*)(x - x^*) \geq 0 \tag{1}$$

for all $x \in C$ such that $\nabla f(x^*)^T (x - x^*) = 0$.

(b) Is the converse statement true in general? If so, prove it; if not, give a counterexample with $C \neq \mathbb{R}^n$.

(c) Does your answer change if the inequality (1) holds strictly for all $x \neq x^*$?

Problem 3.6
Consider the simplex $S = \{x \in \mathbb{R}^n \mid x \geq 0, \quad \sum_{i=1}^n x_i \leq 1\}$.

(a) Develop an algorithm to find the projection of a point z onto S. (*Note:* Your algorithm should be almost as simple as a closed form solution.)

(b) Modify your algorithm from (a) so that it minimizes the function

$$f(x) = \sum_{i=1}^{n} (\alpha_i x_i + \frac{1}{2}\beta_i x_i^2)$$

subject to $x \in S$, where α_i, β_i are scalars with $\beta_i > 0$.

Problem 3.7
Let $\alpha_i > 0$ be fixed scalars, and consider the problem of maximizing $f(x) = \prod_{i=1}^n x_i^{\alpha_i}$ over the simplex S. Find a global maximum and show that it is unique.