EECS 219C: Computer-Aided Verification

Boolean Satisfiability Solving
Part I: Basics

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Project Proposals

- Due Monday, September 17 on bSpace
- Instructions will follow
- Meet me next week to discuss project ideas if you haven't already

Boolean Functions (Formulas) and Propositional Logic

At the core of all Verification Algorithms

Boolean Functions (Formulas) and Propositional Logic

- Variables: $x_1, x_2, x_3, \ldots, x_n \in \{0, 1\}$ (or {false, true})
- $F(x_1, x_2, x_3, \ldots, x_n) \in \{0,1\}$
- F representable as the output (root) of a circuit (expression DAG) constructed with gates (Boolean operators)
  - Standard Boolean operators: And ($\wedge$, $\cdot$), Or ($\vee$, +), Not ($\neg$, ’)
  - Derived operators: Implies ($\rightarrow$) Iff ($\Leftrightarrow$)

The Boolean Satisfiability Problem (SAT)

- Given:

  A Boolean formula $F(x_1, x_2, x_3, \ldots, x_n)$

- Can F evaluate to 1 (true)?
  - Is F satisfiable?
  - If yes, return values to x_i's (satisfying assignment, or "model") that make F true

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Why is SAT important?

- Theoretical importance:
  - First NP-complete problem (Cook, 1971)
- Many practical applications:
  - Model Checking
  - Automatic Test Pattern Generation
  - Combinational Equivalence Checking
  - Planning in AI
  - Automated Theorem Proving
  - Software Verification
  - …

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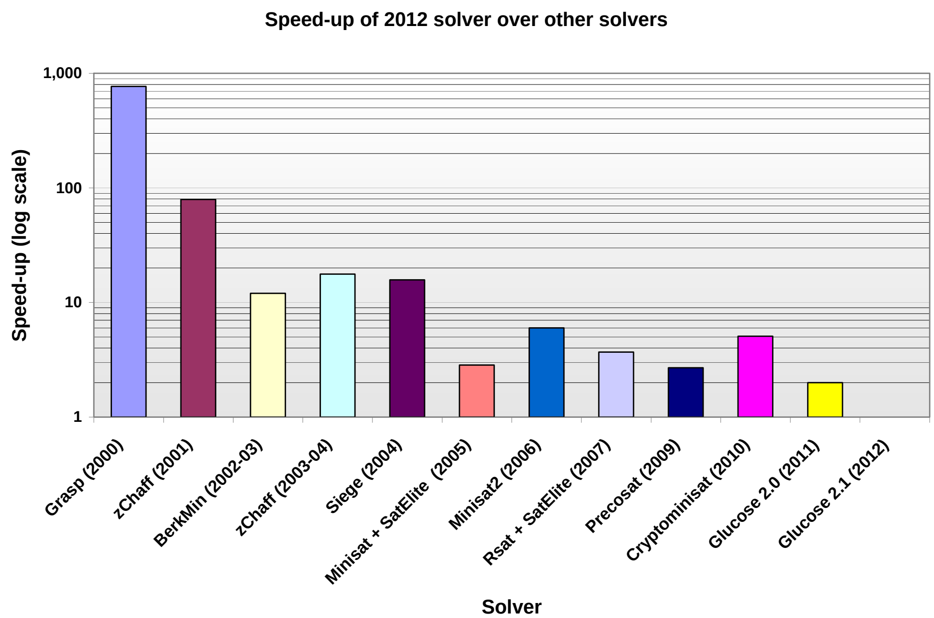
My Experience with SAT Solving



Terminology

- Literal
- Clause
- Conjunctive Normal Form (CNF)
- Disjunctive Normal Form (DNF)
- Tautology
  - Complexity of tautology checking for propositional logic?

An Example

- Inputs to SAT solvers are usually represented in CNF

(a + b + c) (a’ + b’ + c) (a + b’ + c’) (a’ + b + c’)

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Why CNF?

Why CNF?

- Input-related reason
  - Can transform from circuit to CNF in linear time & space (HOW?)
- Solver-related: Most SAT solver variants can exploit CNF
  - Easy to detect a conflict
  - Easy to remember partial assignments that don't work (just add 'conflict' clauses)
  - Other "ease of representation" points?
- Any reasons why CNF might NOT be a good choice?

Complexity Issues

- **k-SAT**: A SAT problem with input in CNF with at most k literals in each clause
- Complexity for non-trivial values of k:
  - 2-SAT: ?
  - 3-SAT: ?
  - > 3-SAT: ?

2-SAT Algorithm

- Linear-time algorithm (Aspvall, Plass, Tarjan, 1979)
  - Think of clauses as implications
  - Think of a graph with literals as nodes
  - Find strongly connected components
  - Variable and its negation should not be in the same component
- Example 1:

  (a’ + b) (b’ + c) (c’ + a)
- Example 2:

  (a’ + b) (b’ + c) (c’ + a) (a + b) (a’ + b’)

3-SAT: Complexity Bounds (circa 2008)

- Obvious upper bound on run-time?
- Best known deterministic upper bound
  1.473^n
- Best known randomized upper bound
  1.324^n
- Best known lower bound
  $n^{2.761}$

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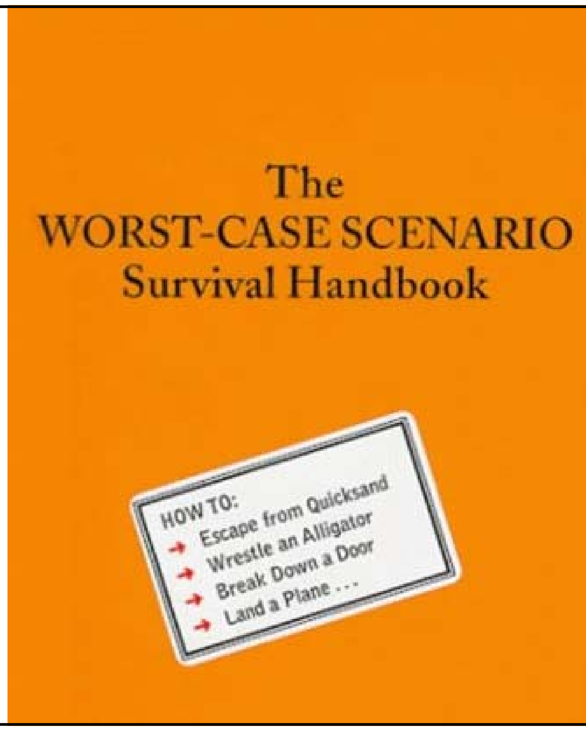
Worst-Case Complexity



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Beyond Worst-Case Complexity

- What we really care about is “typical-case” complexity
- But how can one measure “typical-case”?
- Two approaches:
  – Is your problem a restricted form of 3-SAT? That might be polynomial-time solvable
  – Experiment with (random) SAT instances and see how the solver run-time varies with formula parameters (#vars, #clauses, …)

Special Cases of 3-SAT

- You already know one: 2-SAT
  – T. Larrabee observed that many clauses in ATPG tend to be 2-CNF
- Another useful class: Horn-SAT
  – A clause is a Horn clause if at most one literal is positive
  – If all clauses are Horn, then problem is Horn-SAT
  – E.g. Application:- Simulation checking between 2 finite-state systems

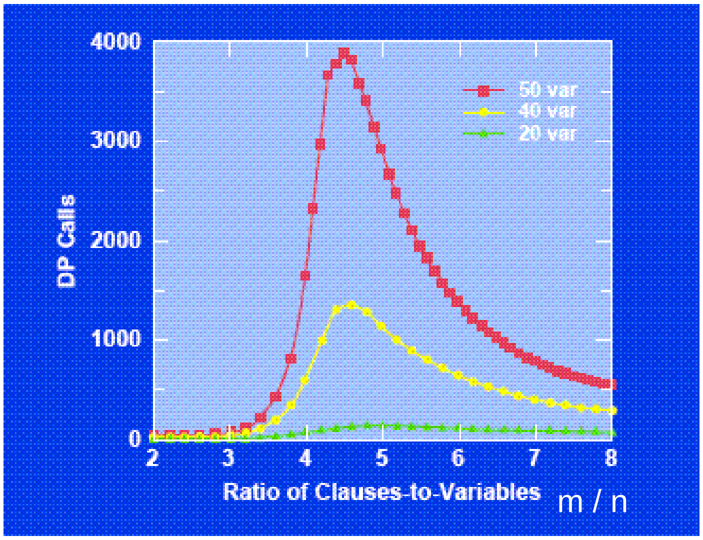
Horn-SAT

- Can we solve Horn-SAT in polynomial time? How? [homework]
  - Hint: view clauses as implications.
- Variants:
  - Negated Horn-SAT: Clauses with at most one literal negative
  - Renamable Horn-SAT: Doesn't look like a Horn-SAT problem, but turns into one when polarities of some variables are flipped

Phase Transitions in k-SAT

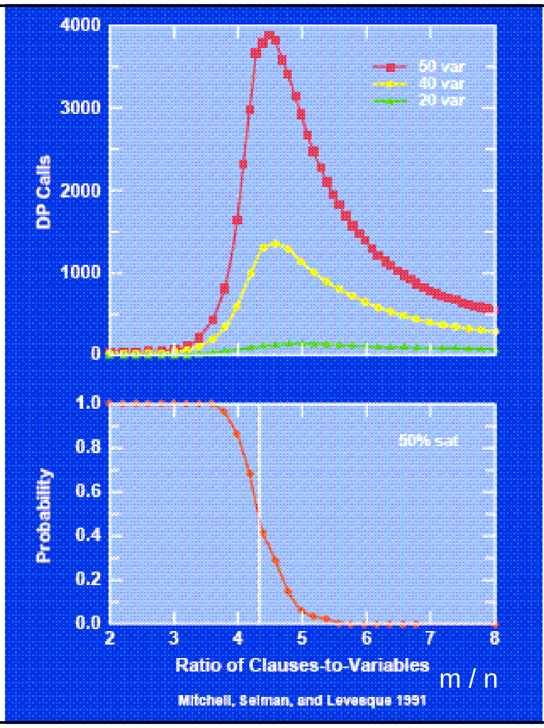
- Consider a fixed-length clause model
  - k-SAT means that each clause contains exactly k literals
- Let SAT problem comprise **m** clauses and **n** variables
  - Randomly generate the problem for fixed k and varying m and n
- Question: How does the problem hardness vary with m/n ?

3-SAT Hardness



As n increases hardness transition grows sharper

Transition at m/n $\simeq$ 4.3



Threshold Conjecture

- For every k, there exists a c* such that
  - For $m/n < c^*$, as $n \rightarrow \infty$, problem is satisfiable with probability 1
  - For $m/n > c^*$, as $n \rightarrow \infty$, problem is unsatisfiable with probability 1
- Conjecture proved true for k=2 and c*=1
- For k=3, current status is that c* is in the range 3.42 – 4.51

The (2+p)-SAT Model

- We know:
  - 2-SAT is in P
  - 3-SAT is in NP
- Problems are (typically) a mix of binary and ternary clauses
  - Let $p \in [0,1]$
  - Let problem comprise (1-p) fraction of binary clauses and p of ternary
  - So-called (2+p)-SAT problem

Experimentation with random (2+p)-SAT

- When $p < \sim 0.41$
  - Problem behaves like 2-SAT
  - Linear scaling
- When $p > \sim 0.42$
  - Problem behaves like 3-SAT
  - Exponential scaling

- Nice observations, but don't help us predict behavior of problems in practice

Backbones and Backdoors

- Backbone [Parkes; Monasson et al.]
  - Subset of literals that must be true in every satisfying assignment (if one exists)
  - Empirically related to hardness of problems
- Backdoor [Williams, Gomes, Selman]
  - Subset of variables such that once you've given those a suitable assignment (if one exists), the rest of the problem is poly-time solvable
  - Also empirically related to hardness
- But no easy way to find such backbones / backdoors! ☹

A Classification of SAT Algorithms

- Davis-Putnam (DP)
  – Based on **resolution**
- Davis-Logemann-Loveland (DLL/DPLL)
  – Search-based
  – Basis for current most successful solvers
- Stalmarck's algorithm
  – More of a "breadth first" search, proprietary algorithm
- Stochastic search
  – Local search, hill climbing, etc.
  – Unable to prove unsatisfiability (incomplete)

Next Class

- Quick review of SAT algorithms; how DPLL/DLL algorithm works in current SAT solvers