UC Berkeley
Department of Electrical Engineering and Computer Science

EECS 227A
Nonlinear and Convex Optimization

# Problem Set 1
Fall 2009

**Issued:** Thursday, August 27 **Due:** Tuesday, September 8, 2009

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The first part of this problem set provides some practice on the mathematical pre-requisites for this course (vector calculus, elementary analysis, and linear algebra) and will calibrate your degree of preparation for the course.
**Reading:** Boyd and Vandenberghe, §3.1, 3.2 and Appendix A for background material.

### Problem 1.1
The symmetric matrix $Q \in \mathbb{R}^{n \times n}$ is *positive semidefinite* means that $x^T Q x \geq 0$ for all $x \in \mathbb{R}^n$. Prove that $Q$ is positive semidefinite if and only all of its eigenvalues are non-negative.

### Problem 1.2
Consider a symmetric matrix $M$ in block-partitioned form

$$M = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}.$$

Prove that the following two statements are equivalent:

(a) $M$ is strictly positive definite (i.e., $x^T M x > 0$ for all $x \neq 0$)

(b) $A$ is strictly positive definite, and $C - B^T A^{-1} B$ is strictly positive definite.

(*Hint:* The following fact may be useful: given a positive definite matrix $Q \in \mathbb{R}^p$ and vector $d \in \mathbb{R}^p$, the function $f(y) = \frac{1}{2} y^T Q y - d^T y$ achieves its unique minimum at $y^* = Q^{-1} d$.)

### Problem 1.3
True or false: either prove the statement or provide a counterexample.

(a) If a non-negative sequence $\{a_i\}$ satisfies $a_i \to 0$, then $s_n = \sum_{i=1}^n a_i$ converges.

(b) If $Q \in \mathbb{R}^{n \times n}$ is a symmetric matrix and $x$ and $y$ are two of its eigenvectors (with distinct eigenvalues $\lambda \neq \mu$), then $x^T Q y = 0$.

(c) For any matrix $M \in \mathbb{R}^{n \times n}$, there is a unitary matrix $U$ (i.e., $U \in \mathbb{R}^{n \times n}$ such that $U^T U = I$) such that $U^T M U$ is diagonal

(d) Consider a differentiable function $f : [0, 1] \to \mathbb{R}$. If $|f'(t)| \leq 0.1$ for all $t \in [0, 1]$, then $|f(t) - f(s)| \leq 0.1\,|t - s|$ for all $s, t \in [0, 1]$.

(e) Given a sequence $\{x_i\}_{i=1}^{\infty}$, suppose that for all $\epsilon > 0$, there exists some $N(\epsilon)$ such that for all $i \geq N(\epsilon)$, we have $\|x_i - x_{i+1}\|_2 < \epsilon$. Then there exists some $x^* \in \mathbb{R}^n$ such that $x_n \to x^*$.

**Problem 1.4**
Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable. Suppose that the point $x^* \in \mathbb{R}^n$ is a local minimum of $f$ along every line that passes through $x^*$. (I.e., the 1-D real-valued function $g_d(\alpha) := f(x^* + \alpha d)$ is minimized at $\alpha = 0$ for all $d \in \mathbb{R}^n$.) Show that $\nabla f(x^*) = 0$. In addition, show by example that $x^*$ need not be a local minimum of $f$. *Hint:* Consider the function $f(x_1, x_2) = (x_2 - px_1^2)(x_2 - qx_1^2)$ with $0 < p < q$, and the point $(0, 0)$.

**Problem 1.5**
Let $Q \in \mathbb{R}^{n \times n}$ be a symmetric matrix, and let $b \in \mathbb{R}^n$. Consider the problem of minimizing the quadratic function $f(x) := \frac{1}{2}x^T Qx - b^T x$.

(a) Show that the following conditions are equivalent:

  (ii) The problem $\inf_{x \in \mathbb{R}^n} f(x)$ has a solution. (I.e., $\exists \;\; x^* \in \mathbb{R}^n$ such that $f(x^*) \leq f(x) \; \forall x \in \mathbb{R}^n$.)
  (ii) The matrix $Q \succeq 0$ and the vector $b$ belongs to $\text{range}(Q) = \{y \mid y = Qu \text{ for some } u \in \mathbb{R}^n\}$.

(b) For each of the following conditions, give an example of a matrix $Q$ and a vector $b$ such that:

  (i) The global optimum is achieved and unique.
  (ii) The conditions in part (a) hold but the optimum is not unique.
  (iii) The global optimum is not achieved.

**Problem 1.6**
Many classical inequalities can be established via optimization theory.

(a) Use induction to prove the following extension of Jensen's inequality: if $f : \mathbb{R}^n \to \mathbb{R}$ is convex, then $f(\sum_{i=1}^n \lambda_i x_i) \leq \sum_{i=1}^n \lambda_i f(x_i)$ for all $x_i \in \mathbb{R}^n$, $\lambda_i \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$.

(b) *Arithmetic-geometric mean inequality:* or any set of positive numbers $x_1, \ldots, x_n$, there holds $(\prod_{i=1}^n x_i)^{\frac{1}{n}} \leq \frac{1}{n}\sum_{i=1}^n x_i$. Prove this inequality using convexity.

(c) *Maximum entropy:* For vectors $(x_1, \ldots, x_n) \geq 0$ such that $\sum_{i=1}^n x_i = 1$, define the entropy function $h(x) = -\sum_{i=1}^n x_i \log x_i$, where we define $0 \log 0 = 0$. Show that $h(x) \leq \log n$.

(d) For $p \in [1, \infty]$, define the $\ell_p$-norm $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$. Prove that $x^T y \leq \|x\|_p \|y\|_q$ for $q$ such that $1/q + 1/p = 1$.

**Problem 1.7**
*Monotone mappings and convex functions:* Let $C \subseteq \mathbb{R}^n$ be an open convex set. The mapping $G : C \to \mathbb{R}^n$ is said to be *monotone* on $C$ if for all $x, y \in C$, we have $[G(x) - G(y)]^T (x - y) \geq 0$.

(a) Let $f : C \to \mathbb{R}$ be differentiable. Show that if $f$ is convex on $C$, then the gradient mapping $\nabla f$ is monotone.

(b) Prove or provide a counterexample: any monotone mapping is the gradient of a convex function.