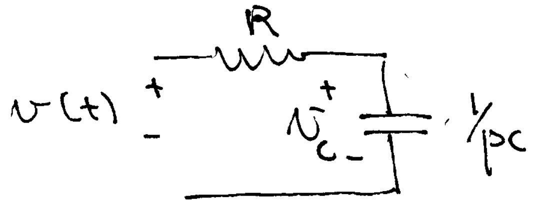
St. State Response of An RC Circuit



$$v_c = \frac{1/pC}{1/pC + R}\, v(t)$$

$$= \frac{1}{1 + pCR}\, v(t)$$

Expand $\frac{1}{1+pCR} = 1 - pCR + (pCR)^2 - \cdots$

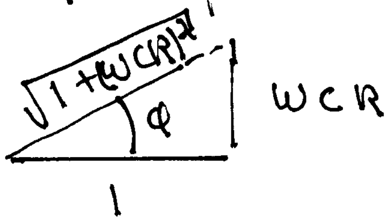
$\therefore$ If $v(t) = V \sin \omega t$

$$v_c(t) = V\,(\sin\omega t - (\omega CR)\cos\omega t + (\omega CR)^2 \sin\omega t$$
$$- (\omega CR)^3 \cos\omega t + (\omega CR)^4 \sin\omega t$$
$$+ \cdots$$

$$= V\,[\sin\omega t\,(1 + (\omega CR)^2 + (\omega CR)^4 + \cdots)$$
$$- \cos\omega t\,(\omega CR)\,(1 + (\omega CR)^2 + (\omega CR)^4 + \cdots)]$$

$$= V\left(\sin\omega t\,\frac{1}{(1+(\omega CR)^2)} - \cos\omega t\,\frac{\omega CR}{1+(\omega CR)^2}\right)$$

Define φ - phase angle



$\tan^{-1}\omega CR = \varphi$

Thus $v_c(t) = \frac{V}{\sqrt{1+(\omega CR)^2}}\,[\sin\omega t \cos\varphi - \cos\omega t \sin\varphi]$

$$= \frac{V}{\sqrt{1+(\omega CR)^2}}\,\sin(\omega t - \varphi)$$

Second Way! Use exponential Notation for $\sin\omega t = e^{j\omega t}/2j - e^{-j\omega t}/2j$

$$\frac{1}{1+pCR}\sin\omega t = \left(\frac{1}{1+pCR}\right)\left(\frac{e^{j\omega t}}{2j} - \frac{e^{-j\omega t}}{2j}\right)$$

$$= \left(1 - pCR + (pCR)^2 + \cdots\right)\left(\frac{e^{j\omega t}}{2j} - \frac{e^{-j\omega t}}{2j}\right)$$

$$= \left(1 - (j\omega CR) + (j\omega CR)^2 + \cdots\right)\frac{e^{j\omega}}{2j}$$

Complex Conjugates

$$- \left(1 - (-j\omega CR) + (j\omega CR)^2 + \cdots\right)\frac{e^{-j\omega t}}{2j}$$

$$= \underbrace{\left(\frac{1}{1+j\omega CR}\right)\frac{e^{j\omega t}}{2j}}_{1} - \underbrace{\frac{1}{1+j\omega CR}\frac{e^{j\omega t}}{2j}}_{2}$$

Note several observations

(A) For term 1 $\frac{1}{1+pCR} \to \frac{1}{1+j\omega CR}$

while term 2 $\frac{1}{1+pCR} \to \frac{1}{1-j\omega CR}$

That is substitute $p = \pm j\omega$ for $e^{\pm j\omega t}$ → very simple

(B) Term 2 is just the complex conjugate of 1. Thus we need only solve for 1, We can always add term 2 (superposition).

(c) when ① and ② are added the result is <u>real</u>. We can write the solution as

$$2\,\mathrm{Re}\left(\frac{V}{1+j\omega CR}\,\frac{e^{j\omega t}}{2j}\right)$$

The imaginery parts <u>cancel</u>.

(D) Since $\frac{1}{1+j\omega CR}$ in polar notation is

$$\frac{1}{\sqrt{1+(\omega CR)^2}}\,e^{-j\,\tan^{-1}\omega CR} \qquad \tan^{-1}\omega CR = \varphi$$

we can write the solution as

$$2\,\mathrm{Re}\left(\frac{V}{\sqrt{1+(\omega CR)^2}}\,\frac{e^{-j\varphi+j\omega t}}{2j}\right)$$

$$= \frac{V}{\sqrt{1+(\omega CR)^2}}\sin(\omega t-\varphi) \qquad \text{As before}$$

Conclusion: For sinusoidal (or cosinusoidal) St. State reponse

1) Only use the $\frac{e^{j\omega t}}{2j}$ term (or $\frac{e^{j\omega t}}{2}$) for sin (or cosine) excitations
2) Use the p-operator technique with $p = j\omega$ to find responses
3) Take 2 Re part of the complex term which results to obtain the real response