TILING AND TRANSLATION INVARIANT HAMILTONIANS

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Abstract. A summary for the paper 'The Quantum and Classical Complexity of Translationally Invariant Tiling and Hamiltonian Problems [1]' by D.Gottesman and S.Irani.

The central idea of the paper is the analogy between the classical tiling problem and the problem of approximating the ground state energy of a system when the Hamiltonian is translationally invariant. We first define the tiling problem and discuss the analogy with translationally invariant Hamiltonians.

Definition 0.1. *The tile set $T = \{t_1, t_2, \cdots, t_m\}$ and rules $H, V \subset T \times T$ specifying tile pairs that can be adjacent horizontally and vertically are fixed. The input is an integer N and the problem is to decide if a $N \times N$ grid can be tiled consistently.*

Tiling rules impose constraints on states of adjacent grid points while 2-local Hamiltonians impose constraints on the states of adjacent particles in a quantum system. Tiling rules are translation invariant so it is natural to consider the r-dimensional translationally invariant Hamiltonian ($rD-TIH$) problem.

Definition 0.2. *2-local Hamiltonians $H_1, H_2, \cdots, H_r$ corresponding to the r dimensions are fixed. The predicate $Adj_i(x, y)$ is true iff. $y = x + e_i$ i.e.(x, y) are adjacent in the ith coordinate for the grid $[N]^r$.*

$$H = \sum_{x,y \in [N]^r} Adj_i(x, y) H_i(x, y)$$

The output for $rD - TIH(N)$ is Yes if $\lambda_0(H) \leq p(N)$ and No if $\lambda_0(H) \geq p(N) + 1/q(N)$ for polynomials $p(N)$ and $q(N)$.

The inputs for both the problems is the integer N, so verifying that a tiling is valid and that a ground state has low energy are problems in EXP and BQP_{EXP} respectively. The paper shows that these problems are complete for the complexity classes $NEXP$ and QMA_{EXP} respectively. The classical and quantum reductions use similar ideas so we sketch them out indicating the similarities.

0.1. **Tiling.** Given a non deterministic turing machine M and input x, we first construct a set of tiling rules and an instance N such that valid tilings encode computations $M(x, \phi)$ for a witness ϕ. An additional rule is added to ensure that only accepting computations yield valid tilings.

The counter Turing machine M_{BC} writes strings in $\{0, 1\}^*$ sequentially such that $f(x)$, the number of steps after which x is written, is computable in time $poly(|x|)$. The input x is encoded in the instance size N by choosing N such that $f(x) = N - 3$.

It is easy to make rules so that valid tilings have special tiles for every boundary direction. Interior tiles have layer 1 and layer 2 types, with layer 1 encoding the computation of M_{BC} and layer 2 encoding the computation of M. The computation proceeds through the following phases:

1. Initialize: Layer 1 tiles below the North boundary are constrained to encode the starting state of M_{BC} and the blank tape.
2. Count: Layer 1 tiling rules encode the transitions of M_{BC} so that layer 1 of the i-th row from the North boundary encodes the configuration of M_{BC} after $i - 1$ steps. Layer 1 tiles above the South boundary encode x, the tape contents after $N - 3$ steps of executing M_{BC}.

(3) Compute: The alphabets on layer 1 and layer 2 tiles above the South boundary must be equal, and layer 2 tiling rules encode the transitions of M. Layer 2 tiles on the i-th row from the bottom encode the possible configurations of $M(x)$ after $i-1$ non deterministic steps.
(4) Accept: Layer 2 tiles below the North boundary are constrained to encode the accept state of M with a blank tape. There is a valid tiling iff. there are non deterministic transitions that make $M(x)$ accept in $N-3$ steps.

We finally note that M can be modified to accept in $2^{c|x|}$ steps via a standard padding argument and N is $2^{c|x|}$ for the standard counter machine.

0.2. **Hamiltonians.** Given a non deterministic quantum Turing machine M and input x, we construct an instance (H_1, N) of $1D{-}TIH$ such that the possible ground states of H are superpositions over computation histories $\sum_t |t\rangle |M_{I,\phi,t}\rangle$ where $M_{I,\phi,t}$ is the state of the computation described in the tiling section with initial state I and witness ϕ at time step t. An additional energy penalty for non accepting computations is added to complete the reduction.

The quantum system consists of N particles on a line with N chosen so that $f(x) = N-3$. The particle dimensions are large enough to support six tracks corresponding to the clocks, work and state tapes for M_{BC} and M and the witness tape. The 2-local hamiltonian H_1 is a sum of energy penalty and propagation constraints on the six tracks.

- Energy Penalty: A term $I \otimes \cdots \otimes I \otimes |ab\rangle\langle ab| \otimes \cdots I$ ensures that states containing the pair ab have high energy. Energy penalty terms are used to constrain the occurrence of adjacent symbols on a track and the symbols at the same location on parallel tracks.
- Propagation: A term $\frac{1}{2}(|ab\rangle\langle ab| + |cd\rangle\langle cd| - |ab\rangle\langle cd| - |cd\rangle\langle ab|)$ ensures that a zero energy ground state has equal amplitude for terms where ab is replaced by cd with all other symbols being invariant.

The object of the construction is to choose suitable energy penalty and propagation terms to constrain the zero energy ground states to be the computation history states for some witness. The first clock track is used to create transitions for the Turing machines while the second clock track keeps track of the number of computation steps. The computation proceeds through the following phases:

(1) Initialize: Propagation of a special symbol $\vec{0}$ on track 1 ensures that all tracks are initialized correctly.
(2) Count/Compute: The track 1 special symbol changes to $\vec{1}$ ($\vec{1}$) and brings about transitions of M_{BC} (M) through its interaction with tracks $3-6$. Track 2 records the number of steps in the computation.
(3) Accept: The final state has special symbols on track 1 and 2, so we add an additional energy penalty term if the final state of M is non accepting.

Computation histories for all witnesses have zero energy wrt constraints 1-3 while only valid witnesses have energy close to 0 for constraint 4. The proof of the spectral gap for the No case follows from results in $KSV02$ [2] presented earlier at the group.

0.3. **Symmetry.** Additional symmetry rules can be imposed on tilings a) Reflection symmetry: $(t_i, t_j) \in H \Leftrightarrow (t_j, t_i) \in H$ (also V) and b) Rotation symmetry: Reflection symmetry with $H = V$. There are constants $N_e(N_o)$ for symmetric tiling rules such that the answer is yes for any even (odd) number more than $N_e(N_o)$ and No otherwise. Partial results about the computability of N_e and N_o for various cases are discussed in the paper.

It is open to determine the complexity of $rD - TIH$ where the Hamiltonians exhibit additional symmetry.

References

[1] D. Gottesman and S.Irani. *The Quantum and Classical Complexity of Translationally Invariant Tiling and Hamiltonian Problems.* FOCS, 2009.

[2] A.Y. Kitaev, A.H. Shen and M.N. Vyalvi. *Classical and Quantum Computation.* AMS, Providence, RI, 2002.

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