EECS 219C: Computer-Aided Verification

# Symbolic Model Checking Part I

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## Today’s Lecture

Symbolic model checking with BDDs

Symbolic: Manipulate sets (of states and transitions) rather than individual elements and represent sets as Boolean formulas

with BDDs: Represent Boolean formulas as BDDs

# Today's Lecture

- Symbolic model checking
  - Basics of symbolic representation
  - Quantified Boolean formulas (QBF)
  - Checking G p
  - Fixpoint theory
  - Checking CTL properties

# Sets as Boolean functions

- Every finite set can be represented as a Boolean function
  - Suppose the set has N (> 0) elements
  - Each element is encoded as a string of at least $\lceil \log M \rceil$ bits, where M is the number of elements in the universe
  - Characteristic Boolean function is the one whose ON-set (satisfying assignments) are those strings
  - Empty set is "False"

# Set Operations as Boolean Operations

- $A \cup B = ?$
- $A \cap B = ?$
- $A \subset B = ?$
- Is A empty?

# Sets of states and transitions

- Set of states → each state s is bit-string comprising values of state variables
- Set of transitions →
  - Transition is a state pair (s, s')
  - View the pair as a combined bit-string
- From now, we will view the set of states S and the transition relation R as Boolean formulas over vector of current state variables v and next state variables v'
  - S(v), R(v, v')

# Quantified Boolean Formulas

- Let F denote a Boolean formula, and v denote one or more Boolean variables
- A quantified Boolean formula $\phi$ is obtained as:

$$\phi ::= F \mid \exists v\, \phi \mid \forall v\, \phi \mid \phi \wedge \phi \mid \phi \vee \phi \mid \neg \phi$$

- How do you express $\exists v_i\, \phi$ and $\forall v_i\, \phi$ in terms of $\phi$'s cofactors and standard Boolean operators?

# Symbolic Model Checking G p

- Given: Set of initial states $S_0$, transition relation R
- Check property G p (or AG p)
- How symbolic model checking will do this:
  - – Compute $S_0, S_1, S_2, \ldots$ where $S_i$ is the set of states reachable from some initial state in at most i steps
    - What kind of search is this: DFS or BFS?
    - When do we stop?
  - – After computing each $S_i$, check whether any element of $S_i$ satisfies $\neg$ p [ How? ]
    - How do we generate a counterexample?

## Reachability Analysis

- The process of computing the set of states reachable from some initial state in 0 or more steps
  - Often characterized as checking (AG true)
  - The resulting set is called “reachable set” or “set of reachable states”
    - This is the “strongest invariant” of the system → WHY? What is a “system invariant”?

## Implementing Reachability Analysis

- How is $S_i$ related to $S_{i+1}$?
  - In words
  - As a recurrence relation using QBF

## Implementing Reachability Analysis

- How is $S_i$ related to $S_{i+1}$?
- $v \in S_{i+1}$ iff $v \in S_i$ or there is a state $x \in S_i$ such that $R(x, v)$
- $S_{i+1}(v) = S_i(v) \vee \exists x \{ S_i(x) \wedge R(x,v) \}$

## Implementing Reachability Analysis

- How is $S_i$ related to $S_{i+1}$?
- $v \in S_{i+1}$ iff $v \in S_i$ or there is a state $x \in S_i$ such that $R(x, v)$
- $S_{i+1}(v) = S_i(v) \vee \exists x \{ S_i(x) \wedge R(x,v) \}$
- $S_{i+1}(v) = S_i(v) \vee (\exists v \{ S_i(v) \wedge R(v,v') \}) [v/v']$
  - F[x/y] means that we substitute x for y in F

# Implementing Reachability Analysis

```
i := 0;
do {
  i++;
  $S_i(v) = S_{i-1}(v) \vee (\exists v \{ S_{i-1}(v) \wedge R(v,v') \}) [v/v']$
} while ($S_i(v) \neq S_{i-1}(v)$)
```

$S_i(v)$ is the set of reachable states

# BDD Issues

- Remember that $S_i$ and R are represented as BDDs
- How large they grow determines the space and time usage of the algorithm

# Backwards Reachability

- Suppose we want to verify G p
- The formula ¬ p characterizes all error states
- We can search backwards for a path to an error state from some initial state
  - Compute $E_0$, $E_1$, $E_2$, … as states reachable from the error states in at most 0, 1, 2, … steps
  - $E_0 = \neg p$
  - How to express $E_{i+1}$ in terms of $E_i$ ?
- Why would we want to do backwards reachability analysis? Is it always better?

# Verification of G p

- Corresponding CTL formula is AGp
- with Forward Reachability Analysis:
  - Check if some $S_i \wedge \neg p$ is true
- with Backward Reachability Analysis:
  - Set $E_0 = \neg p$
  - Check if $E_k \wedge S_0$ is true for any k

# Symbolic Model Checking, General Case

- We will consider properties in CTL
  - As implemented in the original SMV model checker
  - Later we will see how LTL properties can be verified using symbolic techniques

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# Model Checking Arbitrary CTL

- Need only consider the following types of CTL properties:
  - E X p
  - E G p
  - E ( p U q )
- Why? ← all others are expressible using above
  - A G p = ?
  - A G ( p → ( A F q ) ) = ?

# Fixpoint Theory

- Theory about elements/points that are unchanged by application of a function (hence “fixed point”)
- A concept from mathematics and denotational semantics of programming languages
- For us: Theoretical concepts and results that will help us design algorithms for CTL model checking

# Fixpoint (Fixed point)

- Let $\Sigma$ be a set (the “universe”), and $\Sigma' \subseteq \Sigma$
  - In model checking, $\Sigma$ = True
- Let $\tau : P(\Sigma) \rightarrow P(\Sigma)$
  - $P(\Sigma)$ is the power set of $\Sigma$
- Definition: $\Sigma'$ is a fixpoint of $\tau$ if $\tau(\Sigma') = \Sigma'$

# Example of Fixpoint

- Let
  - $\Sigma = \{s_0, s_1\}$
  - $\tau(Z) = Z \cup \{s_0\}, \quad Z \subseteq \Sigma$

- What is a fixpoint of $\tau$? Is there only one?

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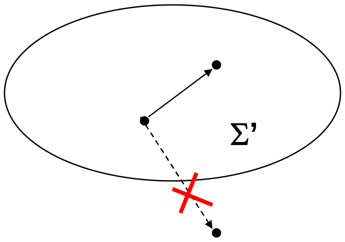
# Model Checking Example

In the context of Reachability Analysis:

- What's an example of a fixpoint we've seen already? What was $\tau$?

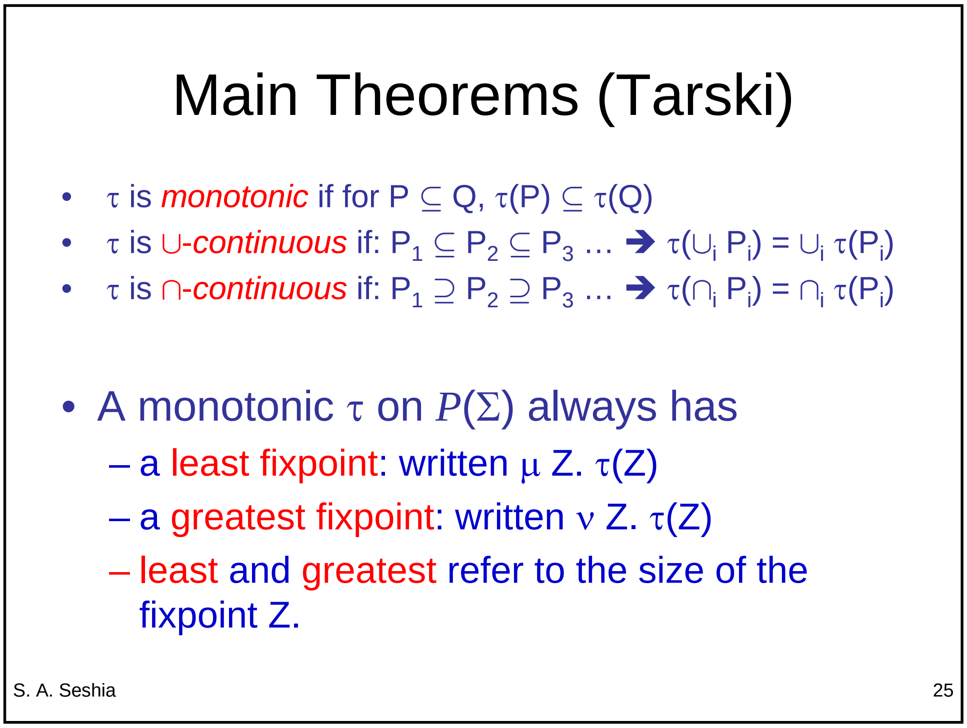


## Model Checking Example

- What's an example of a fixpoint we've seen already? What was $\tau$?
  - A G true can be computed using a fixpoint formulation
  - $\tau$ computes the "next state"
- What we need: a way to generalize this for arbitrary CTL properties: EX, EG, EU
  - Fixpoint theory helps us do this

## More Definitions

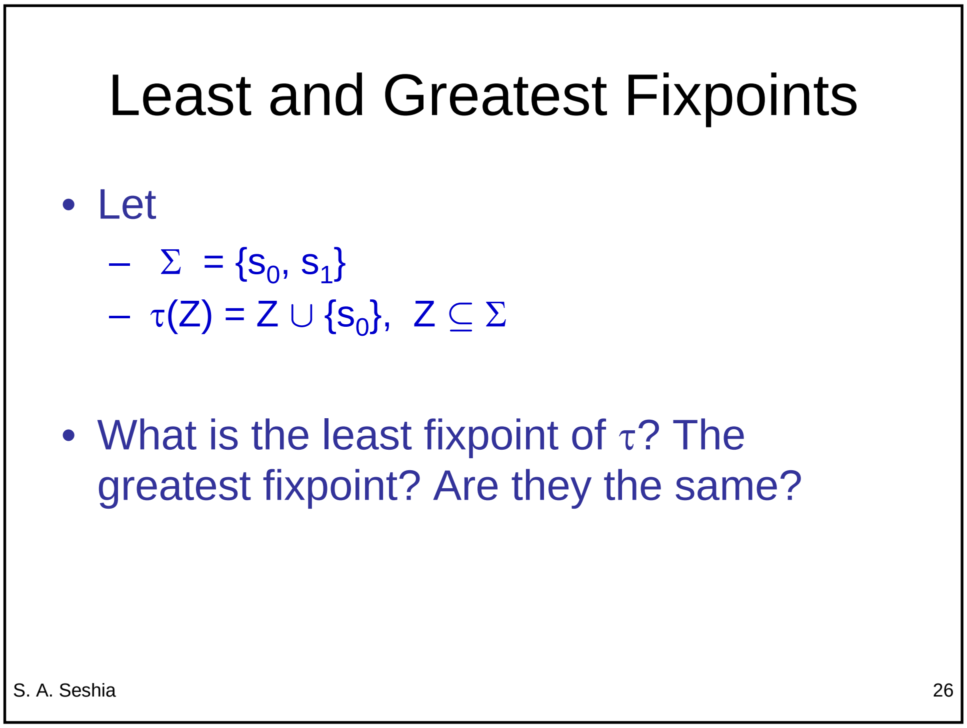
- $\tau$ is *monotonic* if for $P \subseteq Q$, $\tau(P) \subseteq \tau(Q)$
- $\tau$ is $\cup$-*continuous* if: $P_1 \subseteq P_2 \subseteq P_3 \ldots \Rightarrow \tau(\cup_i P_i) = \cup_i \tau(P_i)$
- $\tau$ is $\cap$-*continuous* if: $P_1 \supseteq P_2 \supseteq P_3 \ldots \Rightarrow \tau(\cap_i P_i) = \cap_i \tau(P_i)$



# Main Theorems (Tarski)

- $\tau$ is *monotonic* if for $P \subseteq Q$, $\tau(P) \subseteq \tau(Q)$
- $\tau$ is $\cup$-*continuous* if: $P_1 \subseteq P_2 \subseteq P_3 \ldots \Rightarrow \tau(\cup_i P_i) = \cup_i \tau(P_i)$
- $\tau$ is $\cap$-*continuous* if: $P_1 \supseteq P_2 \supseteq P_3 \ldots \Rightarrow \tau(\cap_i P_i) = \cap_i \tau(P_i)$

- A monotonic $\tau$ on $P(\Sigma)$ always has
  - a least fixpoint: written $\mu Z.\ \tau(Z)$
  - a greatest fixpoint: written $\nu Z.\ \tau(Z)$
  - least and greatest refer to the size of the fixpoint Z.



# Least and Greatest Fixpoints

- Let
  - $\Sigma = \{s_0, s_1\}$
  - $\tau(Z) = Z \cup \{s_0\}, \quad Z \subseteq \Sigma$

- What is the least fixpoint of $\tau$? The greatest fixpoint? Are they the same?

# Main Theorems (Tarski)

- $\tau$ is *monotonic* if for $P \subseteq Q$, $\tau(P) \subseteq \tau(Q)$
- $\tau$ is $\cup$-*continuous* if: $P_1 \subseteq P_2 \subseteq P_3 \ldots \Rightarrow \tau(\cup_i P_i) = \cup_i \tau(P_i)$
- $\tau$ is $\cap$-*continuous* if: $P_1 \supseteq P_2 \supseteq P_3 \ldots \Rightarrow \tau(\cap_i P_i) = \cap_i \tau(P_i)$

- A *monotonic* $\tau$ on $\mathcal{P}(\Sigma)$ always has
  - a least fixpoint: written $\mu Z.\ \tau(Z)$
  - a greatest fixpoint: written $\nu Z.\ \tau(Z)$
  - $\mu Z.\ \tau(Z) = \cap \{ Z \mid \tau(Z) \subseteq Z \}$
  - $\nu Z.\ \tau(Z) = \cup \{ Z \mid \tau(Z) \supseteq Z \}$

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# Main Theorems (Tarski)

- $\tau$ is *monotonic* if for $P \subseteq Q$, $\tau(P) \subseteq \tau(Q)$
- $\tau$ is $\cup$-*continuous* if: $P_1 \subseteq P_2 \subseteq P_3 \ldots \Rightarrow \tau(\cup_i P_i) = \cup_i \tau(P_i)$
- $\tau$ is $\cap$-*continuous* if: $P_1 \supseteq P_2 \supseteq P_3 \ldots \Rightarrow \tau(\cap_i P_i) = \cap_i \tau(P_i)$
- A *monotonic* $\tau$ on $\mathcal{P}(\Sigma)$ always has
  - a least fixpoint: written $\mu Z.\ \tau(Z)$
  - a greatest fixpoint: written $\nu Z.\ \tau(Z)$
  - $\mu Z.\ \tau(Z) = \cap \{ Z \mid \tau(Z) \subseteq Z \}$
  - $\nu Z.\ \tau(Z) = \cup \{ Z \mid \tau(Z) \supseteq Z \}$
  - $\mu Z.\ \tau(Z) = \cup_i \tau^i(\phi)$ when $\tau$ is $\cup$-*continuous*
  - $\nu Z.\ \tau(Z) = \cap_i \tau^i(\Sigma)$ when $\tau$ is $\cap$-*continuous*

# Main Lemma for us

- If $\Sigma$ is finite and $\tau$ is monotonic, then $\tau$ is also $\cup$-continuous and $\cap$-continuous
- Proof? (of $\cup$-continuous)

$\tau$ is $\cup$-*continuous* if: $P_1 \subseteq P_2 \subseteq P_3 \ldots \Rightarrow \tau(\cup_i P_i) = \cup_i \tau(P_i)$

# What's Left?

- We have the needed fixpoint theory
- Now all we need to do is formulate the result of CTL operators as fixpoints
  - We will identify a CTL formula with the set of states that satisfy that formula
    - Remember that CTL formulas start with A or E which are interpreted over states, not runs

# CTL Results as Fixpoints

- $AG\, p = \nu Z.\ p \wedge AX\, Z$
  - $\tau(Z) = p \wedge AX\, Z$
  - Given a point (state) in Z, $\tau$ maps it to another state that
    - Satisfies p
    - Can reach a state in Z along any execution path in one step
    - So what happens when we reach $\tau$'s fixpoint?
  - Remember: $\nu$ fixpoint computation starts with the universal set $\Sigma$ and works 'downward'

# Other Fixpoint Formulations

- $EF\, p = \mu Z.\ p \vee EX\, Z$
- $EG\, p = \nu Z.\ p \wedge EX\, Z$
- $E(p\ U\ q) = \mu Z.\ q \vee (p \wedge EX\, Z)$

- Intuitively:
  - Eventualities → least fixpoints
  - Always/Forever → greatest fixpoints

# Model Checking CTL Properties

- We define a general recursive procedure called "Check" to do the fixpoint computations
- Definition of Check:
  - Input: A CTL property $\Pi$ (and implicitly, R)
  - Output: A Boolean formula B representing the set of states satisfying $\Pi$
- If $S_0(v) \rightarrow B(v)$, then $\Pi$ is true

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# The "Check" procedure

Cases:

- If $\Pi$ is a Boolean formula, then Check($\Pi$) = $\Pi$
- Else:
  - $\Pi$ = EX p, then Check($\Pi$) = CheckEX(Check(p))
  - $\Pi$ = E(p U q), then
    Check($\Pi$) = CheckEU(Check(p), Check(q))
  - $\Pi$ = E G p, then Check($\Pi$) = CheckEG(Check(p))
- Note: What are the arguments to CheckEX, CheckEU, CheckEG? CTL properties or Boolean formulas?

# CheckEX

- CheckEX(p) returns a set of states such that p is true in their next states

- How to write this?

$\exists$ x [ p(x) . R (s, x) ]

# CheckEU

- CheckEU(p, q) returns a set of states, each of which is such that
  - Either q is true in that state
  - Or p is true in that state and you can get from it to a state in which p U q is true

# CheckEU

- CheckEU(p, q) returns a set of states, each of which is such that
  - Either q is true in that state
  - Or p is true in that state and you can get from it to a state in which p U q is true

- Let $Z_0$ be our initial approximation to the answer to CheckEU(p, q)

- $Z_k(v) = \{ q(v) + [ p(v) . \exists v' \{ R(v, v') . Z_{k-1}(v') \} ] \}$

- What's $Z_0$? Why will this terminate?

# Summary

- EGp computed similarly

- Definition of Check:
  - Input: A CTL property $\Pi$ (and implicitly, R)
  - Output: A Boolean formula B representing the set of states satisfying $\Pi$

- All Boolean formulas represented "symbolically" as BDDs
  - "Symbolic Model Checking"

# Counterexample/Witness Generation for CTL

- Counterexample = run showing how the property is violated
  - Formulas with universal path quantifier A
- Witness = run showing how the property is satisfied
  - Formulas with existential path quantifier E
  - Can also view as counterexample for the negated property
    - E.g. E G p and A F $\neg$ p

# Witness Generation for EG p

- Fixpoint formulation for E G p:
  - $\nu Z . p \wedge EX Z$
  - $\tau(Z) = p \wedge EX Z$
- Fixpoint computation yields sequence $Z_0, Z_1, \ldots, Z_k$
  - $Z_0$ = True (universal set)
  - $Z_1 = \tau(\text{True}) = ?$
  - each $Z_i$ is a BDD representing a set of states
  - How would you describe an element of $Z_i$ ?
- We need to generate the counterexample from $S_0, R, Z_0, Z_1, \ldots, Z_k$

# Witness Generation for EG p

- Fixpoint computation yields sequence $Z_0, Z_1, \ldots, Z_k$
  - A state in $Z_i$ ($i > 0$) satisfies p and there is a path of length i-1 from that state comprising states satisfying p
  - How would you describe an element of $Z_k$ ?
    - Remember: it's the fixpoint

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# Witness Generation for EG p

- Fixpoint computation yields sequence $Z_0, Z_1, \ldots, Z_k$
  - A state in $Z_i$ satisfies p and there is a path of length i-1 from that state comprising states satisfying p
  - How would you describe an element of $Z_k$ ?
    - State in $Z_k$ has path from it of length k-1 or more (including a cycle) with all states satisfying p
    - If $S_0$ is contained in $Z_k$, any initial state has such a path

# Witness Generation for EG p

- Let $s_0$ be an initial state with a desired witness path
  - – We need to reproduce one such witness
  - – How can we do this?

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# Witness Generation for EG p

- Let $s_0$ be an initial state with a desired witness path
  - – We need to reproduce one such witness
  - – How can we do this?
    - Main insight: desired successor of $s_0$ also satisfies EG p, and so on
    - Look for a cycle in such a computed chain
      - – Why should there be a cycle?

# Fairness

- A computation path is defined as fair if a fairness constraint p is true infinitely often along that path
  - Fairness constraint is a state predicate
  - Generalized to set of fairness constraints $\{p_1, p_2, \ldots, p_k\}$ by requiring each element of the subset to be true infinitely often
- Example: Every process in an asynchronous composition must be scheduled infinitely often