CS 172: Computability and Complexity

# Regular Expressions

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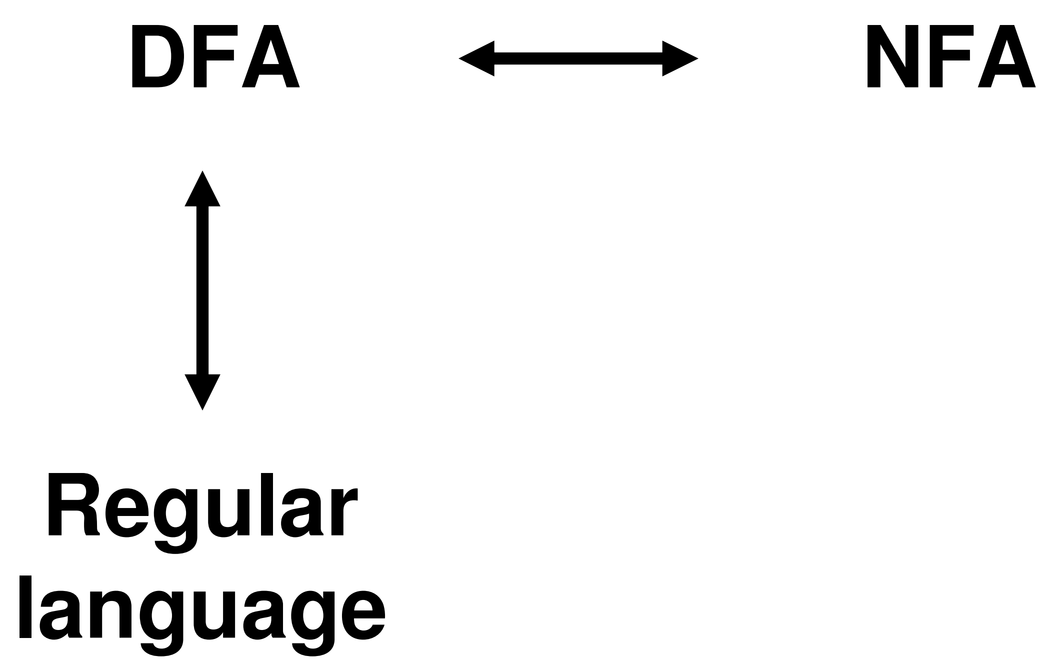
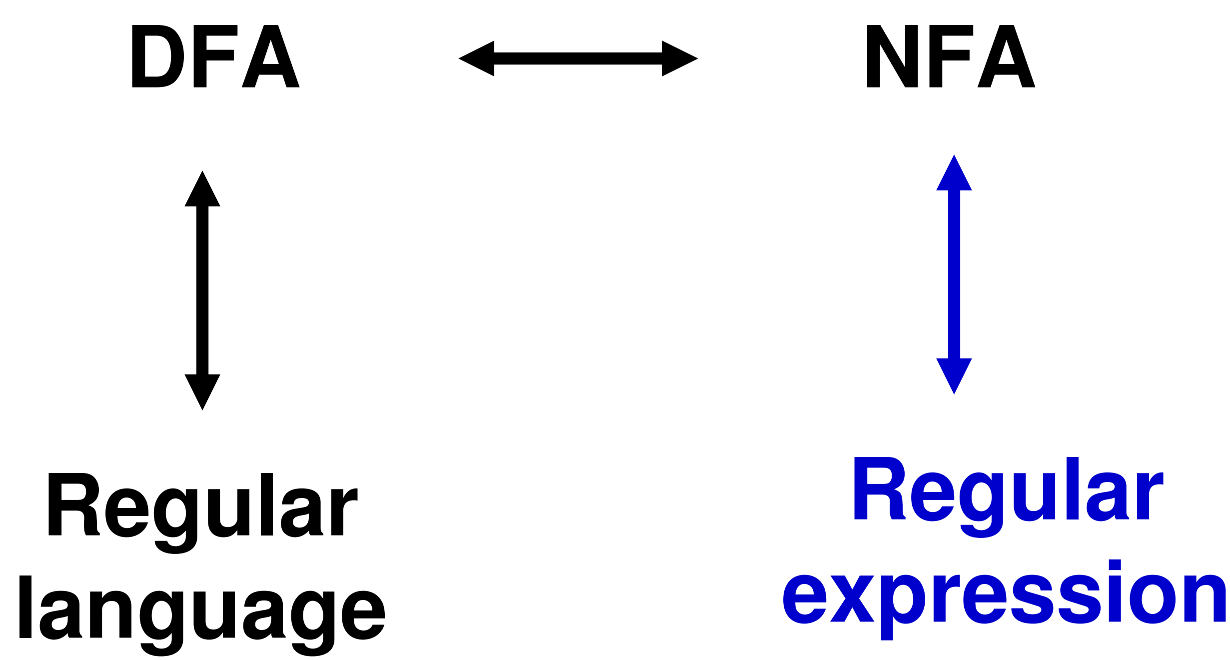
# The Picture So Far



# Today's Lecture



# Announcement

- HW1 must be dropped off in 283 Soda before 5 pm today

# Regular Expressions

- What is a regular expression?

# Regular Expressions

- Q. What is a regular expression?
- A. It’s a “textual”/ “algebraic” representation of a regular language
  - A DFA can be viewed as a “pictorial” / “explicit” representation

- We will *prove* that a regular expressions (regexps) indeed represent regular languages

# Regular Expressions: Definition

**$\sigma$ is a regular expression representing $\{\sigma\}$**
**( $\sigma \in \Sigma$ )**

**$\varepsilon$ is a regular expression representing $\{\varepsilon\}$**

**$\emptyset$ is a regular expression representing $\emptyset$**

**If $R_1$ and $R_2$ are regular expressions representing $L_1$ and $L_2$ then:**

**$(R_1 R_2)$ represents $L_1 \cdot L_2$**

**$(R_1 \cup R_2)$ represents $L_1 \cup L_2$**

**$(R_1)^*$ represents $L_1^*$**

# Operator Precedence

1. $*$
2. $\cdot$ ← ( often left out; $a \cdot b \rightarrow ab$ )
3. $\cup$

# Example of Precedence

$$\mathbf{R_1{}^*R_2 \cup R_3 = ((R_1{}^*)R_2) \cup R_3}$$

# What’s the regexp?

**{ w | w has exactly a single 1 }**

**0*10***

# What language does $\emptyset^*$ represent?

$\{\varepsilon\}$

# What's the regexp?

**{ w | w has length ≥ 3 and its 3rd symbol is 0 }**

$$\Sigma^2 \mathbf{0} \Sigma^*$$

$$\Sigma = (0 \cup 1)$$

# Some Identities

Let R, S, T be regular expressions

- $R \cup \emptyset = ?$
- $R \cdot \emptyset = ?$
- Prove: $R ( S \cup T ) = R S \cup R T$

  (what's the proof idea?)

# Some Applications of Regular Expressions

- String matching & searching
  - Utilities like grep, awk, …
  - Search in editors: emacs, …
- Programming Languages
  - Perl
  - Compiler design: lex/yacc
- Computer Security
  - Virus signatures

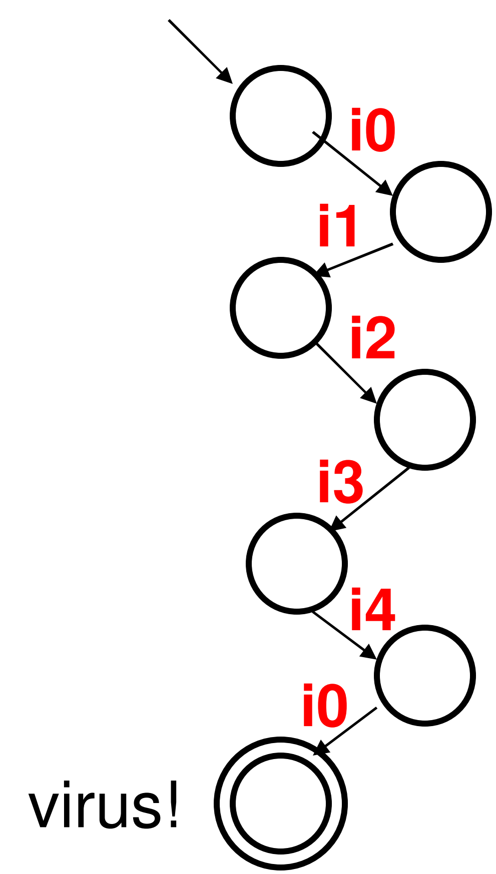
# Virus Signature as String

```
…
pop ecx
jecxz SFModMark
mov esi, ecx
mov eax, 0d601h
pop edx
pop ecx
…
```

Chernobyl virus
code fragment

Sequence of words, one
for each instruction:

i0
i1
i2
i3
i4
i0



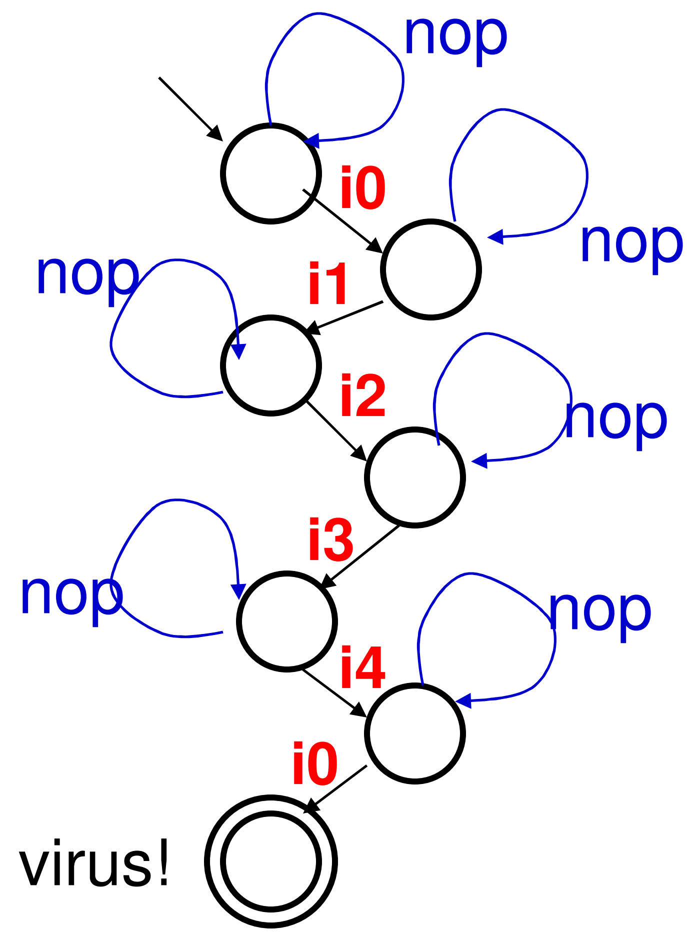
# Virus Signature as Regexp

```
…
nop
pop ecx
nop
jecxz  SFModMark
mov esi,  ecx
nop
nop
mov eax,  0d601h
pop edx
pop ecx
…
```

Simple obfuscated Chernobyl virus code fragment

Sequence of words doesn't work!

i0
i1
i2
i3
i4
i0



# Equivalence Theorem

A language is regular

$\Uparrow$ if and only if $\Downarrow$

some regular expression describes it

# Part I (“if part”)

Some regular expression R
describes a language
$\Rightarrow$
That language is regular

There exists NFA N such that R describes L(N)

# Given regular expression R, we show there exists NFA N such that R represents L(N)

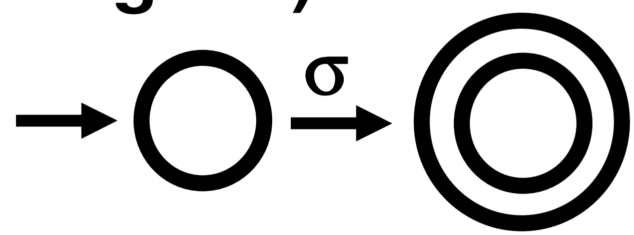
**Proof idea?**

**Given regular expression R, we show there exists NFA N such that R represents L(N)**

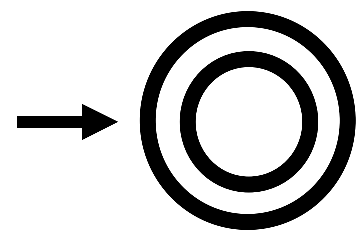
**Proof Idea: Induction on the length of R:**

**Base Cases (R has length 1):**

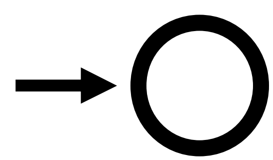
**R = $\sigma$**




**R = $\varepsilon$**



**R = $\emptyset$**



## Inductive Step:

**Assume R has length $k > 1$ and that any regular expression of length $< k$ represents a language that can be recognized by an NFA**

**What might R look like?**

$$R = R_1 \cup R_2$$

$$R = R_1 R_2$$

$$R = (R_1)^*$$

(remember: we have NFAs for $R_1$ and $R_2$)

# Part I (“if part”)

Some regular expression R
describes a language

$\Rightarrow$

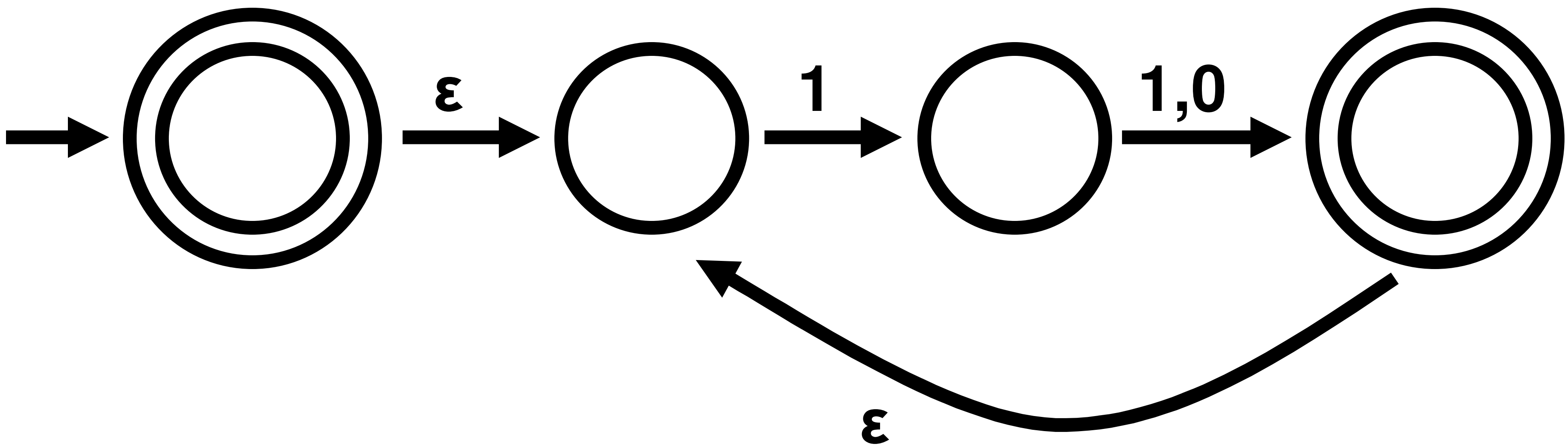
That language is regular

There exists NFA N such that R describes L(N)

**DONE !**

# An Example

**Transform $(1(0 \cup 1))^*$ to an NFA**



# Part II (“only if part”)

A language is regular

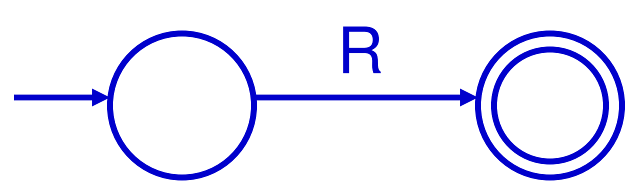
$\Rightarrow$

Some regular expression R
describes it

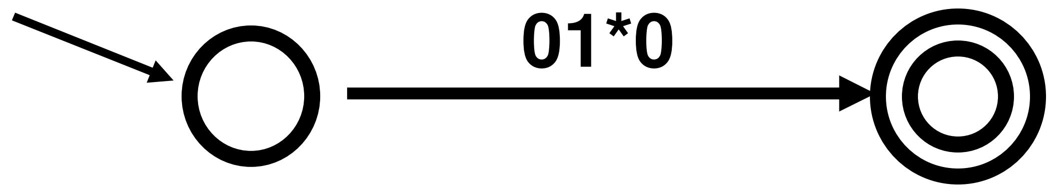
Turn DFA into equivalent regular expression

# Proof Sketch

1. DFA → Generalized NFA
   - NFA with edges labeled by regexps, 1 start state, and 1 accept state
2. GNFA with k states → GNFA with 2 states
   - k > 2; delete states but maintain equivalence
3. 2-state GNFA → regular expression R



# GNFA Example & Definition



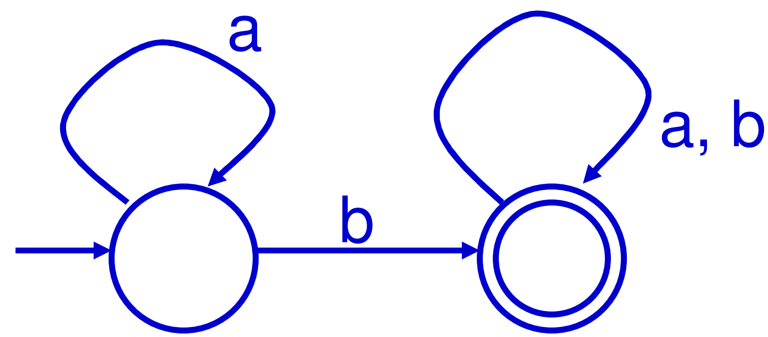

A GNFA is a tuple $(Q, \Sigma, \delta, q_{start}, q_{accept})$

- $Q$ – set of states
- $\Sigma$ – finite alphabet (not regexps)
- $q_{start}$ – initial state (unique, no incoming edges)
  - $\varepsilon$ transitions to old start state
- $q_{accept}$ – accepting state (unique, no outgoing edges)
  - $\varepsilon$ transitions from old accept states
- $\delta : (Q \setminus q_{accept}) \times (Q \setminus q_{start}) \rightarrow R$

  R – set of all regexps over $\Sigma$.

  Example: Any string matching $01^*0$ can cause the transition.

# Step 1: DFA to GNFA




What's the corresponding GNFA?

# Step 1: DFA to GNFA


**Add unique and distinct start and accept states**

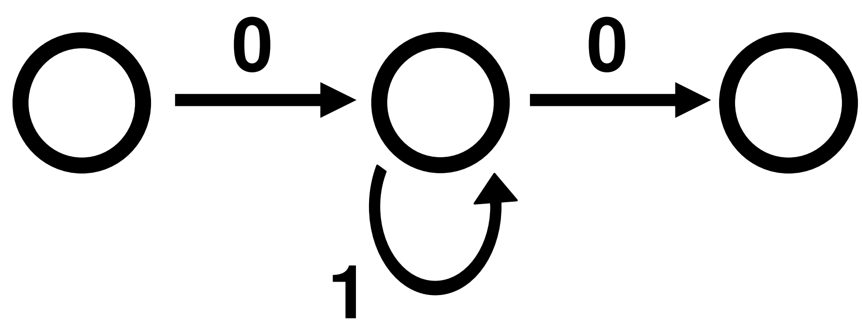
**Edges with multiple labels → regexp labels**

**If internal states ($q_1$, $q_2$) don't have an edge between them, add one labeled with ∅**

# Step 2: Eliminate states from GNFA

**While machine has more than 2 states:**

**Pick an internal state, rip it out and re-label the arrows with regular expressions to account for the missing state**



# Step 2: Eliminate states from GNFA

**While machine has more than 2 states:**

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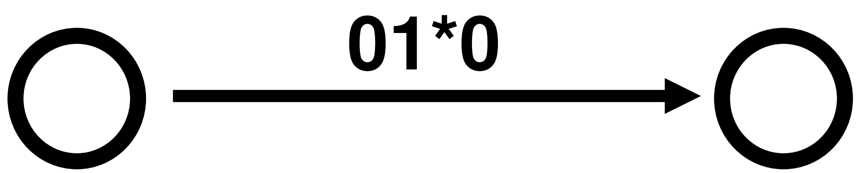
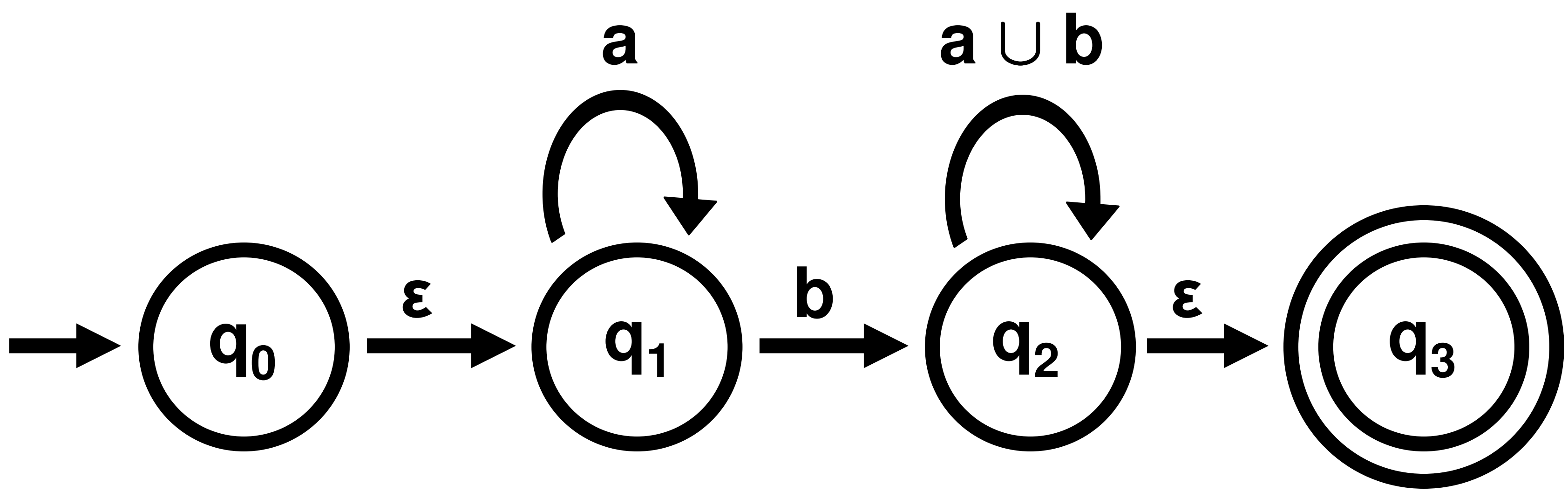
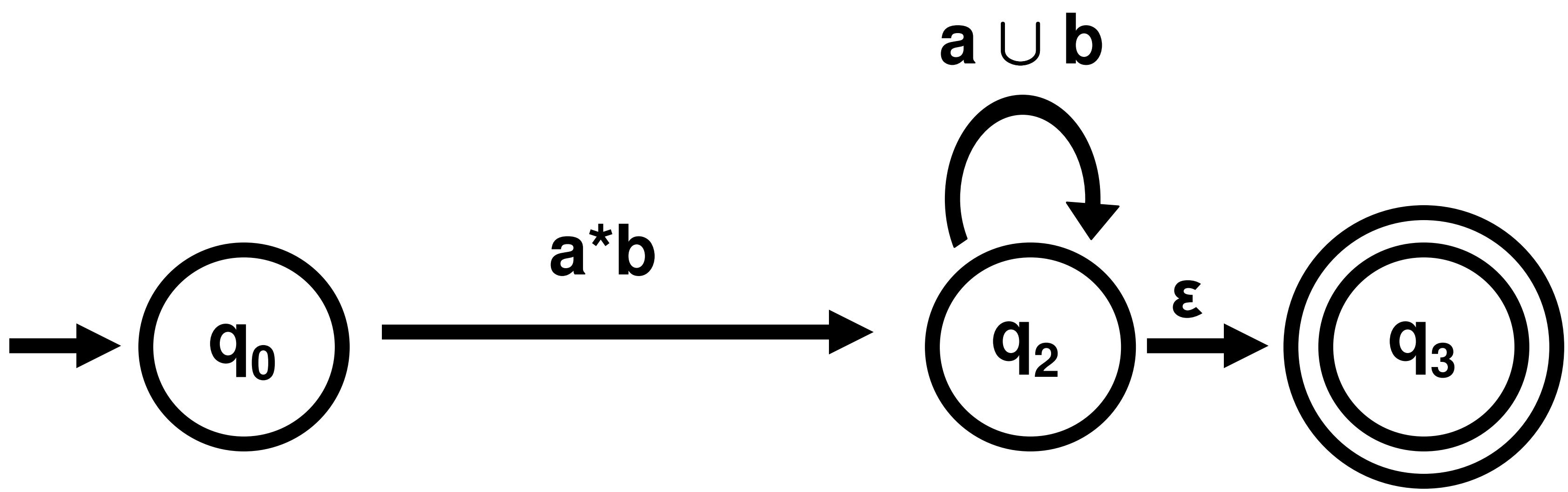
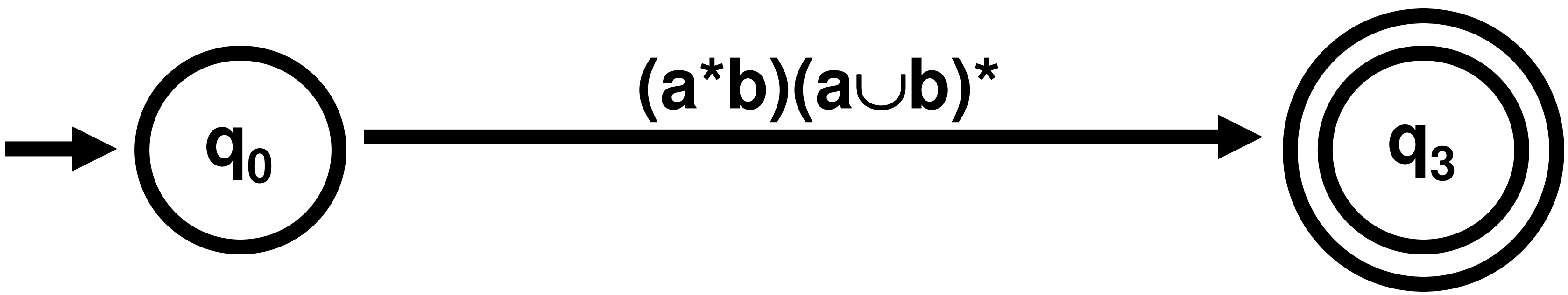



a
a ∪ b
ε
b
ε
q0
q1
q2
q3


a ∪ b
a*b
ε
q0
q2
q3




$$\delta(q_0,q_3) = (a^*b)(a\cup b)^*$$

**Formally: Add $q_{start}$ and $q_{accept}$ and create GNFA G**

**Run CONVERT(G) to eliminate states & get regexp:**

**If #states = 2**

**return the expression on the arrow going from $q_{start}$ to $q_{accept}$**

**If #states > 2**

**?**

**Formally: Add $q_{start}$ and $q_{accept}$ to create G**

**Run CONVERT(G):**

**If #states > 2**

**select $q_{rip} \in Q$ different from $q_{start}$ and $q_{accept}$**

**define $Q' = Q - \{q_{rip}\}$**

**define $\delta'$ as:**

$$\delta'(q_i,q_j) = \delta(q_i,q_{rip})\delta(q_{rip},q_{rip})^*\delta(q_{rip},q_j) \cup \delta(q_i,q_j)$$

**return CONVERT(G′) /* recursion */**

(what does this look like, pictorially?)

## Prove: CONVERT(G) is equivalent to G

**Proof *by induction on k* (number of states in G)**

**Base Case:**

- ✓ **k = 2**

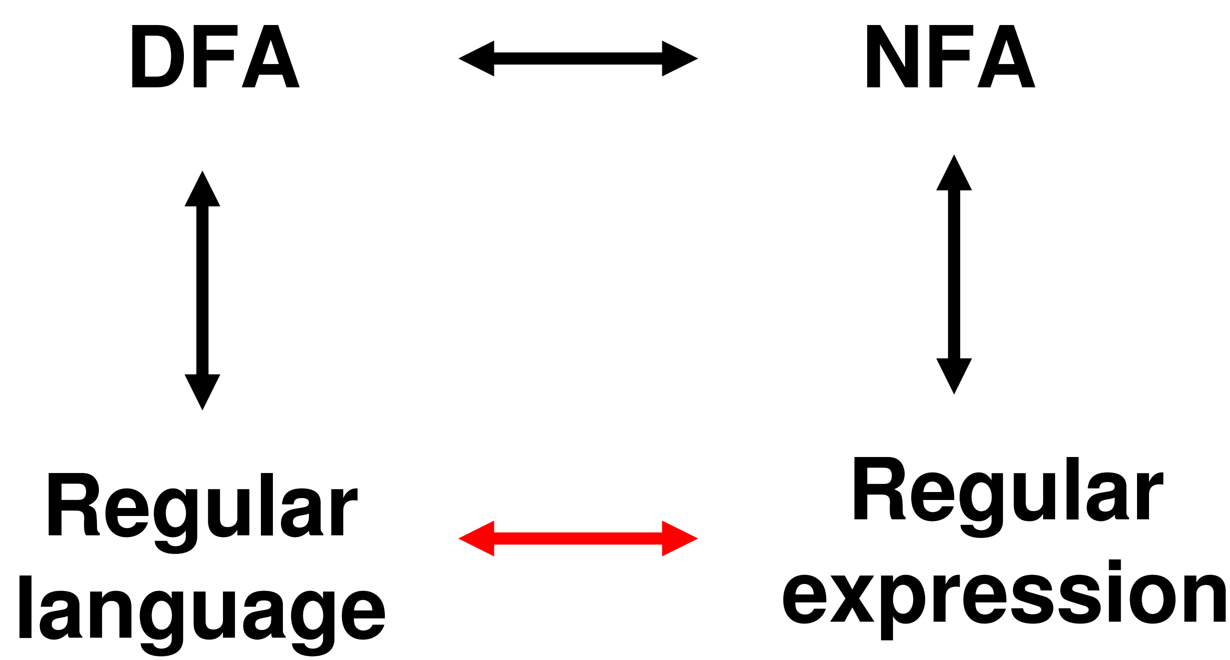
**Inductive Step:**

**Assume claim is true for k-1 states**

**Prove that G and G′ are equivalent**

**By the induction hypothesis, G′ is equivalent to CONVERT(G′)**

# The Complete Picture



# Which language is regular?

**C = { w | w has equal number of 1s and 0s}**

**NOT REGULAR**

**D = { w | w has equal number of occurrences of 01 and 10}**

**REGULAR!**

# Next Steps

- Read Sipser 1.4 in preparation for next lecture