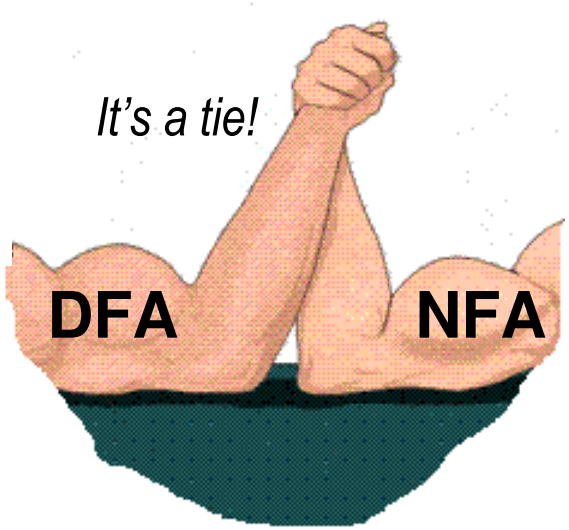
CS 172: Computability and Complexity

Equivalence of DFAs and NFAs



Sanjit A. Seshia
EECS, UC Berkeley

Acknowledgments: L.von Ahn, L. Blum, M. Blum

What we'll do today

- Prove that DFAs and NFAs are equally expressive
- Use that to prove closure of other regular operations
- Introduction to regular expressions

Recap: Closure

- If you perform an operation on one/more regular languages, is the result also a regular language?

Operations on Regular Languages

Given: Two regular languages A and B

- ✓ Union: $A \cup B = \{w \mid w \in A \text{ or } w \in B\}$
- ✓ Intersection: $A \cap B = ?$
- ✓ Complementation: $\overline{A} = \{w \mid w \notin A\}$
- Reverse: $A^R = \{w_1 w_2 \ldots w_k \mid w_k w_{k-1} \ldots w_1 \in A\}$
- Concatenation:

  $A \cdot B = \{vw \mid v \in A \text{ and } w \in B\}$
- Star:

  $A^* = \{w_1 w_2 \ldots w_k \mid k \geq 0 \text{ and each } w_i \in A\}$

Closure under Reverse

- Reverse: $A^R = \{w_1 w_2 \ldots w_k \mid w_k w_{k-1} \ldots w_1 \in A\}$

- Regular languages are closed under reverse. Here's an attempt to prove it:
  - Given M that recognizes A
  - What if you could "run it backwards"?
  - Construct M^R as M with all arrows reversed & accept state interchanged with start state
  - M^R is an NFA

A non-deterministic finite automaton (NFA) is also a 5-tuple $M = (Q, \Sigma, \delta, q_0, F)$

Q is the set of states

Σ is the alphabet

$\delta : Q \times \Sigma_\varepsilon \rightarrow 2^Q$ is the transition function

$q_0 \in Q$ is the start state

$F \subseteq Q$ is the set of accept states

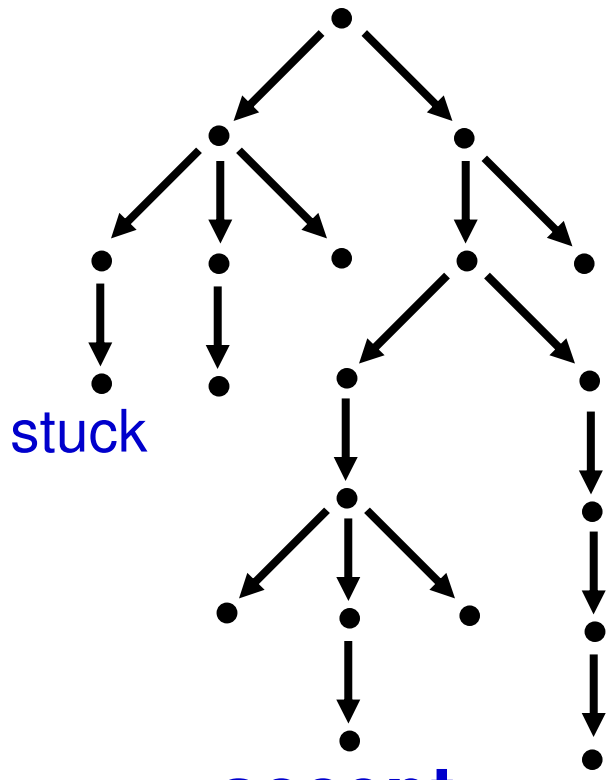
2^Q is the set of subsets of Q and $\Sigma_\varepsilon = \Sigma \cup \{\varepsilon\}$

Tree of Computations

Deterministic Computation

accept or reject

Non-Deterministic Computation



accept

reject

Equivalence

- Two automata are equivalent if their languages are the same
  - For M_1, M_2, $L(M_1) = L(M_2)$
- DFAs and NFAs:
  - For every NFA there is an equivalent DFA (we'll prove this) and vice-versa (this is easy, why?)

Theorem: **Every NFA has an equivalent DFA**

Corollary: A language is regular iff it is recognized by an NFA

Corollary: L is regular iff L^R is regular
(need to also prove that M^R recognizes L^R)

From NFA to DFA

- Proof Hints:
  - Proof by construction

    Given an arbitrary NFA N, construct an equivalent DFA M

  - Proof by induction

    N accepts a word w iff M accepts w

FROM NFA TO DFA

Input: $N = (Q, \Sigma, \delta, q_0, F)$

Output: $M = (Q', \Sigma, \delta', q_0', F')$

$Q' = ?$

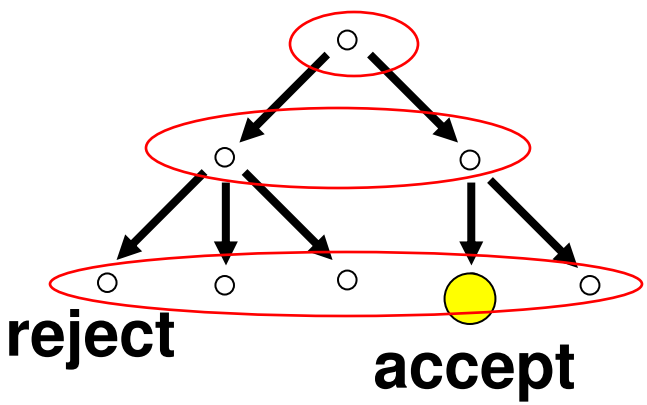
From NFA to DFA

Input: $N = (Q, \Sigma, \delta, q_0, F)$

Output: $M = (Q', \Sigma, \delta', q_0', F')$

Idea:

$$Q' = 2^Q$$

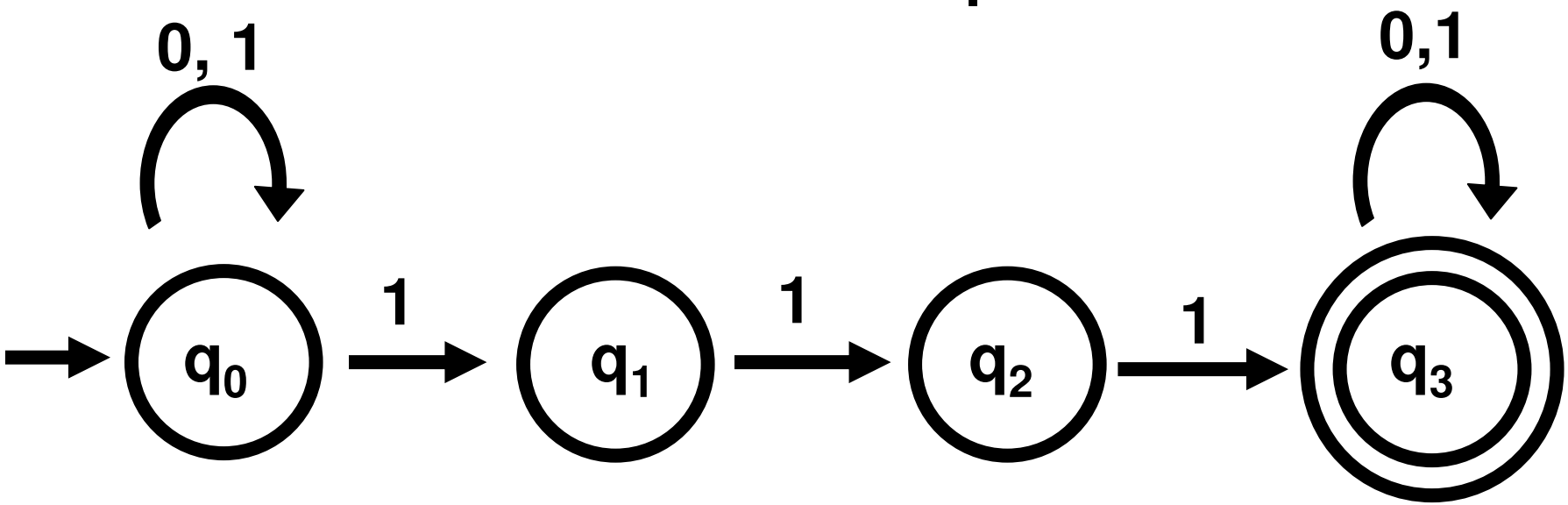


Assume (for now) that there are no ε-transitions

Each non-stuck path in the computation tree is of equal length

Do a BFS (breadth-first search) on this tree, tracking the "set of states" transitioned to

NFA Example



Run on 1110

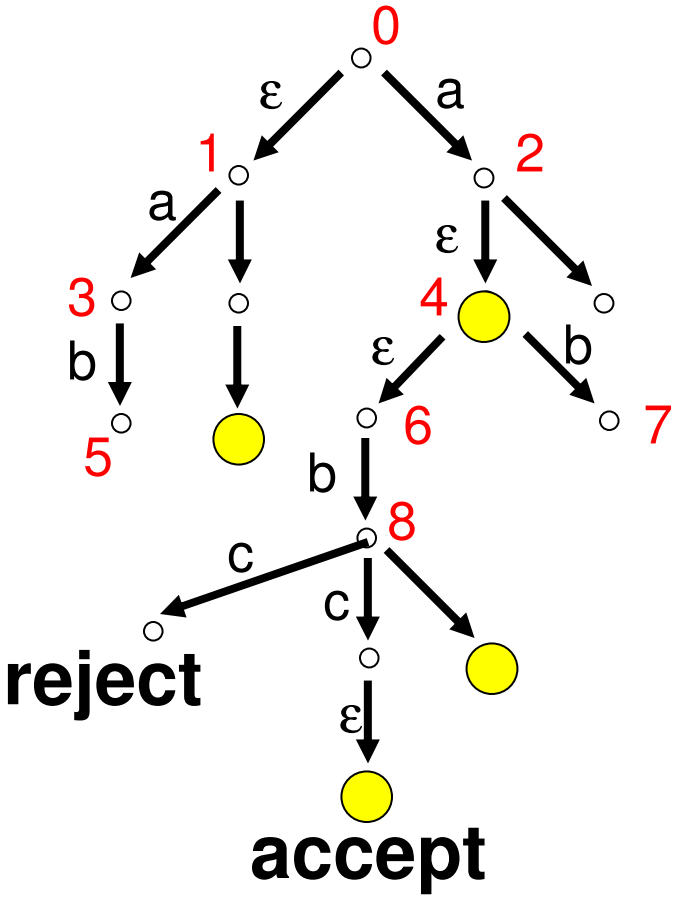
From NFA to DFA

Input: $N = (Q, \Sigma, \delta, q_0, F)$

Output: $M = (Q', \Sigma, \delta', q_0', F')$

Idea:

$$Q' = 2^Q$$



What if we had ε-transitions?

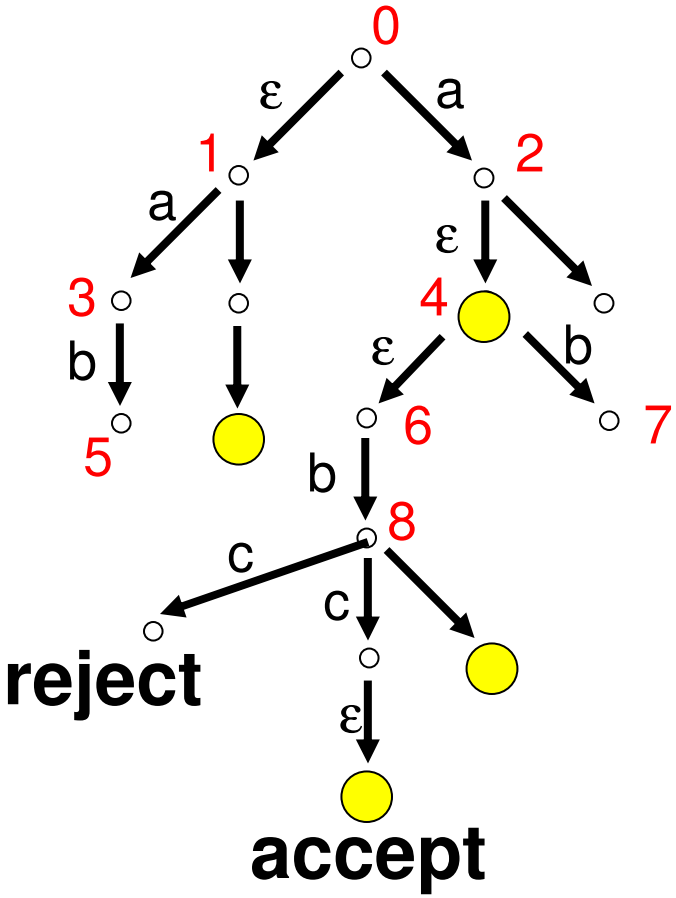
From NFA to DFA

Input: $N = (Q, \Sigma, \delta, q_0, F)$

Output: $M = (Q', \Sigma, \delta', q_0', F')$

Idea:

$$Q' = 2^Q$$



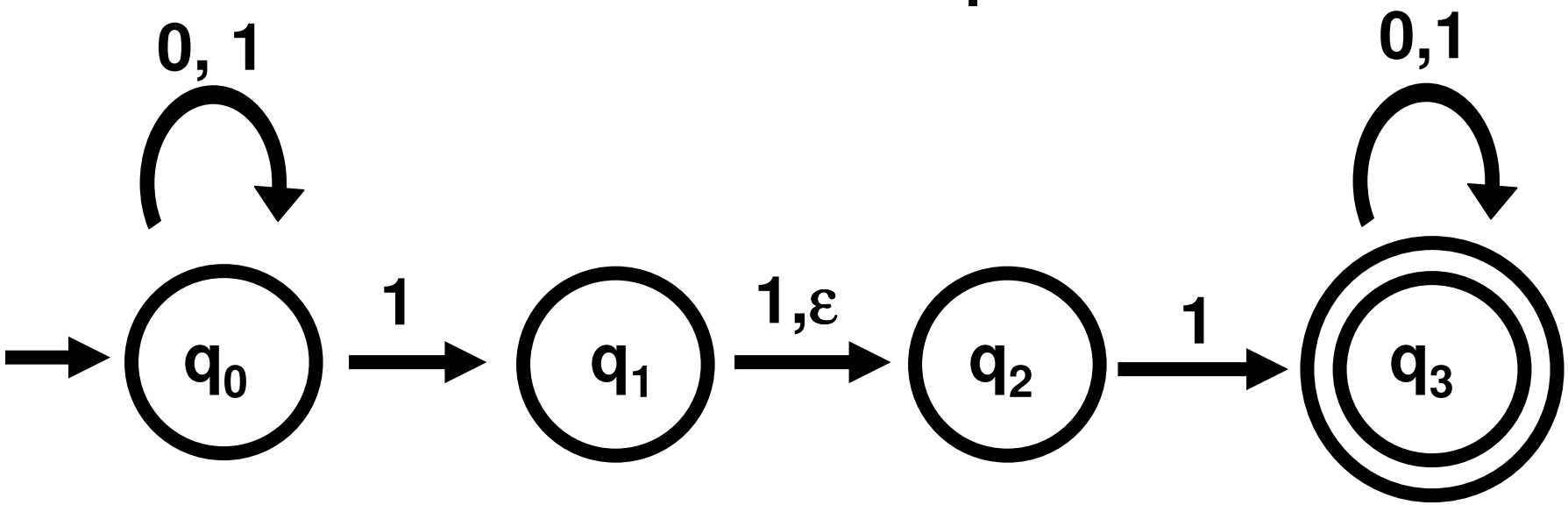
What if we had ε-transitions?

After reading an input symbol, follow ε-transitions until you can't any more

Given a set S in 2^Q, E(S) is the set of all states reached from S by following ε-transitions

- E(S) is called the ε-closure of S

NFA Example



Run on 1110

From NFA to DFA

Input: $N = (Q, \Sigma, \delta, q_0, F)$

Output: $M = (Q', \Sigma, \delta', q_0', F')$

$$Q' = 2^Q$$

$$\delta' : Q' \times \Sigma \rightarrow Q'$$

$$\delta'(R,\sigma) = \bigcup_{r \in R} E(\ \delta(r,\sigma)\ )$$

$$q_0' = ?$$

$$F' = ?$$

From NFA to DFA

Input: $N = (Q, \Sigma, \delta, q_0, F)$

Output: $M = (Q', \Sigma, \delta', q_0', F')$

$$Q' = 2^Q$$

$$\delta' : Q' \times \Sigma \rightarrow Q'$$

$$\delta'(R,\sigma) = \bigcup_{r \in R} E(\ \delta(r,\sigma)\ )$$

$$q_0' = E(\{q_0\})$$

$$F' = \{\ R \in Q' \mid f \in R \text{ for some } f \in F\ \}$$

(read details of the construction in Sipser)

From NFA N to DFA M

- Construction is complete
- But the proof isn't: Need to prove

  N accepts a word w iff M accepts w

- Use *structural induction* on the length of w, |w|
  - Base case: |w| = 0
  - Induction step: Assume for |w| = n, prove for |w| = n+1

Useful Definition

- Let $w \in \Sigma^*$
- For an NFA N:
  - $\hat{\delta}(q, w)$ = set of states reached by executing N on w starting from q
  - Note that a state of N is in Q
- For *the corresponding DFA* M:
  - $\hat{\delta}'(q', w)$ = state reached by executing M on w starting from q'
  - Note that a state of M is in 2^Q

NFA to DFA: Complexity

- If the original NFA N has n states, how large *can* the corresponding DFA M be?

NFA to DFA: Complexity

- If the original NFA N has n states, how large *can* the corresponding DFA M be?
  - Answer: 2^n states
  - Exercise: construct an example where N has n states and M has $\Theta(2^n)$ states

Remaining Operations

Given: Two regular languages A and B

- Concatenation:

  $A \cdot B = \{vw \mid v \in A \text{ and } w \in B\}$

- Star:

  $A^* = \{w_1 w_2 \ldots w_k \mid k \geq 0 \text{ and each } w_i \in A\}$

Closure under Concatenation

Given DFAs M_1 and M_2, how can we construct an NFA N for $L(M_1) \cdot L(M_2)$?

Closure under Concatenation

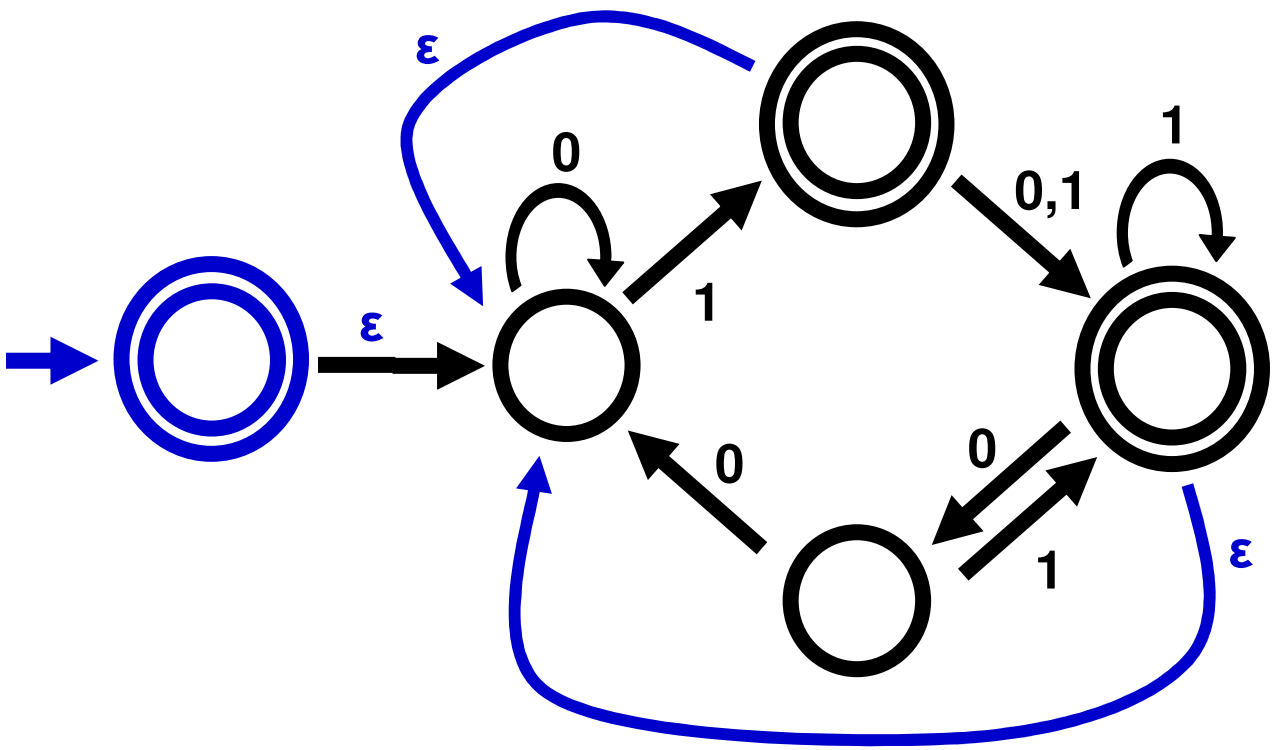
Given DFAs M_1 and M_2, construct NFA N by connecting all accept states in M_1 to start states in M_2

- **What are accept states of N?**

Closure under Star

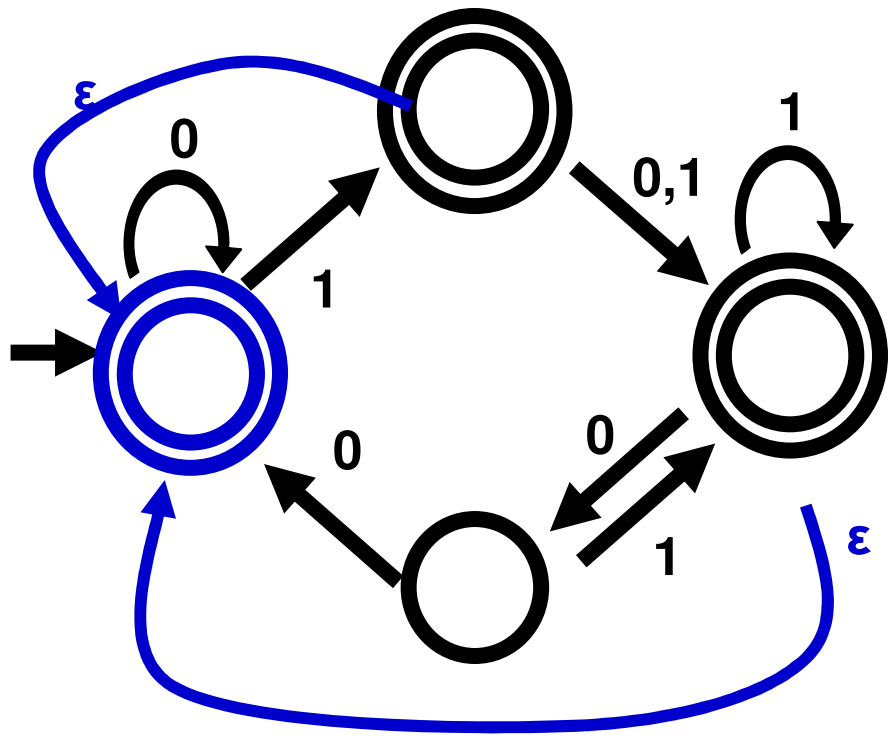
Let L be a regular language and M be a DFA for L

How do we construct an NFA N that recognizes L* ?



Closure under Star

Why not the following?



Formally:

Input: DFA $M = (Q, \Sigma, \delta, q_1, F)$

Output: NFA $N = (Q', \Sigma, \delta', \{q_0\}, F')$

$$Q' = Q \cup \{q_0\}$$

$$F' = F \cup \{q_0\}$$

$$\delta'(q,a) = \begin{cases} \{\delta(q,a)\} & \text{if } q \in Q \text{ and } a \neq \varepsilon \\ \{q_1\} & \text{if } q \in F \text{ and } a = \varepsilon \\ \{q_1\} & \text{if } q = q_0 \text{ and } a = \varepsilon \\ \varnothing & \text{if } q = q_0 \text{ and } a \neq \varepsilon \\ \varnothing & \text{else} \end{cases}$$

REGULAR LANGUAGES ARE CLOSED UNDER **REGULAR OPERATIONS**

Union: $A \cup B = \{ w \mid w \in A \text{ or } w \in B \}$

Intersection: $A \cap B = \{ w \mid w \in A \text{ and } w \in B \}$

Reverse: $A^R = \{ w_1 \ldots w_k \mid w_k \ldots w_1 \in A \}$

Complementation: $\overline{A} = \{ w \mid w \notin A \}$

Concatenation: $A \cdot B = \{ vw \mid v \in A \text{ and } w \in B \}$

Star: $A^* = \{ w_1 \ldots w_k \mid k \geq 0 \text{ and each } w_i \in A \}$

Regular Expressions

What does Language D look like?

D = { w | w has equal number of occurrences of 01 and 10}

w should "toggle" between 0 and 1 an equal number of times

How about: 0, 1, 011, 0110, ε -- are they in D?

What does Language D look like?

D = { w | w has equal number of occurrences of 01 and 10}

= { w | w = 1, w = 0, w = ε or
w starts with a 0 and ends with a 0 or
w starts with a 1 and ends with a 1 }

$$\mathbf{1} \cup \mathbf{0} \cup \boldsymbol{\varepsilon} \cup (\mathbf{0}\Sigma^*\mathbf{0}) \cup (\mathbf{1}\Sigma^*\mathbf{1})$$

$\Sigma = \{0,1\}$

REGULAR **EXPRESSIONS**

σ is a regular expression representing $\{\sigma\}$ ($\sigma \in \Sigma$)

ε is a regular expression representing $\{\varepsilon\}$

$\varnothing$ is a regular expression representing $\varnothing$

If R_1 and R_2 are regular expressions representing L_1 and L_2 then:

(R_1R_2) represents $L_1 \cdot L_2$

$(R_1 \cup R_2)$ represents $L_1 \cup L_2$

$(R_1)^*$ represents L_1^*

Next Steps

- Read Sipser 1.3 in preparation for next lecture