Today

Experts/Zero-Sum Games Equilibrium.

Boosting and Experts.

Routing and Experts.

Linear Programming Introduction (Gentle)

Games and experts

Again: find (x^*, y^*), such that

$(\max_y x^* Ay) - (\min_x x^* Ay^*) \leq \varepsilon$

$C(x^*) \quad - \quad R(y^*) \leq \varepsilon$

Experts Framework:
n Experts, T days, L^* -total loss of best expert.

Multiplicative Weights Method yields loss L where

$L \leq (1+\varepsilon)L^* + \frac{\log n}{\varepsilon}$

Games and Experts.

Assume: A has payoffs in $[0,1]$.

For $T = \frac{\log n}{\varepsilon^2}$ days:

1) m pure row strategies are experts.
Use multiplicative weights, produce row distribution.
Let x_t be distribution (row strategy) x_t on day t.

2) Each day, adversary plays best column response to x_t.
Choose column of A that maximizes row's expected loss.
Let y_t be indicator vector for this column.

Let $y^* = \frac{1}{T}\sum_t y_t$ and $x^* = \text{argmin}_{x_t} x_t A y_t$.

Let $y^* = \frac{1}{T}\sum_t y_t$ and $x^* = \frac{1}{T}\sum_t x_t$.

Approximate Equilibrium: slightly different!

Experts: x_t is strategy on day t, y_t is best column against x_t.

Let $x^* = \frac{1}{T}\sum_t x_t$ and $y^* = \frac{1}{T}\sum_t y_t$.

Claim: (x^*, y^*) are 2ε-optimal for matrix A.

Column payoff: $C(x^*) = \max_y x^* Ay$.
Let y_r be best response to $C(x^*)$.
Day t, $x_t A y_t \geq x_t A y_r$. Since y_t is best response to x_t.
Algorithm loss: $\sum_t x_t A y_t \geq \sum_t x_t A y_r$
$L \geq T \times C(x^*)$.

Best expert: L^*- best row against all the columns played.

best row against $\sum_t Ay_t$ and $Ty^* = \sum_t y_t$
$\rightarrow$ best row against TAy^*.
$\rightarrow L^* \leq T \times R(y^*)$.

Multiplicative Weights: $L \leq (1+\varepsilon)L^* + \frac{\ln n}{\varepsilon}$

$TC(x^*) \leq (1+\varepsilon)TR(y^*) + \frac{\ln n}{\varepsilon} \rightarrow C(x^*) \leq (1+\varepsilon)R(y^*) + \frac{\ln n}{\varepsilon T}$
$\rightarrow C(x^*) - R(y^*) \leq \varepsilon R(y^*) + \frac{\ln n}{\varepsilon T}$.

$T = \frac{\ln n}{\varepsilon^2}$, $R(y^*) \leq 1 \rightarrow C(x^*) - R(y^*) \leq 2\varepsilon$.

Comments

For any ε, there exists an ε-Approximate Equilibrium.

Does an equilibrium exist? Yes.

Something about math here? Fixed point theorem.

Later: will use geometry, linear programming.

Complexity?
$T = \frac{\ln n}{\varepsilon^2} \rightarrow O(nm\frac{\log n}{\varepsilon^2})$. Basically linear!

Versus Linear Programming: $O(n^3 m)$ Basically quadratic.
(Faster linear programming: $O(\sqrt{n+m})$ linear solution solves.)
Still much slower ... and more complicated.

Dynamics: best response, update weight, best response.

Also works with both using multiplicative weights.

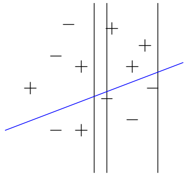
"In practice."

Boosting...

Learning

Learning just a bit.

Example: set of labelled points, find hyperplane that separates.



Looks hard.

$1/2$ of them? Easy.
Arbitrary line. And Scan.

Useless. A bit more than $1/2$ Correct would be better.

Weak Learner: Classify $\geq \frac{1}{2}+\varepsilon$ points correctly.

Not really important but ...

Weak Learner/Strong Learner

Input: n labelled points.

Weak Learner:
produce hypothesis correctly classifies $\frac{1}{2}+\varepsilon$ fraction

Strong Learner:
~~produce hyp. correctly classifies $1+\mu$ fraction~~

That's a really strong learner!
produce hypothesis correctly classifies $1-\mu$ fraction

Same thing?

Can one use weak learning to produce strong learner?

Boosting: use a weak learner to produce strong learner.

Poll.

Given a weak learning method (produce ok hypotheses.)
produce a great hypothesis.

Can we do this?

(A) Yes
(B) No

If yes. How?

Multiplicative Weights!

The endpoint to a line of research.

Boosting/MW Framework

Experts are points. "Adversary" weak learner.

Points (experts) suffer loss when classified correctly.

Learner (adversary) wants to maximize probability
of classifying random point correctly.
Strong learner algorithm will come from adversary.

Do $T=\frac{2}{\gamma^2}\ln\frac{1}{\mu}$ rounds

1. Row player: multiplicative weights($1-\gamma$) on points.
2. Column: run weak learner on row distribution.
3. Hypothesis $h(x)$: majority of $h_1(x), h_2(x), \ldots, h_T(x)$.

Claim: $h(x)$ is correct on $1-\mu$ of the points ! ! !

Cool!

Really? Proof?

Adaboost proof.

Claim: $h(x)$ is correct on $1-\mu$ of the points ! ! !

Let S_{bad} be the set of points where $h(x)$ is incorrect.
majority of $h_t(x)$ are wrong for $x \in S_{bad}$.

$x \in S_{bad}$ is a good expert – loses less than $\frac{1}{2}$ the time.

$W(T) \geq (1-\varepsilon)^{\frac{T}{2}}|S_{bad}|$

Each day, weak learner gets $\geq \frac{1}{2}+\gamma$ payoff.

$\to L_t \geq \frac{1}{2}+\gamma.$

$\to W(T) \leq n(1-\varepsilon)^L \leq ne^{-\varepsilon L} \leq ne^{-\varepsilon(\frac{1}{2}+\gamma)T}$

Combining

$|S_{bad}|(1-\varepsilon)^{T/2} \leq W(T) \leq ne^{-\varepsilon(\frac{1}{2}+\gamma)T}$

Calculation..

$|S_{bad}|(1-\varepsilon)^{T/2} \leq ne^{-\varepsilon(\frac{1}{2}+\gamma)T}$

Set $\varepsilon=\gamma$, take logs.

$\ln\left(\frac{|S_{bad}|}{n}\right)+\frac{T}{2}\ln(1-\gamma) \leq -\gamma T(\frac{1}{2}+\gamma)$

Again, $-\gamma-\gamma^2 \leq \ln(1-\gamma)$,

$\ln\left(\frac{|S_{bad}|}{n}\right)+\frac{T}{2}(-\gamma-\gamma^2) \leq -\gamma T(\frac{1}{2}+\gamma) \to \ln\left(\frac{|S_{bad}|}{n}\right) \leq -\frac{\gamma^2 T}{2}$

And $T=\frac{2}{\gamma^2}\log\mu$,

$\to \ln\left(\frac{|S_{bad}|}{n}\right) \leq \log\mu \to \frac{|S_{bad}|}{n} \leq \mu.$

The misclassified set is at most μ fraction of all the points.

The hypothesis correctly classifies $1-\mu$ of the points ! ! !

Claim: Multiplicative weights: $h(x)$ is correct on $1-\mu$ of the points ! ! !

Some details...

Weak learner learns over distributions of points not points.

Make copies of points to simulate distributions.

Used often in machine learning.
Blending learning methods.

Toll/Congestion

Given: $G = (V, E)$.
Given $(s_1, t_1) \ldots (s_k, t_k)$.
Row: choose routing of all paths.
Column: choose edge.
Row pays if column chooses edge on any path.

Matrix:
row for each routing: r
column for each edge: e

$A[r, e]$ is congestion on edge e by routing r

Offense: (Best Response.)
Router: route along shortest paths.
Toll: charge most loaded edge.

Defense: Toll: maximize shortest path under tolls.
Route: minimize max congestion on any edge.

Two person game.

Row is router.

An exponential number of rows!

Two person game with experts won't be so easy to implement.

Version with row and column flipped may work.

$A[e, r]$ - congestion of edge e on routing r.

m rows. Exponential number of columns.

Multiplicative Weights only maintains m weights.

Adversary only needs to provide best column each day.

Runtime only dependent on m and T (number of days.)

Congestion minimization and Experts.

Will use gain and $[0, \rho]$ version of experts:

$G \geq (1-\varepsilon)G^* - \frac{\rho \log n}{\varepsilon}$.

Let $T = \frac{k \log n}{\varepsilon^2}$

1. Row player runs multiplicative weights on edges:
   $w_i = w_i(1+\varepsilon)^{g_i/k}$.
2. Column routes all paths along shortest paths.
3. Output the average of all routings: $\frac{1}{T}\sum_t f(t)$.

Claim: The congestion, c_{max} is at most $C^* + 2k\varepsilon$.

Proof:
$G \geq G^*(1-\varepsilon) - \frac{k \log n}{\varepsilon T} \rightarrow G^* - G \leq \varepsilon G^* + \frac{k \log n}{\varepsilon}$

$G^* = T * c_{\max}$ – Best row payoff against average routing (times T).

$G \leq T \times C^*$ – each day, gain is avg. congestion $\leq$ opt congestion.

$T = \frac{k \log n}{\varepsilon^2} \rightarrow Tc^*_{\max} - TC \leq \varepsilon TC^* + \frac{k \log n}{\varepsilon} \rightarrow$
$c_{max} - C^* \leq \varepsilon C^* + \varepsilon$ □

Better setup.

Runtime: $O(km \log n)$ to route in each step (using Dijkstra's)
$O(\frac{k \log n}{\varepsilon^2})$ steps
to get $c_{\max} - C^* < \varepsilon C^*$ (assuming $C^* > 1$) approximation.
To get constant c error.
$\rightarrow O(k^2 m \log n / \varepsilon^2)$ to get a constant approximation.
(Similar to homework 2 bound that you will get.)

Homework 3: $O(km \log n)$ algorithm ! ! !

Fractional versus Integer.

Did we (approximately) solve path routing?
Yes? No?

No! Average of T routings.
We approximately solved fractional routing problem.

No solution to the path routing problem that is $(1+\varepsilon)$ optimal!

Homework 2. Problem 1.

Decent solution to path routing problem?

For each s_i, t_i, choose path p_i uniformly at random from "daily" paths.

Congestion $c(e)$ edge has expected congestion, $\tilde{c}(e)$, of $c(e)$.

"Concentration" (law of large numbers)
$c(e)$ is relatively large ($\Omega(\log n)$)
$\rightarrow \tilde{c}(e) \approx c(e)$.

Concentration results? later.

Portfolio Management.

Every day, choose one of n stocks to invest all your money in.

c_i^t - price of stock on day t, and end of day for $t-1$.

If invest P in stock i, on day t.

Have $\frac{c_i^{(t)}}{c_i}P$ next day $\quad (r_i^{(t)} = \frac{c_i^{(t)}}{c_i}.)$

Experts/multiplicative weights: loss/gains are additive.
Loss/Gain is $\log r$.
Total loss is $\sum_t r^{(t)}$ where $r^{(t)}$ is return on day t.

MW: Gives bound on **expected** loss.

$\sum_t \sum_i P_i^{(t)} \log r(t)_i$ where $P_i^{(t)}$ is MW distribution on day t.

$\frac{\log x + \log y}{2} \leq \log(\frac{x+y}{2}) \implies \sum_i P_i^{(t)} \log r_i^{(t)} \leq \log \sum_i P_i^{(t)} r_i^{(t)}$.

Thus expected log of the ratio of the algorithm to the best stock is within $O(\sqrt{\frac{\log n}{T}})$ of the best. ($\log r \leq 1$).

See you on Tuesday.