Problem Set 9 for CS 170

Note

When asked for an algorithm you must give (1) a brief informal description of the algorithm, (2) a precise description using pseudo-code, (3) an argument for termination and correctness of the algorithm, and (4) an analysis of the running time of the algorithm. Be clear about what the input to the algorithm is, how you measure the size of the input, and what constitutes a "step" in your running-time analysis.

Problem 1. [Simplex] (5 points)

Apply the Simplex method to solve the following linear program: minimize $x_1 + x_2 - 4x_3$ subject to the constraints

$$\begin{aligned} x_1 + x_2 + 2x_3 &\leq 9 \\ x_1 + x_2 - x_3 &\leq 2 \\ -x_1 + x_2 + x_3 &\leq 4 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

What is the minimal value of the objective function and at which points $(x_1, x_2, x_3) \in R^3$ is it achieved?

Problem 2. [Duality] (5 points)

You are going to go on holiday and are making up a reading list. Your favourite authors are Somerset Maugham, P.G. Wodehouse, Isaac Asimov, R.K. Narayan, Erich Segal, and Leo Tolstoy.

Reading a Maugham book gives you 14 units of information, 20 units of irony, and 5 units of joy. Wodehouse gives you 0 units information, 36 units of irony, and 50 units of joy. Asimov gives you 0 units each of information and irony, and 30 units of joy. Narayan gives you 2 units of information, 1 unit of irony, and 10 units of joy. Segal gives 0 units of information, 0 units of irony, and 38 units of joy. Tolstoy gives you 34 units of information, 1 unit of irony, and 3 units of joy.

You want to get at least 100 units of information, 36 units of irony, and 150 units of joy. Reading a Maugham book takes 26 hours, Wodehouse takes 7, Asimov 15, Narayan 9, Segal 15, and Tolstoy 39. You can read any fraction of a book and get a proportionate amount of information, irony, and joy, in a proportionate amount of time.

(a) You want to figure out how to make a reading plan that fulfils your requirements and takes the least time. Frame the problem as a linear-programming problem.

(b) Write down the dual of the linear program. How would you interpret it?

(c) If the feasible region of a linear program is empty, what can you say about the feasible region of the dual? Justify your answer.

Problem 3. [Integer Programming] (5 points)

Consider the following linear program: maximize $y - x$ subject to the constraints

$$\begin{aligned} 4x + 3y - 26 &\leq 0 \\ y - 5x &\leq 0 \\ x, y &\geq 0. \end{aligned}$$

(a) Find the optimal values of x and y assuming that x and y can be any reals. What is the corresponding value of the objective function?

(b) Find the optimal values of x and y assuming that x and y must be integers. What is the corresponding value of the objective function?

(c) Can you find a linear program where all optimal vertices of the feasible region have at least one non-integral coordinate, but there is an optimal integer solution?

You may solve this problem by any method (e.g., graphically).

Problem 4. [Star Wars] (5 points)

Luke Skywalker has discovered the following statistics about the star fleet of the Galactic Empire:

- The emperor has d star destroyers: $D_1, \ldots, D_d$.
- Darth Vader has g generals under his command: $G_1, \ldots, G_g$.
- The empire is divided into p planetary systems for administrative purposes: $P_1, \ldots, P_p$.
- Each star destroyer is operated by a general. A general can operate at most one star destroyer at a time, and he can only operate star destroyers he is familiar with. Each general G_i is familiar with a set $f_i \subseteq \{D_1, \ldots, D_d\}$ of star destroyers.
- Each general is given a peace time assignment to work in the government of a particular planetary system. Each planetary system P_j can spare at most s_j generals for war time duties in an emergency.

Can you help Luke figure out the maximum number of star destroyers the emperor can mobilize?

(a) Frame the problem as a network-flow problem.

(b) Frame the problem as a linear-programming problem.

(c) Would you choose Ford-Fulkerson or Simplex to solve the problem? (Ignore the run times but consider the validity of non-integral solutions.)

Problem 5. [Hollywood] (5 points)

A film producer wants to make a movie. For this she needs to choose actors. There are n available actors $A_1, \ldots, A_n$. Each actor A_i demands a payment of s_i dollars to participate in the movie.

The funding for the movie will come from m investors $I_1, \ldots, I_m$. Each investor I_k will pay the director p_k dollars, but only under the following condition: I_k has a list of actors $\ell_k \subseteq \{A_1, \ldots, A_n\}$, and she will only invest in the movie if all actors on her list appear in the movie. (If even one of the actors on the list doesn't appear, she pays nothing.)

The profit of the director is the sum of payments she gets from the investors minus the sum of payments she makes to the actors. Find an algorithm for the director to maximize her profit. (*Hint*: frame the problem as a network-flow problem on a network with nodes $s, t, A_1, \ldots, A_n, I_1, \ldots, I_m$.)

Problem 6. [Camp David] (5 points)

In the Camp David talks there were n Americans, n Israelis, and n Palestinians. Every American has a list of Israelis and Palestinians she knows. The "work" at Camp David was done in groups of 3, each consisting of one American, one Israeli, and one Palestinian such that the American knows both the Israeli and the Palestinian. Find an algorithm that divides the $3n$ participants into n disjoint work groups, or answers that such a division is impossible.