CS 294-2 Hilbert Spaces, Tensor Products, Teleportation 9/7/04
Fall 2004 Lecture 3

0.1 Tensor Products

Consider two quantum systems - the first with k distinguishable (classical) states (associated Hilbert space $\mathscr{C}^k$), and the second with l distinguishable states (associated Hilbert space $\mathscr{C}^l$). What is the Hilbert space associated with the composite system? We can answer this question as follows: the number of distinguishable states of the composite system is kl — since for each distinct choice of basis (classical) state $|i\rangle$ of the first system and basis state $|j\rangle$ of the second system, we have a distinguishable state of the composite system. Thus the Hilbert space associated with the composite system is $\mathscr{C}^{kl}$.

The tensor product is a general construction that shows how to go from two vector spaces V and W of dimension k and l to a vector space $V \otimes W$ (pronounced "V tensor W") of dimension kl. Fix bases $|v_1\rangle, \ldots, |v_k\rangle$ and $|w_1\rangle, \ldots, |w_l\rangle$ for V, W respectively. Then a basis for $V \otimes W$ is given by

$$\{|v_i\rangle \otimes |w_j\rangle : 1 \le i \le k, 1 \le j \le l\},$$

so that $\dim(V \otimes W) = kl$. So a typical element of $V \otimes W$ will be of the form $\sum_{ij} \alpha_{ij}(|v_i\rangle \otimes |w_j\rangle)$. We can define an inner product on $V \otimes W$ by

$$(|v_1\rangle \otimes |w_1\rangle, |v_2\rangle \otimes |w_2\rangle) = (|v_1\rangle, |v_2\rangle) \cdot (|w_1\rangle, |w_2\rangle),$$

which extends uniquely to the whole space $V \otimes W$.

For example, consider $V = \mathscr{C}^2 \otimes \mathscr{C}^2$. V is a Hilbert space of dimension 4, so $V \cong \mathscr{C}^4$. So we can write $|00\rangle$ alternatively as $|0\rangle \otimes |0\rangle$. More generally, for n qubits we have $\mathscr{C}^2 \otimes \cdots$ (n times) $\otimes \cdots \mathscr{C}^2 \cong \mathscr{C}^{2^n}$. A typical element of this space is of the form

$$\sum_{x \in \{0,1\}^n} \alpha_x |x\rangle.$$

A word of caution: Not all elements of $V \otimes W$ can be written as $|v\rangle \otimes |w\rangle$ for $|v\rangle \in V$, $|w\rangle \in W$. As an example, consider the Bell state $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

0.2 The Significance of Tensor Products

Classically, if we put together a subsystem that stores k bits of information with one that stores l bits of information, the total capacity of the composite system is $k + l$ bits.

From this viewpoint, the situation with quantum systems is extremely paradoxical. We need k complex numbers to describe the state of a k-level quantum system. Now consider a system that consists of a k-level subsystem and an l-level subsystem. To describe the composite system we need kl complex numbers. One might wonder where nature finds the extra storage space when we put these two subsystems together.

An extreme case of this phenomenon occurs when we consider an n qubit quantum system. The Hilbert space associated with this system is the n-fold tensor product of $\mathscr{C}^2 \equiv \mathscr{C}^{2^n}$. Thus nature must "remember" of 2^n complex numbers to keep track of the state of an n qubit system. For modest values of n of a few hundred, 2^n is larger than estimates on the number of elementary particles in the Universe.

This is the fundamental property of quantum systems that is used in quantum information processing.

Finally, note that when we actually a measure an n-qubit quantum state, we see only an n-bit string - so we can recover from the system only n, rather than 2^n, bits of information.

0.3 Tensor product of operators

Suppose $|v\rangle$ and $|w\rangle$ are unentangled states on $\mathscr{C}^m$ and $\mathscr{C}^n$, respectively. The state of the combined system is $|v\rangle \otimes |w\rangle$ on $\mathscr{C}^{mn}$. If the unitary operator A is applied to the first subsystem, and B to the second subsystem, the combined state becomes $A|v\rangle \otimes B|w\rangle$.

In general, the two subsystems will be entangled with each other, so the combined state is not a tensor-product state. We can still apply A to the first subsystem and B to the second subsystem. This gives the operator $A \otimes B$ on the combined system, defined on entangled states by linearly extending its action on unentangled states.

(For example, $(A\otimes B)(|0\rangle \otimes |0\rangle) = A|0\rangle \otimes B|0\rangle$. $(A\otimes B)(|1\rangle \otimes |1\rangle) = A|1\rangle \otimes B|1\rangle$. Therefore, we define $(A\otimes B)(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle)$ to be $\frac{1}{\sqrt{2}}(A\otimes B)|00\rangle + \frac{1}{\sqrt{2}}(A\otimes B)|11\rangle = \frac{1}{\sqrt{2}}\left(A|0\rangle \otimes B|0\rangle + A|1\rangle \otimes B|1\rangle\right)$.)

Let $|e_1\rangle, \ldots, |e_m\rangle$ be a basis for the first subsystem, and write $A = \sum_{i,j=1}^m a_{ij}|e_i\rangle\langle e_j|$ (the i,jth element of A is a_{ij}). Let $|f_1\rangle, \ldots, |f_n\rangle$ be a basis for the second subsystem, and write $B = \sum_{k,l=1}^n b_{kl}|f_k\rangle\langle f_l|$. Then a basis for the combined system is $|e_i\rangle \otimes |f_j\rangle$, for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. The operator $A\otimes B$ is

$$\begin{aligned}
A\otimes B &= \left(\sum_{ij} a_{ij}|e_i\rangle\langle e_j|\right) \otimes \left(\sum_{kl} b_{kl}|f_k\rangle\langle f_l|\right) \\
&= \sum_{ijkl} a_{ij}b_{kl}|e_i\rangle\langle e_j| \otimes |f_k\rangle\langle f_l| \\
&= \sum_{ijkl} a_{ij}b_{kl}(|e_i\rangle \otimes |f_k\rangle)(\langle e_j| \otimes \langle f_l|) \ .
\end{aligned}$$

Therefore the $(i,k),(j,l)$th element of $A\otimes B$ is $a_{ij}b_{kl}$. If we order the basis $|e_i\rangle \otimes |f_j\rangle$ lexicographically, then the matrix for $A\otimes B$ is

$$\begin{pmatrix} a_{11}B & a_{12}B & \cdots \\ a_{21}B & a_{22}B & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} ;$$

in the i,jth subblock, we multiply a_{ij} by the matrix for B.

1 Quantum Teleportation

The *No Cloning Theorem* states that no quantum system can copy a qubit; that is, there is no transform sending $|\psi\rangle \otimes |0\rangle \mapsto |\psi\rangle \otimes |\psi\rangle$. However, if we are willing to destroy the original, we can transmit a qubit, even to a remote location.

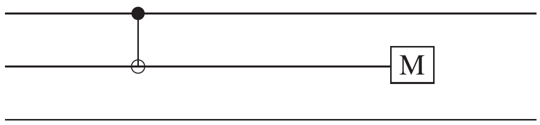
Suppose Alice (A) has access to a quantum state $|\psi\rangle = a_0|0\rangle + a_1|1\rangle$, which she wants to transmit to a remote party Bob (B). She can accomplish this by transmitting only classical bits of information, provided A and B share the entangled two-qubit state

$$|\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

The technique is known as *quantum teleportation*.

The basic idea is this. A controls $|\psi\rangle$ and the first qubit of $|\phi\rangle$. A's strategy, roughly speaking, is to forcibly entangle $|\psi\rangle$ with the first qubit $|\phi\rangle$. A then measures the first qubit of $|\phi\rangle$, resolving it completely, and hopes this will cause $|\psi\rangle$ to become entangled with the *second* qubit of $|\phi\rangle$. Presumably, B could then transfer $|\psi\rangle$ to the second qubit of $|\phi\rangle$.

As a first try, consider the following diagram. The top line represents $|\psi\rangle$; the bottom two represent the two qubits of $|\phi\rangle$.



That is, A passes $|\psi\rangle$ and the first qubit of $|\phi\rangle$ through a CNOT gate, and then measures the first qubit of $|\phi\rangle$. Now the input into the system as a whole is

$$|\phi\rangle \otimes |\psi\rangle = \sum_{i=0,1} a_i |i\rangle \otimes \sum_{j=0,1} \frac{1}{\sqrt{2}} |j,j\rangle.$$

After passing through the CNOT gate this becomes

$$\sum_{i,j} a_i \big| i, i\oplus j, j\big\rangle.$$

Now A measures the middle qubit. Suppose it is measured as l; then $l = i \oplus j$. The state is now

$$\sum_{j} a_{j\oplus l} \big| j\oplus l, j\big\rangle.$$

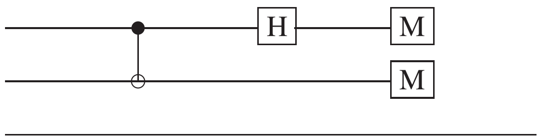
Next, A transmits l to B. If $l = 0$, B takes no action, while if $l = 1$, then B performs a bit flip on his qubit (the bottom qubit in the diagram.) A bit flip is just the transformation $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Thus we have

$$\sum_{j} a_{j\oplus l} \big| j\oplus l, j\oplus l\big\rangle = \sum_j a_j \big| j, j\big\rangle.$$

This is almost exactly what we want. The only problem is that now, the qubit corresponding to $|\psi\rangle$ is entangled with B's qubit. The entanglement that was necessary to get the whole process started is now a liability. One way to disentangle them would be for A to measure her remaining qubit. But this would destroy B's qubit as well.

The ideal solution would be to send the entangle qubits through a CNOT gate—but A controls the first qubit and B controls the second. This would require quantum communication between A and B, which is prohibited.

The correct solution is to go back and modify the original diagram, inserting a Hadamard gate and an additional measurement:



Now the algorithm proceeds exactly as before. However A's application of the Hadamard gate now induces the transformation

$$\sum_j a_j \big| j, j \big\rangle \longrightarrow \tfrac{1}{\sqrt{2}} \sum_{ij} a_j (-1)^{ij} \big| i, j \big\rangle.$$

Finally A measures i and sends the measurement to B. The state is now:

$$\sum_j a_j (-1)^{ij} | j \rangle.$$

If $i = 0$ then we are done; if $i = 1$ then B applies a phase flip. In either case the state is now $a_0|0\rangle + a_1|1\rangle$. So A has transported the quantum state to B simply by sending two classical bits.
