CS 172: Computability and Complexity

Equivalence of CFGs and PDAs & CFL Pumping Lemma

Sanjit A. Seshia
EECS, UC Berkeley

Acknowledgments: L.von Ahn, L. Blum, M. Blum

Formal Definition of **Acceptance**

PDA $P = (Q, \Sigma, \Gamma, \delta, q_0, F)$ accepts a word $w \in \Sigma^*$

where $w = w_1 w_2 w_3 \ldots w_m$ with $w_i \in \Sigma_\varepsilon$

if *there exists* a sequence

$(q_0, s_0) \rightarrow (q_1, s_1) \rightarrow (q_2, s_2) \rightarrow \ldots \rightarrow (q_m, s_m)$

where

$s_i \in \Gamma^*$ (represent the stack), with $s_0 = \varepsilon$,

$q_m \in F$,

$(q_{i+1}, b) \in \delta(q_i, w_{i+1}, a)$

where $s_i = at$ and $s_{i+1} = bt$, $a,b \in \Gamma_\varepsilon$, $t \in \Gamma^*$

Theorem

Suppose L is generated by a CFG G = (V, Σ, R, S)

Construct P = $(Q, \Sigma, \Gamma, \delta, q, F)$ that recognizes L

A Language is generated by a CFG

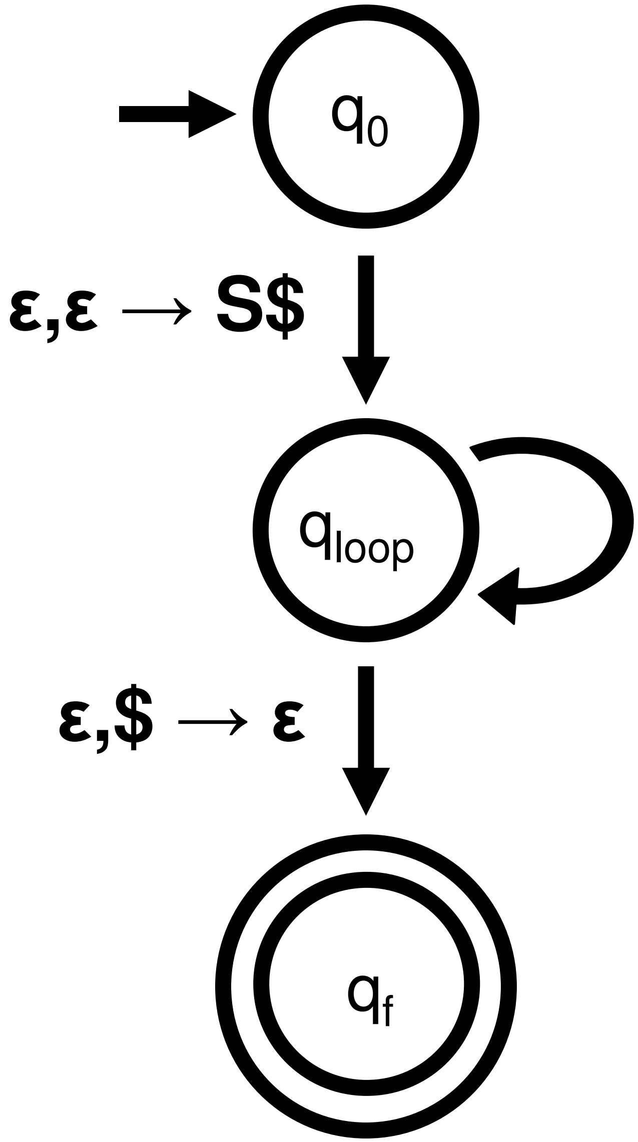
$\Leftrightarrow$

It is recognized by a PDA

A Language is generated by a CFG

$\Rightarrow$

It is recognized by a PDA



ε,A → w for rule A → w

a,a → ε for terminal a

Theorem

Suppose PDA P = $(Q, \Sigma, \Gamma, \delta, q, F)$ recognizes L

Construct CFG G = (V, Σ, R, S) that generates L

A Language is generated by a CFG

$\Leftarrow$

It is recognized by a PDA

Proof Ideas

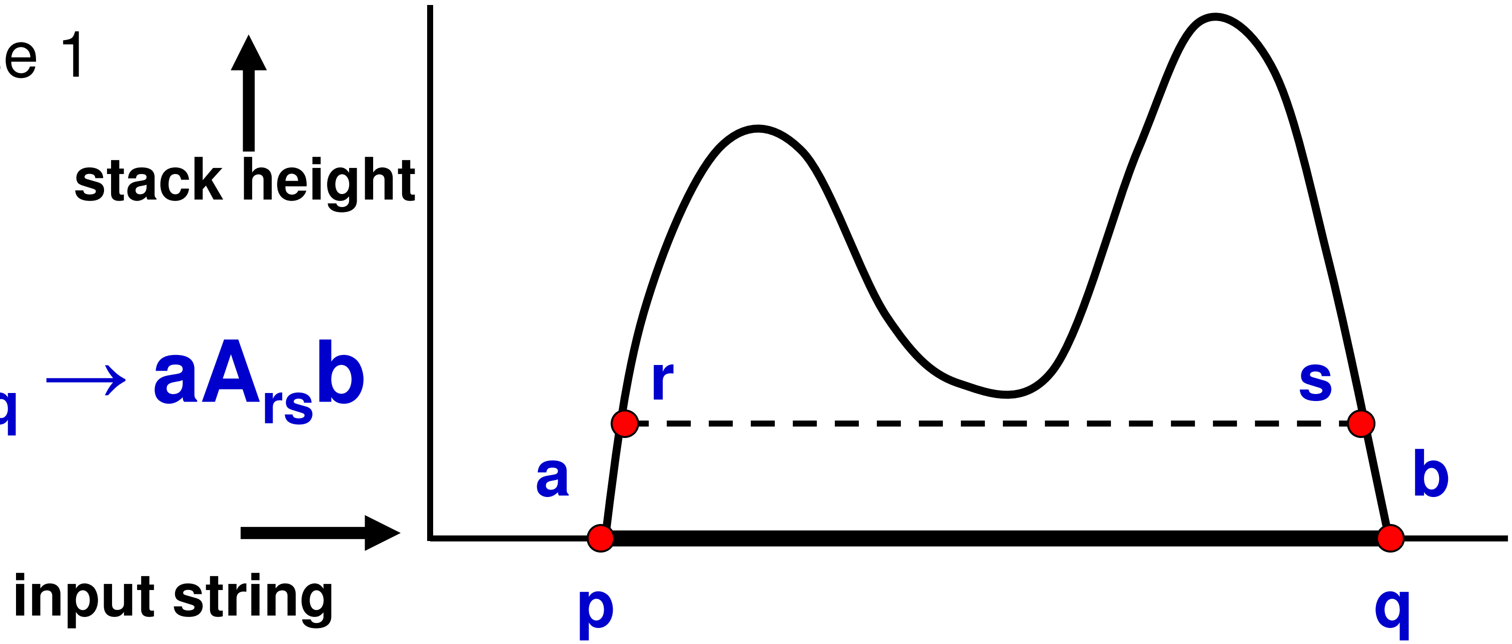
- A_{pq} = variable generating all x that takes P from (p, ε) to (q, ε)
- Formal construction:
  - $V = \{ A_{pq} \mid p, q \in Q \}$
  - $S = A_{q0qf}$
  - Defining R:

    Intuition: Derivations correspond to computations of P;

    There are two cases for a derivation from A_{pq}

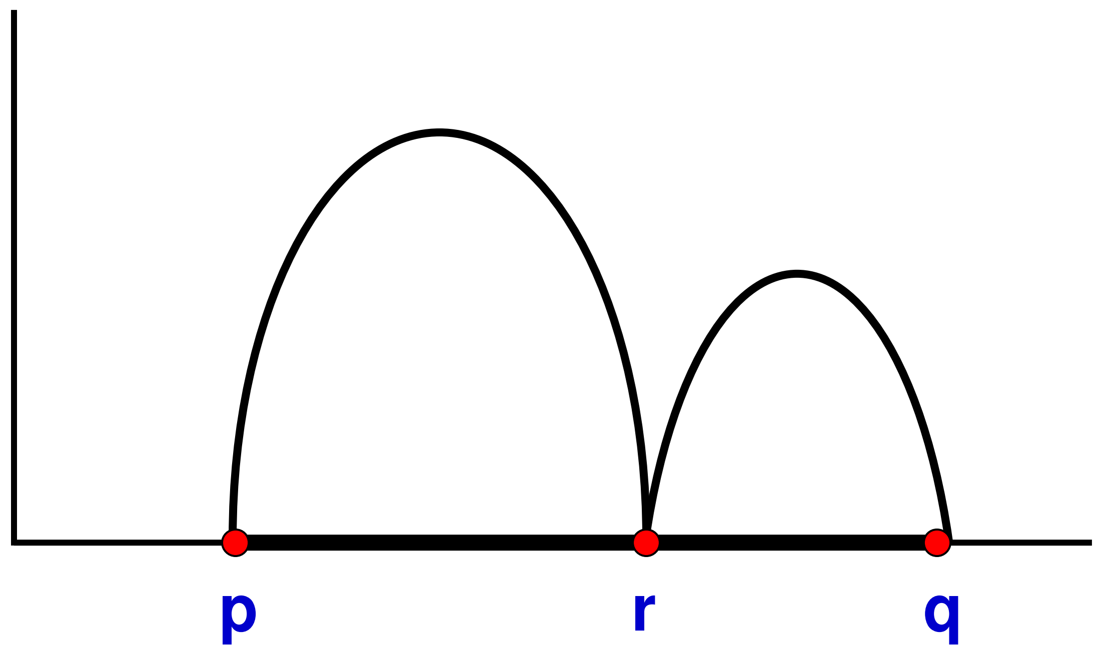
    1. Stack is empty only at the beginning and end of the derivation from A_{pq}
    2. Stack becomes empty somewhere in between

Case 1



$A_{pq} \rightarrow aA_{rs}b$

Case 2



$A_{pq} \rightarrow A_{pr}A_{rq}$

Recap of Proof Ideas

- A_{pq} = variable generating all x that takes P from (p, ε) to (q, ε)
- Formal construction:
  - $V = \{ A_{pq} \mid p, q \in Q \}$
  - $S = A_{q0qf}$
  - R defined as follows:
    - $A_{pp} \rightarrow \varepsilon \quad \forall p \in Q$
    - $A_{pq} \rightarrow a A_{rs} b \quad \forall p, q, r, s \in Q$ s.t. $(r, t) \in \delta(p, a, \varepsilon)$, $(q, \varepsilon) \in \delta(s, b, t)$
    - $A_{pq} \rightarrow A_{pr} A_{rq} \quad \forall p, q, r \in Q$

CFL Pumping Lemma

Let L be a context-free language

Then there exists p such that
for all $w \in L$ and $|w| \geq p$

we can write $w = uvxyz$, where:

1. **$uv^ixy^iz \in L$ for any $i \geq 0$**
2. **$|vy| > 0$**
3. **$|vxy| \leq p$**

“Surgery” on Parse Trees

Idea: If w is long enough, then any parse tree for w must have a path that contains a variable more than once