6.8 CAVITY PERTURBATIONS

In practical applications cavity resonators are often modified by making small changes in their shape, or by the introduction of small pieces of dielectric or metallic materials. For example, the resonant frequency of a cavity can be easily tuned with a small screw (dielectric or metallic) that enters the cavity volume, or by changing the size of the cavity with a movable wall. Another application involves the determination of dielectric constant by measuring the shift in resonant frequency when a small dielectric sample is introduced into the cavity.

In some cases, the effect of such perturbations on the cavity performance can be calculated exactly, but often approximations must be made. One useful technique for doing this is the perturbational method, which assumes that the actual fields of a cavity with a small shape or material perturbation are not greatly different from those of the unperturbed cavity. Thus, this technique is similar in concept to the perturbational method introduced in Section 2.7 for treating loss in good conductors, where it was assumed that there was not a significant difference between the fields of a component with good conductors and one with perfect conductors.

In this section we will derive expressions for the approximate change in resonant frequency when a cavity is perturbed by small changes in the material filling the cavity, or by small changes in its shape.

Material Perturbations

Figure 6.25 shows a cavity perturbed by a change in the permittivity ($\Delta\epsilon$), or permeability ($\Delta\mu$), of all or part of the material filling the cavity. If $\bar{E}_0, \bar{H}_0$ are the fields of the original cavity, and $\bar{E}, \bar{H}$ are the fields of the perturbed cavity, then Maxwell's curl equations can be written for the two cases as

$$\nabla \times \bar{E}_0 = -j\omega_0 \mu \bar{H}_0, \qquad 6.97a$$

$$\nabla \times \bar{H}_0 = j\omega_0 \epsilon \bar{E}_0, \qquad 6.97b$$

$$\nabla \times \bar{E} = -j\omega(\mu + \Delta\mu)\bar{H}, \qquad 6.98a$$

$$\nabla \times \bar{H} = j\omega(\epsilon + \Delta\epsilon)\bar{E}, \qquad 6.98b$$

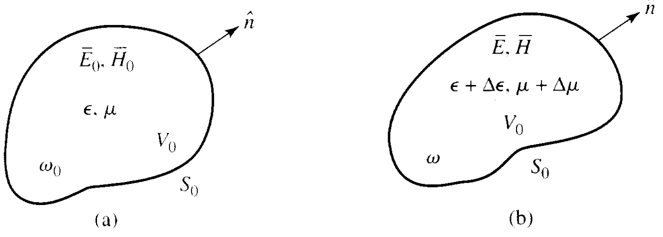


FIGURE 6.25 A resonant cavity perturbed by a change in the permittivity or permeability of the material in the cavity. (a) Original cavity. (b) Perturbed cavity.

vhere ω_0 is the resonant frequency of the original cavity and ω is the resonant frequency f the perturbed cavity.

Now multiply the conjugate of (6.97a) by $\bar{H}$ and multiply (6.98b) by $\bar{E}_0^*$ to get

$$\bar{H} \cdot \nabla \times \bar{E}_0^* = j\omega_0 \mu \bar{H} \cdot \bar{H}_0^*,$$

$$\bar{E}_0^* \cdot \nabla \times \bar{H} = j\omega(\epsilon + \Delta\epsilon)\bar{E}_0^* \cdot \bar{E}.$$

ubtracting these two equations and using the vector identity (B.8) that $\nabla \cdot (\bar{A} \times \bar{B}) = \bar{B} \cdot \nabla \times \bar{A} - \bar{A} \cdot \nabla \times \bar{B}$ gives

$$\nabla \cdot (\bar{E}_0^* \times \bar{H}) = j\omega_0 \mu \bar{H} \cdot \bar{H}_0^* - j\omega(\epsilon + \Delta\epsilon)\bar{E}_0^* \cdot \bar{E}. \tag{6.99a}$$

imilarly, we multiply the conjugate of (6.97b) by $\bar{E}$ and multiply (6.98a) by $\bar{H}_0^*$ to get

$$\bar{E} \cdot \nabla \times \bar{H}_0^* = -j\omega_0 \epsilon \bar{E}_0^* \cdot \bar{E},$$

$$\bar{H}_0^* \cdot \nabla \times \bar{E} = -j\omega(\mu + \Delta\mu)\bar{H}_0^* \cdot \bar{H}.$$

ubtracting these two equations and using vector identity (B.8) gives

$$\nabla \cdot (\bar{E} \times \bar{H}_0^*) = -j\omega(\mu + \Delta\mu)\bar{H}_0^* \cdot \bar{H} + j\omega_0 \epsilon \bar{E}_0^* \cdot \bar{E}. \tag{6.99b}$$

ow add (6.99a) and (6.99b), integrate over the volume V_0, and use the divergence eorem to obtain

$$\int_{V_0} \nabla \cdot (\bar{E}_0^* \times \bar{H} + \bar{E} \times \bar{H}_0^*) dv = \oint_{S_0} (\bar{E}_0^* \times \bar{H} + \bar{E} \times \bar{H}_0^*) \cdot d\bar{s} = 0$$

$$= j \int_{V_0} \{[\omega_0 \epsilon - \omega(\epsilon + \Delta\epsilon)]\bar{E}_0^* \cdot \bar{E} + [\omega_0 \mu - \omega(\mu + \Delta\mu)]\bar{H}_0^* \cdot \bar{H}\} dv, \tag{6.100}$$

here the surface integral is zero because $\hat{n} \times \bar{E} = 0$ on S_0. Rewriting gives

$$\frac{\omega - \omega_0}{\omega} = \frac{-\int_{V_0} \left(\Delta\epsilon \bar{E} \cdot \bar{E}_0^* + \Delta\mu \bar{H} \cdot \bar{H}_0^*\right) dv}{\int_{V_0} \left(\epsilon \bar{E} \cdot \bar{E}_0^* + \mu \bar{H} \cdot \bar{H}_0^*\right) dv}. \tag{6.101}$$

This is an exact equation for the change in resonant frequency due to material rturbations, but is not in a very usable form since we generally do not know $\bar{E}$ and , the exact fields in the perturbed cavity. But, if we assume that $\Delta\epsilon$ and $\Delta\mu$ are small, en we can approximate the perturbed fields $\bar{E}, \bar{H}$ by the original fields $\bar{E}_0, \bar{H}_0$, and ω in denominator of (6.101) by ω_0, to give the fractional change in resonant frequency as

$$\frac{\omega - \omega_0}{\omega_0} \simeq \frac{-\int_{V_0} \left(\Delta\epsilon |\bar{E}_0|^2 + \Delta\mu |\bar{H}_0|^2\right) dv}{\int_{V_0} \left(\epsilon |\bar{E}_0|^2 + \mu |\bar{H}_0|^2\right) dv}. \tag{6.102}$$

This result shows that any increase in ϵ or μ at any point in the cavity will decrease resonant frequency. The reader may also observe that the terms in (6.102) can be ated to the stored electric and magnetic energies in the original and perturbed cavities, that the decrease in resonant frequency can be related to the increase in stored energy the perturbed cavity.

EXAMPLE 6.7 Material Perturbation of a Rectangular Cavity

A rectangular cavity operating in the TE_{101} mode is perturbed by the insertion of a thin dielectric slab into the bottom of the cavity, as shown in Figure 6.26. Use the perturbational result of (6.102) to derive an expression for the change in resonant frequency.

Solution

From (6.42a–c), the fields for the unperturbed TE_{101} cavity mode can be written as

$$E_y = A \sin \frac{\pi x}{a} \sin \frac{\pi z}{d},$$

$$H_x = \frac{-jA}{Z_{TE}} \sin \frac{\pi x}{a} \cos \frac{\pi z}{d},$$

$$H_z = \frac{j\pi A}{k\eta a} \cos \frac{\pi x}{a} \sin \frac{\pi z}{d}.$$

In the numerator of (6.102), $\Delta\epsilon = (\epsilon_r - 1)\epsilon_0$ for $0 \leq y \leq t$, and zero elsewhere. The integral can then be evaluated as

$$\int_V (\Delta\epsilon |\bar{E}_0|^2 + \Delta\mu |\bar{H}_0|^2) dv = (\epsilon_r - 1)\epsilon_0 \int_{x=0}^{a} \int_{y=0}^{t} \int_{z=0}^{d} |E_y|^2 dz\, dy\, dx$$

$$= \frac{(\epsilon_r - 1)\epsilon_0 A^2 atd}{4}.$$

The denominator of (6.102) is proportional to the total energy in the unperturbed cavity, which was evaluated in (6.43), thus,

$$\int_V (\epsilon |\bar{E}_0|^2 + \mu |\bar{H}_0|^2) dv = \frac{abd\epsilon_0}{2} A^2.$$

Then (6.102) gives the fractional change (decrease) in resonant frequency as

$$\frac{\omega - \omega_0}{\omega_0} = \frac{-(\epsilon_r - 1)t}{2b}.$$

○

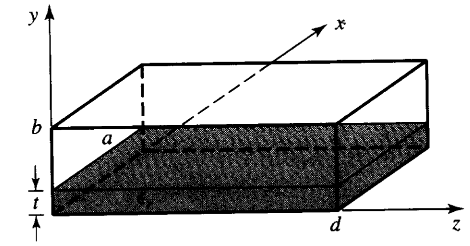


FIGURE 6.26 A rectangular cavity perturbed by a thin dielectric slab.

Shape Perturb

Changing the si
change in the shape
perturbation techniqu
shape; we will deriv

As in the case c
frequency of the orig
the perturbed cavity.

Now multiply the co

Subtracting these tw

Similarly, we multip

Subtracting and appl

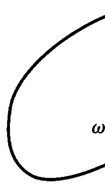


FIGURE 6.27 A res
turbe

Now add (6.105a) and (6.105b), integrate over the volume V, and use the divergen
theorem to obtain

$$\int_V \nabla\cdot(\bar{E}\times\bar{H}_0^* + \bar{E}_0^*\times\bar{H})dv = \oint_S (\bar{E}\times\bar{H}_0^* + \bar{E}_0^*\times\bar{H})\cdot d\bar{s}$$

$$= \oint_S \bar{E}_0^*\times\bar{H}\cdot d\bar{s} = -j(\omega-\omega_0)\int_V(\epsilon\bar{E}\cdot\bar{E}_0^* + \mu\bar{H}\cdot\bar{H}_0^*)dv, \qquad 6.10$$

since $\hat{n}\times\bar{E}=0$ on S.

Since the perturbed surface $S = S_0 - \Delta S$, we can write

$$\oint_S \bar{E}_0^*\times\bar{H}\cdot d\bar{s} = \oint_{S_0}\bar{E}_0^*\times\bar{H}\cdot d\bar{s} - \oint_{\Delta S}\bar{E}_0^*\times\bar{H}\cdot d\bar{s} = -\oint_{\Delta S}\bar{E}_0^*\times\bar{H}\cdot ds,$$

because $\hat{n}\times\bar{E}_0 = 0$ on S_0. Using this result in (6.106) gives

$$\omega-\omega_0 = \frac{-j\oint_{\Delta S}\bar{E}_0^*\times\bar{H}\cdot d\bar{s}}{\int_V(\epsilon\bar{E}\cdot\bar{E}_0^* + \mu\bar{H}\cdot\bar{H}_0^*)dv}, \qquad 6.107$$

which is an exact expression for the new resonant frequency, but not a very usable one since we generally do not initially know $\bar{E}, \bar{H}$, or ω. If we assume ΔS is small, and approximate $\bar{E}, \bar{H}$ by the unperturbed values of $\bar{E}_0, \bar{H}_0$, then the numerator of (6.107) can be reduced as follows:

$$\oint_{\Delta S}\bar{E}_0^*\times\bar{H}\cdot d\bar{s} \simeq \oint_{\Delta S}\bar{E}_0^*\times\bar{H}_0\cdot d\bar{s} = -j\omega_0\int_{\Delta V}(\epsilon|\bar{E}_0|^2 - \mu|\bar{H}_0|^2)dv, \qquad 6.108$$

where the last identity follows from conservation of power, as derived from the conjugate of (1.87) with σ, $\bar{J}_s$, and $\bar{M}_s$ set to zero. Using this result in (6.107) gives an expression for the fractional change in resonant frequency as

$$\frac{\omega-\omega_0}{\omega_0} \simeq \frac{\int_{V_0}(\mu|\bar{H}_0|^2 - \epsilon|\bar{E}_0|^2)dv}{\int_{V_0}(\mu|\bar{H}_0|^2 + \epsilon|\bar{E}_0|^2)dv}, \qquad 6.109$$

where we have also assumed that the denominator of (6.107), which represents the total energy stored in the perturbed cavity, is approximately the same as that for the unperturbed cavity.

Equation (6.109) can be written in terms of stored energies as follows:

$$\frac{\omega-\omega_0}{\omega_0} = \frac{\Delta W_m - \Delta W_e}{W_m + W_e}, \qquad 6.110$$

where ΔW_m and ΔW_e are the changes in the stored magnetic energy and electric energy, respectively, after the shape perturbation, and $W_m + W_e$ is the total stored energy in the cavity. These results show that the resonant frequency may either increase or decrease, depending on where the perturbation is located and whether it increases or decreases the cavity volume.

EXAMPLE 6.8 Shape Perturbation of a Rectangular Cavity

A thin screw of radius r_0 extends a distance ℓ through the center of the top wall of a rectangular cavity operating in the TE_{101} mode, as shown in Figure 6.28.

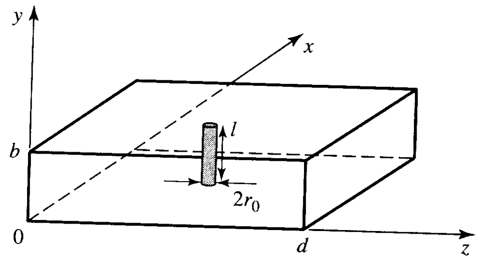


FIGURE 6.28 A rectangular cavity perturbed by a tuning post in the center of the top wall.

If the cavity is air-filled, use (6.109) to derive an expression for the change in resonant frequency from the unperturbed cavity.

Solution

From (6.42a–c), the fields for the unperturbed TE_{101} cavity can be written as

$$E_y = A \sin \frac{\pi x}{a} \sin \frac{\pi z}{d},$$

$$H_x = \frac{-jA}{Z_{TE}} \sin \frac{\pi x}{a} \cos \frac{\pi z}{d},$$

$$H_z = \frac{j\pi A}{k\eta a} \cos \frac{\pi x}{a} \sin \frac{\pi z}{d}.$$

Now if the screw is thin, we can assume that the fields are constant over the cross-section of the screw and can be represented by the fields at $x = a/2, z = d/2$:

$$E_y\left(x = \frac{a}{2}, y, z = \frac{d}{2}\right) = A,$$

$$H_x\left(x = \frac{a}{2}, y, z = \frac{d}{2}\right) = 0,$$

$$H_z\left(x = \frac{a}{2}, y, z = \frac{d}{2}\right) = 0.$$

Then the numerator of (6.109) can be evaluated as

$$\int_{\Delta V} (\mu|\bar{H}_0|^2 - \epsilon|\bar{E}_0|^2)dv = -\epsilon_0 \int_{\Delta V} A^2 dv = -\epsilon_0 A^2 \Delta V,$$

where $\Delta V = \pi \ell r_0^2$ is the volume of the screw. The denominator of (6.109) is, from (6.43),

$$\int_{V_0} (\mu|\bar{H}_0|^2 + \epsilon|\bar{E}_0|^2)dv = \frac{abd\epsilon_0 A^2}{2} = \frac{V_0 \epsilon_0 A^2}{2},$$

where $V_0 = abd$ is the volume of the unperturbed cavity. Then (6.109) gives

$$\frac{\omega - \omega_0}{\omega_0} = \frac{-2\ell\pi r_0^2}{abd} = \frac{-2\Delta V}{V_0},$$

which indicates a lowering of the resonant frequency. ○

REFERENCES

[1] R. E. Collin, *Foundations for Microwave Engineering*, Second Edition, McGraw-Hill, N.Y., 1992.

[2] S. B. Cohn, "Microwave Bandpass Filters Containing High-Q Dielectric Resonators," *IEEE Trans. Microwave Theory and Techniques*, vol. MTT-16, pp. 218–227, April 1968.

[3] M. W. Pospieszalski, "Cylindrical Dielectric Resonators and Their Applications in TEM Line Microwave Circuits," *IEEE Trans. Microwave Theory and Techniques*, vol. MTT-27, pp. 233–238, March 1979.

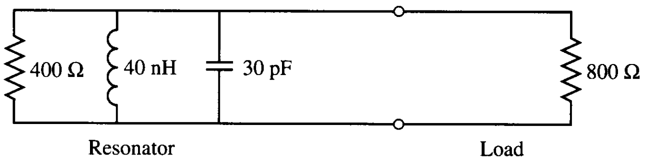
[4] J. E. Degenford and P. D. Coleman, "A Quasi-Optics Perturbation Technique for Measuring Dielectric Constants," *Proc. IEEE*, vol. 54, pp. 520–522, April 1966.

[5] S. Ramo, J. R. Whinnery, and T. Van Duzer, *Fields and Waves in Communication Electronics*, John Wiley & Sons, N.Y., 1965.

[6] R. E. Collin, *Field Theory of Guided Waves*, McGraw-Hill, N.Y., 1960.

PROBLEMS

6.1 Consider the loaded parallel resonant RLC circuit shown below. Compute the resonant frequency, unloaded Q, and loaded Q.



6.2 Derive an expression for the Q of a transmission line resonator consisting of a short-circuited transmission line 1λ long.

6.3 A transmission line resonator is fabricated from a $\lambda/4$ length of open-circuited line. Find the Q of this resonator if the complex propagation constant of the line is $\alpha + j\beta$.

6.4 Consider the resonator shown below, consisting of a $\lambda/2$ length of lossless transmission line shorted at both ends. At an arbitrary point z on the line, compute the impedances Z_L and Z_R seen looking to the left and to the right, and show that $Z_L = Z_R^*$. (This condition holds true for any lossless resonator and is the basis for the transverse resonance technique discussed in Section 3.9.)