UC Berkeley
Department of Electrical Engineering and Computer Science

EECS 227A
Nonlinear and Convex Optimization

Review 1
Fall 2009

Problem 1.1
Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by:

$$f(x, y) = \left(x^2 - y\right)^2 - 10x^2.$$

(a) Is f convex or not?

(b) Show that f has exactly one stationary point over $\mathbb{R}^2$, and characterize whether it is a local minimum, a local maximum, or neither.

(c) Now consider the constrained optimization of f over the set

$$X = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1\}.$$

Show that a global minimum exists, and find all points at which it is attained.

Problem 1.2
Consider the inequality-constrained problem

$$\begin{aligned} \min_{x \in \mathbb{R}^n} f(x) & \qquad \text{such that } x \in C \\ C \;\; &= \;\; \{x \in \mathbb{R}^n \mid g_j(x) \leq 0, \; j = 1, \ldots, m\}, \end{aligned}$$

where f and $g_j, j = 1, \ldots m$ are convex and differentiable functions on $\mathbb{R}^n$. Suppose that x^* is feasible, and the pair (x^*, μ^*) satisfy the Karush-Kuhn-Tucker necessary conditions, including the complementary slackness condition.

Using the convexity and KKT conditions, show that $[\nabla f(x^*)]^T \, [x - x^*] \geq 0$ for all $x \in C$.

Problem 1.3
Let K be a non-empty, closed and convex cone, and let $y \in \mathbb{R}^n$ be a fixed vector. In this problem we consider the projection $\Pi_K(y)$ of y onto K, which is defined by

$$\Pi_K(y) \quad = \quad \arg \min_{x \in K} \|x - y\|^2,$$

where $\| \cdot \|$ denotes the ordinary Euclidean or ℓ_2 norm.

Show that the projection $\Pi_K(y) \in K$ satisfies the following properties:

(a) First show that $\left[y - \Pi_K(y)\right]^T \Pi_K(y) = 0$.

(b) Show that the difference $\Pi_K(y) - y$ belongs to the dual cone K^*.
(*Hint:* You can assume the result of part (a) in proving this.)

(c) Use part (b) to characterize the projections onto the following cones:

   (i) The orthant cone $K = \{x \in \mathbb{R}^n \mid x_i \geq 0\}$.

   (ii) The semidefinite cone $K = \{X \in S^n \mid X \succeq 0\}$.

Problem 1.4
In many applications, it is of interest to find a sparse "eigenvector" of a matrix $\Gamma \succeq 0$. That is, we would like to solve the optimization problem, with vector $x \in \mathbb{R}^n$:

$$a^* = \max x^T \Gamma x \quad \text{such that } \|x\|_2 = 1, \text{ and } \|x\|_1 \leq C. \tag{1}$$

(a) Is optimization problem (1) an instance of a conic program? Why or why not?

(b) Consider the semidefinite program with matrix variable $X \in S^n_+$:

$$b^* = \max_{X \in S^n} \text{trace}(\Gamma X) \quad \text{such that } X \succeq 0, \quad \text{trace}(X) = 1, \text{ and } \textstyle\sum_{i,j} |X_{ij}| \leq C^2. \tag{2}$$

What is the relation between the two optimal values a^* and b^*?

(c) Compute the Lagrangian dual of the SDP (2). (In doing so, define the cost function $f(X) = \text{trace}(\Gamma X)$ with $\text{dom}(f) = \{X \succeq 0, \text{trace}(X) = 1\}$, so that these constraints are handled directly, and then add a Lagrange multiplier μ only for the inequality constraint $\sum_{i,j} |X_{ij}| \leq 1$.)

(d) Set-up and describe the steps involved in implementing a barrier method to solve the SDP (2).

Problem 1.5
Consider the following questions:

(i) Show that f is convex. Calculate its domain $\text{dom}(f)$, and the set $\partial f(x)$ for each $x \in \text{dom}(f)$.

(ii) Calculate the conjugate dual f^*. Specify its domain $\text{dom}(f^*)$ explicitly.

(iii) Calculate the bi-conjugate f^{**}. Explain why it is (or is not) equal to f.

Answer these questions for each of the following functions:

(a) $f(x) = \exp(x)$ for $x \in \mathbb{R}$.

(b) $f(x) = x \log(x) - x$ for $x > 0$ and ∞ otherwise.

(c) $f(x) = \frac{1}{2} x^T \Gamma x$ where $\Gamma \succeq 0$.

(d) $f(x) = \sup_{y \in C} y^T x$, where C is a non-empty, closed and convex set.

Problem 1.6
True or false: justify your answer with an argument, or an explicit counterexample if false. You may use results stated in class in order to justify your answer.

(a) The hyperbolic set $C = \{x \in \mathbb{R}^n \mid \prod_{i=1}^n x_i \geq 1\}$ is convex.

(b) Consider the optimization problem $\min_{x \in C} f(x)$, where f is continuous and convex, and C is a non-empty, closed and bounded polyhedron. Strong duality always holds in this case.

(c) Consider the optimization problem $\min f(x) : x \in \mathbb{R}^n, h(x) = 0$, where f, h are differentiable functions on $\mathbb{R}^n$, then an optimal point must satisfy the Lagrange multiplier rule, i.e. x^* is optimal only if there exists λ such that $\nabla f(x^*) - \lambda \nabla h(x^*) = 0$

(d) Newton's method (simplified to the problem of finding root of a differentiable function) converges to a stationary point if the starting point is sufficiently close.